How to Squander Your Endowment Pitfalls and Remedies.pdf

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How to Squander Your Endowment: Pitfalls and Remedies

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Abstract

University donors choose to contribute to endowment if they want to make a permanent contribution to the university. It is consequently viewed as a responsibility of the university to preserve capital when choosing the investments and spending rule of endowments. Practitioners commonly model the preservation-of-capital constraint by looking at the excess of expected return over the spending rate, but this criterion involves an incorrect application of the law of large numbers based on products instead of sums. The measure can be corrected by looking expected log return net of spending, which is less by approximately half the variance of returns if period returns are not too volatile. Even if the correct target spending rule is applied, the common practice of smoothing spending using a partial adjustment model for spending tends to make spending unstable in bad times and in fact the probability of eventual ruin is one. However, we show that a simple modification to the traditional smoothing rule does preserve capital.

Preliminary and incomplete

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1 Introduction

Donors who wish to contribute to universities have a number of options depending on when they want their giving to have an impact. For example, donors wanting to have an immediate impact can contribute through annual giving, donors who want to have an impact on an intermediate time frame can give funds for a building, and donors who want to have a permanent impact can contribute to endowment. Since contributions to endowment are supposed to have a permanent impact, the university has a responsibility to make sure that the spending rule and investment strategy for endowment, taken together, preserve capital, and preservation of capital is viewed as a constraint on universities’ choice of policy. This paper takes a look at preservation of capital with a focus on existing practice. We find that the usual criterion (spending less than expected real return) for preservation of capital is incorrect because it is based on an incorrect application of the law of large numbers, and is consistent with policies for which wealth always tends to zero over time. We provide a corrected formula based on positivity of the log return net of spending. We also show that a stylized version of the practice of smoothing spending implies that the endowment never preserves capital, and we show how to modify the smoothing rule to preserve capital.

A spending rate less than the expected return on assets, calculated in real terms, has long been used as a criterion for whether an endowment preserves capital. This criterion is based on the intuition of the law of large numbers, since it means that on average the expected return on the portfolio should cover spending. However, this intuition implicitly represents a mis-application of the law of large numbers, since the law of large numbers applies to sums but the portfolio problem involves products. In particular, the proportional change in value in a period is one plus the return less spending, and these returns multiply over time. However, the law of large numbers does not apply to products. The difference is significant: in continuous time or approximately in discrete

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1It is possible but complicated to write down an optimization problem that includes preservation of capital as a constraint. However, the traditional definition of preservation of capital used in this paper is insufficient for that task, as we show in Section 5.
time models if the noise in returns is close to zero, the traditional criterion says the spending rate should be less than the mean return, while the correct criterion says the spending rate should be no larger than the mean less half the variance.

We say a policy destroys (preserves) capital if the value of a unit of the endowment in real terms goes to zero (infinity) over time in probability. These definitions are motivated by the intuition for the traditional criterion.

If we incorrectly apply the rule of large numbers to products, spending less than the mean return would imply that wealth grows without bound over time, while spending more than the mean return would send wealth to zero over time. Our definitions incorporate two reasonable features normally used in practice: (1) we use real “inflation-adjusted” returns since capital must be preserved in terms of spending power and not just nominal units, and (2) we look at the value of a unit of endowment and do not include future contributions: a particular donor’s contribution to endowment must make a permanent contribution, not just a temporary contribution that will be replaced by other temporary contributions in the future.

While the traditional criterion does not ensure that capital is preserved, we provide a simple alternative criterion that does. Taking logarithms converts products into sums, and capital is preserved if the expected log return net of spending, defined as the expected log of one plus the return less the spending rate, is positive. This criterion preserves capital since it implies that the value of the endowment arising from an initial investment grows without limit over time if this assumption is true. Noticeably, we provide a reasonable example in which the admissible spending rate is reduced by 1%, which implies that endowments may need to cut off around 20% of their spending if they currently spend about 5% of their capital.

Besides looking at the basic spending criterion, we also look at the common practice of overlaying smoothing on the basic spending rule. Smoothing of spending is supposed to prevent the damage done by large fluctuations in spending. This is a reasonable idea: sudden decreases in spending are disruptive, and sudden increases may be used carelessly. Unfortunately, the usual partial adjustment rule of moving
only a fixed fraction of the way toward the target spending level never preserves capital in the endowment if the target spending rate is positive (even if very small) and the portfolio choice is constant\(^2\). This result is based on a continuous-time model in which that portfolio returns are randomly drawn from the same distribution and are independent over time. Intuitively, random fluctuations imply that the spending rate will eventually be very large, and when the spending rate is large, the high spending depletes capital relatively more quickly than the smoothing reduces spending, and as a result the portfolio ends up in a “death spiral” plunging towards zero.

Since smoothing is a good idea and the traditional smoothing rule does not preserve capital, we have proposed a possible solution. A simple modified smoothing rule that adds a new term that changes spending to compensate for the expected change in spending rate given the excess of current spending over the expected return of assets. For this rule, we have a characterization of the parameter values for which capital is preserved. Moreover, an interest rate environment like the current one where inflation exceeds the nominal rate is a special challenge, but there is a simple result: given some stationarity, expected log return of spending does not have to be positive every period, and only has to be positive on average.

The traditional endowment decision making process does not see an obvious link between the asset allocation strategies and spending rules (Dybvig 1999). On one hand, endowments adopt simple spending rules which let spending to be proportional to the wealth, or more advanced rules, e.g., moving average smooth spending rule. One the other hand, endowments usually have either target proportions in different asset classes or target ranges of proportions. Basic on the simples rules, parameters are chosen to preserve capital using the traditional criterion. In this paper, we intend to propose correct rules which easily fits into the practical decision environment. As a result, we model a fixed proportion of wealth in the risky asset, which is qualitatively similar to what endowments do in practice.

\(^{2}\)Having a portfolio choice that does not respond to the current spending rate is probably not optimal, see Dybvig (1999), but this is the usual practices.
This paper investigates rules violating presentation of capital. The focus is on the necessary conditions endowments need to meet, i.e., preserving capital as they promise to the donors when donating money. This contrasts to the usual optimal investment approach which maximizes a utility function subjects to some wealth constraints. Optimization approaches seem more ambitious and less realistic than just studying conditions for satisfying constraints. However, unfortunately, any useful optimization model for endowments tends to be too complicated and intractable. Since such a useful model should at least captures two key important characteristics for endowments: preservation of capital and the smoothed spending. While incorporating these two features simultaneously leads to models which are hard to solve and interpret.

We prove that our definition of preservation of capital is not adequate when using in an optimization model as a constraint. Since the constraint only imposes the condition needs to meet in the infinite future, without setting any restrictions on the wealth process on the intermediate dates. To obtain a positive amount of wealth at infinity, one can employ strategies that guarantee a nonnegative value at any finite date $T$, and from that date, invests part of the wealth into the riskless asset. Theoretically, this amount of wealth converges to infinity over time exponentially with riskless rate of return.

The rest of the paper is arranged as follows. Section 2 documents the problem with the traditional rule for preserving capital and provides the new correct rule. Section 3 shows that traditional smoothing implies capital is not preserved. We provide a modified smooth spending rule that preserves capital. Section 4 comes up with the condition for preserving capital with temporarily negative risk-free rate. Section 5 discusses optimization model of spending and investment that preserve capital not written yet. Section 6 closes the paper.
2 Spending Rate Less Than Expected Return

In the following subsection, we present a reasonable definition of preservation of capital that will be used in most of the paper.

2.1 Definition of Preservation of Capital

To characterize preservation of capital, we require a formal definition of what this means. Fortunately, most of our results will be robust to a range of reasonable choices for how we define preservation of capital. We study the management of a unit of endowment, with a relative change equalling the investment return less the spending rate, but not including any new contributions. Looking at a unit without credits for subsequent contributions is standard in practice and it is important because we are looking for a contribution to have a permanent impact. It is annual giving, not a permanent contribution to endowment, if we spend the entire contribution this year and replace it using future contributions.

We let $W_t$ be the real (inflation-adjusted) value of wealth in the unit at time $t$ with spending $S_t$. We will consider both continuous and discrete time. In discrete time, we model wealth dynamics as $W_t = W_{t-1}(1+r_t-s_t)$, where $r_t$ is the real rate of return and $s_t$ is the spending rate (as a fraction of $W_{t-1}$) at time $t$.\(^3\) We will not concern ourselves with valuation issues such as what price index to use or how to value illiquid assets, so that given the investment and spending policy for the endowment, the processes $W_t$, $r_t$, and $s_t$ are well-defined. We also abstract from parameter uncertainty about the distribution of returns.

For most of the paper, we will use the following definitions:\(^4\)

\(^3\)This convention amounts to having consumption $c_t$ taking place at the end of the period just before $w_t$ is measured. It is straightforward to change our results for other conventions. For example, if consumption takes place at the beginning of the period just after $w_{t-1}$ is measured, we would have $w_t = w_{t-1}(1-s_t)(1+r_t)$ with obvious changes in the statements of our results.

\(^4\)As above, we also need minor technical assumptions, that the mean exists and that we have sufficient independence over time and the expected log return less the spending rate does not got to zero too quickly. For example, it suffices to have iid returns and constant spending.
Definition 1 Endowment wealth is said to be preserved if the value of a unit $W_t$ becomes arbitrarily large over time: $\text{plim}_{t \to \infty} W_t = \infty$.\footnote{As is conventional, plim indicates convergence in probability. By definition plim$W_t = \infty$ if for all $X > 0$, $\text{prob}(W_t > X) \to 1$ as $t \to \infty$.}

Definition 2 Endowment wealth is said to be destroyed if the value of a unit $W_t$ vanishes over time: $\text{plim}_{t \to \infty} W_t = 0$.

The form of this definition looks the same in both continuous and discrete time although the implicit set of possible times is different. We think of the definition of destroying capital as relatively conservative, since no reasonable rule for preserving capital would say we are preserving capital if wealth is almost always close to 0 when $t$ is large. This is what we need for our main results that the traditional rules are not sufficient to preserve capital. The definition of preserving capital should be strengthened because it puts little restriction on capital at intermediate dates; we will return to this issue in Section 5.

2.2 Preserving Capital in Discrete Time

One traditional criterion says that a spending rate of no more than the average return on the endowment will preserve its value. This traditional criterion is widely adopted and clearly stated in the spending policy statements of many university endowments. For example, the spending policy statement of UCSD Foundation (2014) states that its objective is to “achieve an average total annual net return equivalent to the endowment spending rate adjusted for inflation”. Moreover, the endowment of Henderson State University (2014) even employs a concrete example to illustrate its objective of achieving that the inflation-adjusted average return equals to the spending rate: “Total return objective 7.00%, spending rate 4.00%, administration fee 1.50%, and inflation rate 1.50%”. This criterion is also mentioned by Rice, Dimeo, and Porter (2012), which gives as a hypothetical example: “the primary objective of the Great State University Endowment fund is to preserve the purchasing power of the endowment after spending.
This means that the Great State University Endowment must achieve, on average, an annual total rate of return equal to inflation plus actual spending”. Despite its wide use, the traditional criterion is not sufficient to preserve capital.

Absent risk, this criterion makes perfect sense. Suppose the real portfolio return \( r_t = r \) and the spending rate \( s = S_t/W_t \) are both riskless and constant over time. The traditional criterion says that the spending is less than the return on the portfolio, that is, \( S_t/W_t < r \), then capital is preserved. We have that

\[
W_{t+1} = W_t + rW_t - S_t = W_t (1 + r - s) = W_0 (1 + r - s)^{t+1}. \tag{1}
\]

In this riskless case, spending less than the return on the endowment implies the endowment increases without bound, so we have preservation of capital. So far so good. In the traditional criterion, the next step says we can use the same analysis if spending is less than the expected return on the endowment in an uncertain world, “you know, because of the law of averages”. The traditional criterion has an implication that is hard to believe. Consider investing in a riskless asset with mean return \( r \) and a risky asset with a mean return \( \mu > r \). Then if we put a proportion \( \theta \) in the stock (\( \theta \) could be larger than one for a levered position), the traditional criterion says we preserve capital if \( r + \theta(\mu - r) > s \). However, this implies that we can spend at as high a rate \( s \) as we want, so long as we take on enough risk by choosing \( \theta \) to be high enough! Not surprisingly, there is a problem here.

The application of law of the large numbers is fallacious because the criterion applies to sums, not products. Now that the return is random, we write \( r_{t+1} \) for the return from time \( t \) to \( t + 1 \), and (1) becomes

\[
W_{t+1} = W_t + r_{t+1}W_t - S_t = W_t (1 + r_{t+1} - s_{t+1}), \tag{2}
\]

which implies that

\[
W_t = W_0 \prod_{i=1}^{t} (1 + r_i - s_i). \tag{3}
\]
However, the law of large number applies to sums not products. Even if $1 + r_i - s_i$ has mean larger than 1 and is i.i.d. over time, the wealth (3) needs not grow over time and indeed destroys capital as shown in the following Example 1.

**Example 1 (Destroying capital by following traditional criterion):** Assume an endowment has a spending rate of 0% and an investment of 1 which has half probability of tripling and going to zero respectively:

$$
1 \xleftarrow{\text{3 Prob 0.5,}} \xrightarrow{\text{0 Prob 0.5.}}
$$

Hence, the expected return is (50%) greater than the spending rate (0%), i.e., the. According to the traditional criterion, capital should be preserved. However, in each year there is a 50% probability the endowment will be wiped out and the probability of surviving for $T$ years is $2^{-T}$ which approaches 0 rapidly as $T$ increases. Having no endowment at all with probability close to one certainly does not preserve capital but it satisfies the traditional criterion. Having the possibility of the portfolio value dropping to zero is not critical in this example, as we will see in Example 2 in the text.

Example 1 may seem extreme because the wealth can actually reach 0; the following example shows that the traditional criterion can destroy capital even if wealth is always positive:

**Example 2** Assume an endowment has a spending rate of 0% and an investment of 1 which has half probability of tripling and going to 1/9 respectively:

$$
1 \xleftarrow{\text{3 Prob 0.5,}} \xrightarrow{\text{1/9 Prob 0.5.}}
$$

Hence, the expected return is $5/9 > 0\%$, i.e., the spending rate, but the whole endowment still vanishes in probability. So the traditional criterion fails.
Proof: Let $1 + r_i - s_i$ be drawn i.i.d. over time as 3 with probability 1/2 and 1/9 with probability 1/2. Then

$$W_t = W_0 \prod_{i=1}^{t} (1 + r_i - s_i) = W_0 \exp \left( \sum_{i=1}^{t} \log (1 + r_i - s_i) \right).$$

Then we have

$$E \left[ \log (1 + r_i - s_i) \right] = \frac{1}{2} \log 3 + \frac{1}{2} \log \left( \frac{1}{9} \right) = \left( \frac{1}{2} + \frac{1}{2} \times (-2) \right) \log 3 = -\frac{1}{2} \log 3 < 0.$$

Therefore, by the law of large number,

$$\text{plim} \sum_{i=1}^{t} \log(1 + r_i - s_i) = -\infty \implies \text{plim} W_t = 0.$$

To solve the problem of mistakenly applying the criterion of large number, we need to first convert the multiplication to a sum by taking logarithms:

$$\log(W_t) = \log(W_0) + \sum_{i=1}^{t} \log(1 + r_i - s_i),$$

and now we can use the law of averages (i.e., the law of large numbers or the central limit theorem). Assume that $r_i$ is independent over time and has the same distribution in each period (the usual assumption\(^6\) for these calculations), and that $\log(1+r_{t+1}-S_t/W_t)$ has a finite mean. This leads to the following theorem.

**Theorem 1** If the random return is \(\{r_t\}_{t=1,\infty}\) and the spending \(s_t = S_t/W_{t-1}\) at the end of the period is a fraction of wealth at the beginning of the period. Given \(E[\log (1 + r_t - s_t)] < \infty\), \(\text{Var}[\log (1 + r_t - s_t)] < \infty\), and \(1 + r_t - s_t\) is i.i.d. over time, then (1) endowment capital is preserved according to definition 1 if and only if \(E[\log (1 + r_t - s_t)] > 0\); (2) endowment capital is destroyed according to definition 2 if and only if \(E[\log (1 + r_t - s_t)] < 0\).

\(^6\)There are many generalizations of the law of large numbers that can be used more generally, we consider that in Section 4.
Moreover, Jensen’s inequality and concavity of the logarithm, we have

\[ E[\log(1 + r_t - s_t)] < \log(E[1 + r_t - s_t]), \quad (4) \]

which demonstrates that the correct criterion \( E[\log(1 + r_t - s_t)] > 0 \) is stricter than the traditional criterion \( E[r_t - s_t] > 0 \).

In Theorem 1, we have assumed that returns are i.i.d. and that \( S_t/W_t = E[r_t] \), but by continuity the result still holds for \( S_t/W_t < E[r_t] \) but sufficiently close to \( E[r_t] \). It also obviously holds if instead of i.i.d. we have the sort of mixing property that implies the law of large numbers, as we discuss further in Section 5.

Applying the correct criterion in Example 2 shows that the capital is not preserved as the expectation of log rate of return net of spending is negative. And we will see later with appropriate assumption of return volatility, the correct criterion gives the condition for preserving capital in Case 2. Moreover, we have following Corollary which makes the criterion easily apply to some different cases.

**Corollary 1** Consider two investment and spending policies, A and B. If the return net of spending for policy A is larger than for policy B for some state, i.e., \( r^A_i - s^A_i > r^B_i - s^B_i \) for some \( i \), and for \( \forall j \neq i, r^A_j - s^A_j \geq r^B_j - s^B_j \), then (1) if policy A destroys capital, so does policy B, and (2) if policy B preserves capital, so does policy A.

The Corollary tells that if the set of investment opportunities and spending criterion A dominates (is dominated) the set B in a sense that return net of spending of A is no worse (better) than B in each state, then spending criterion in B preserves (destroys) capital implies spending criterion in A preserves (destroys) capital. For example, make the investment opportunity in Example 2 a bit worse by changing 1/9 into 0, which is just investment opportunity in Example 1, the spending criterion does not preserve capital. Since the return net of spending is less than destroying capital in some state.

The results of the correct criterion are economically significant. Since according to the correct criterion, the endowment may need to cut off around 20% of their spending...
if they currently plan to spending around 5% of the capital. By Taylor series expansion, we have

\[ E[\log(1 + r - s)] \approx E[r - s] - (1/2) \times \text{Var}[r - s], \]

take \( \sigma = 15\% \), we have

\[ E[s] < E[r] - \sigma^2/2 = E[r] - 1.125\%, \]

which means endowment should reduce the spending rate by 1.125%.

Knowing that we have to deal with a sum in logs, we make a statement about log returns. We provide the following Theorem describing the condition needed for preserving capital.

**Remark 1 (Jensen’s Inequality Argument)** Mathematically we can view the problem in terms of concavity of the logarithm. By Jensen’s inequality and concavity of the logarithm,

\[ E[\log(1 + r_t - s_t)] < \log(E[1 + r_t - s_t]). \]  

(5)

Positivity of the right-hand side is the traditional criterion \( E[r_t - s_t] > 0 \), and positivity of the left-hand side is the correct criterion, which require a low level of spending to preserve capital. For example, assume \( \log(1 + r_t - s_t) \sim N(\mu - \sigma^2/2, \sigma^2) \), then we have

\[ \log(E[1 + r_t - s_t]) = \mu \quad \text{and} \quad E[\log(1 + r_t - s_t)] = \mu - \sigma^2/2. \]

However, the traditional criterion fails if \( \sigma > 0, \mu \simeq 0 \) but still positive, since the right-hand side of (5) is positive, but capital is not preserved \( (\mu - \sigma^2/2 < 0) \). To obtain a positive left-hand side of (5) with the same investment opportunity \( 1 + r_t \), one needs to decrease the spending rate \( s_t \) to \( \hat{s}_t \) to obtain \( E[\log(1 + r_t - \hat{s}_t)] = \hat{\mu} > 0. \)

A couple of qualifications are in order for the positive result for the riskless case and are also relevant for the risky case. First, we should work with real returns,
that is, returns in excess of inflation. This adjustment is normally done correctly in practice when using the traditional criterion: we are not really preserving capital if we are keeping the same dollar amount in an inflationary environment. The second qualification is a little trickier but probably not too big a deal. The assumption in (1) is that spending takes place at the end of the period, so the wealth relative $W_t / W_{t-1} = 1 + r_t - s_t$. However, the actual timing depends on the local convention. For example, if budgeted spending for the year is taken out of the endowment and placed in a separate account at the beginning of the year, the wealth relative would be $(1 - s_t)(1 + r_t)$. Calculation given other convention are straightforward but can be messy. For example, if the spending $S_t = s_t W_{t-1}$, is computed at the beginning of the year but taken out in two parts, half at the start of the year and half in the middle, the wealth relative is $(1 - s_t/2)(1 + r_t^{H1} - s_t/2)(1 + r_t^{H2})$, where $r_t^{H1}$ is the return on the assets in the first half of the year and $r_t^{H2}$ is the return in the second half. In general, we want to compute $W_{t+1} / W_t$.

2.3 Preserving Capital in Continuous Time

Suppose a constant return on the endowment with local mean $\mu$ and local standard deviation $\sigma$, and spending rate $s$. The wealth dynamic of the endowment follows

$$dW_t / W_t = \mu dt + \sigma dZ_t - s dt,$$

with spending rate $s$. Here we assume exogenous fixed portfolio proportions since these portfolio weights are usually determined by a committee in practice. More importantly, there has being a lack of linkage between spending and investment strategies in the real world. This problem is investigated in Dybvig (1995), emphasizing a stronger consideration of planning investment and spending together are in need.

Consequently,

$$W_t = W_0 e^{(\mu - \sigma^2/2 - s)t + \sigma Z_t}.$$

(6)
If the log return $\mu - \sigma^2/2$ is larger than $s$, $\lim_{t \to \infty} W_t = \infty$, capital is preserved, while if it is smaller than $s$, $\lim_{t \to \infty} W_t = 0$, and capital is not preserved. The traditional result fails if $\mu - \sigma^2/2 < s < \mu$. With equality, probably we would say capital is not preserved, but that depends on what definition we use.

We can also look at this in terms of Itô’s lemma (and concavity of the logarithm because of the second derivative in the Itô term)). We have that

$$d \log (W_t) = (\mu - \frac{\sigma^2}{2} - s)dt + \sigma dZ_t,$$

so the drift is positive if the coefficient of $dt$ is positive. Only with a positive drift can the capital be preserved. We summarize the results in the following Theorem.

**Theorem 2** Given the endowment can invest in a risky asset continuously, the capital is preserved if the expected log growth rate is larger than the spending rate, i.e., $\mu - \sigma^2/2 > s$.

### 3 Preserving Capital with Smooth Spending

Instead of making spending strictly proportional to the size of the endowment, it is common to smooth spending using a moving-average (partial adjustment) rule to move from current spending towards a spending target. Probably there is some economic sense to smoothing, since a sudden decrease in a budget can cause distress, while a sudden increase can invite waste. As a result, there are many endowments which are employing some kinds of smooth spending formulas. For instance, several universities in the UC system use smooth spending policy (Mercer Investment Consulting (2015)): UC Berkeley, UC Irvine, and UC Santa Cruz plan to spend about 4.5% of a twelve-quarter (three year) moving average market value of the endowment pool. Another example: Grinnell College Endowment (2014) states that endowment distribution is calculated as 4.0% of the 12-quarter moving average endowment market value determined annually as of the December 31 immediately prior to the beginning of the fiscal year. Actually,
according to Commonfund (2005), 63 per cent of institutions in the US report ‘they employ either a three-year or 12-quarter moving average of market value as a smoothing mechanism in their spending formula; 38 per cent use the three-year and 25 per cent use a 12-quarter moving average (Also see page 112, Chapter 4, Acharya and Dimson (2007)).

However, the moving average rule tends to destabilize the endowment. We illustrate this with a riskless example for which an initial high spending rate sends the fund into a "death spiral" with the wealth going to zero for sure at a known finite time. Then we give a result for risky i.i.d. returns. When stock returns are bad, wealth goes down but spending is slow to adjust so the spending rate goes up. This pushes wealth down and at some point the fall in wealth becomes unstable because the adjustment is not fast enough to keep the spending rate from getting large as wealth (in the denominator) falls. Over time, this scenario will play out sooner or later, with probability one the fund’s wealth will reach zero at some (random) future time.

3.1 Benchmark: Traditional Moving Average Spending Rule with Only A Riskless Bond

A traditional moving average spending rule assumes the dynamic of spending to be\(^7\)

\[ dS_t = \kappa (\tau W_t - S_t) dt. \tag{8} \]

We will assume \( \tau < r \), which implies that the target spending rate would preserve capital, so our policy has a fighting chance. If the endowment only invests in a riskless bond with constant risk-free rate \( r \), then the wealth process is given as

\[ dW_t = rW_t dt - S_t dt. \tag{9} \]

\(^7\)Often practitioners use a moving average rule, e.g., a 10-year average, in place of this autoregressive rule, the distinction is not important for us.
We have the following result.

**Proposition 1** When the endowment invests in only the riskless asset, the moving average spending rule does not preserve capital when the initial spending rate $S_0/W_0$ is sufficiently high. Specifically, given the dynamic (8) and (9), wealth $W_t$ converges to 0 over time if $S_0/W_0$ is larger enough, and the spending rate increases and converges to a positive limit over time.

If the endowment starts with high spending under the moving average rule, capital will be wiped out quickly. The intuition is that with a high initial spending, the wealth declines dramatically. Decline in wealth is faster than decline in spending under moving average rule, hence, the spend rate becomes increasingly higher and much higher then return of the riskless investment. As a results, wealth converges to zero in a "death spiral". The spending rate which is designed to be mean-reverting tends to increase and approach a limit. Here is a illustration.

**Case 3 (Increasing spending rate):** Assume $W_0 = 100$, $S_0 = 15$, $r = 5\%$, $\tau = 4\%$, and the adjustment rate $\kappa = 20\%$ each year, where target rate is intentionally set to be less than the interest rate to indicate a relatively good investment opportunity and a potential to preserve capital. However, given a high enough initial spending rate, the wealth declines dramatically comparing to the drop in the mean-reverting spending. Note the wealth at the next year is

$$W_1 = W_0 (1 + r - s) = 100 \times (100\% + 5\% - 15\%) = 90,$$

hence the wealth drops by 10. However, the adjustment of spending is

$$\Delta S = 20\% \times (4\% \times 100 - 15) = -2.2,$$

much less than the decrease in wealth. The disproportional change leads to a higher spending rate in the next year: $s_1 = (20 - 2.2)/90 = 19.8\% > 15\% = s_0$, even spending
is declining. As time evolves, the spending rate becomes higher and converge to a limit. The reason is that the endowment spends not only the interest, but also a part of the principal, which accelerates the decline in wealth.

3.2 Traditional Moving Average Spending Rule with Fixed Risky Portfolio Proportions

In practice, the portfolio choice is not often link to the current spending rate. Usually, the portfolios in different asset classes look fixed or around fixed. As a result, it is reasonable to model that the endowment invest in a single risky asset with constant mean and variance. Given the moving average spending rule still follows (8), if the endowment has return with constant mean and volatility, then the wealth process is given as

\[ dW_t = W_t (\mu dt + \sigma dZ) - S_t dt = (W_t \mu - S_t) dt + W_t \sigma dZ, \]

so long as wealth as positive. Also assume that if \( W_t \) reaches zero, then the endowment is shut down and \( W_t \) and \( S_t \) are both zero forever afterwards if wealth reaches zero. We have the following result.

**Proposition 2** When the endowment invests in only a risky asset (10), the moving average spending rule (8) cannot preserve capital and survival forever has zero probability, i.e., for any initial positive wealth \( W_0 \) and spending \( S_0 \), \( \lim_{t \to \infty} \text{prob}(W_t = 0) = 1 \).

In other words, always reaches zero in finite time.

Sketch of proof: Given the dynamic of wealth and spending, we can write the dynamics of wealth over spending (which is Markov). Then find a function \( F \) of the variable \( W_t/S_t \) that is a local martingale (by deriving the dynamics of \( F \) using Itô’s Lemma, and set the drift term of \( F \) equal to zero). Note that \( F(0) \) is finite and \( F(\infty) = \infty \). Considered \( F(W_t/S_t) \) stopped at the first time it reaches \( F(0) \) or \( K \) (where \( K \) is chosen larger than \( F(W_0/S_0) \)). This is a bounded martingale, so it must converge over time, and since the volatility is positive on the interior, it must converge to either \( F(0) \) or
The martingale condition gives the probability that \( F(W_t/S_t) \) converges to the two boundaries. Computing the probability that \( F(W_t/S_t) \to F(0) \), and taking the limit as \( K \to \infty \) gives us the results.

See the Appendix for the proof.

When wealth declines a lot over a short time, smoothed spending does not change much but the wealth changes quickly so that the spending rate now exceeds the return on investments. This causes further decline in wealth. For a sufficiently large initial decline, the mean reversion towards the target spending rate is too slow to overcome the current loss due to spending too much. Luck might increase wealth enough to save the endowment from falling to zero, but sooner or later we will encounter a large enough shock, without subsequent offsetting good luck, that will pull wealth down to zero in finite time.

3.3 A Smooth Spending Rule that Preserves Capital

The problem with the traditional mean reverting spending rule is that the endowment can spending too much when wealth is low and, thus, target spending moves away more quickly than spending can adjust and capital is not preserved. Hence, to keep the target within a reasonable distance, we need to change the smoothing rule. To give the rule a fighting chance, we assume the target spending rate would preserve capital, i.e., \( \mu - \sigma^2/2 \). We propose the smooth spending rule which has potential to preserve capital as

\[
\frac{dS_t}{S_t} = \kappa \left( \tau - \log \left( \frac{S_t}{W_t} \right) \right) dt + \mu - \frac{\sigma^2}{2} - \frac{S_t}{W_t} \quad (11)
\]

where the wealth process follows

\[
dW_t = W_t (\mu dt + \sigma dZ) - S_t dt = (W_t \mu - S_t) dt + W_t \sigma dZ. \quad (12)
\]

We have the following theorem.
Theorem 3 The smooth spending rule given by (11) preserves capital in the sense that

\[
\lim_{t \to \infty} \Pr (W_t < W_0) = 0,
\]

provided the parameters satisfy the following condition

\[
\mu - \frac{\sigma^2}{2} - \exp \left[ \tau + \frac{\sigma^2}{4\kappa} \right] = Q > 0.
\]  

(13)

However, if

\[
\mu - \frac{\sigma^2}{2} - \exp \left[ \tau + \frac{\sigma^2}{4\kappa} \right] < 0,
\]

then the smooth spending rule does not preserves capital, i.e.,

\[
\lim_{t \to \infty} \Pr (W_t < W_0) = 1.
\]

Sketch of proof: Given the proposed dynamics of spending and the wealth dynamics, we can derive the dynamic of \( \log \left( \frac{S_t}{W_t} \right) \) by Itô’s Lemma, which turns out to be a Gaussian and stationary process if assuming the starting point follow a specific distribution. Furthermore, we can prove that \( \frac{S_t}{W_t} \) is a stationary and mean-square ergodic process by mean-square ergodic theorem (Finite Autocovariance Time). Then turn to the expression of the wealth. To prove the capital is preserved, we only need to prove \( \text{plim}_{t \to \infty} (\log W_t/W_0) = \infty \), which can be proved if \( \text{plim}_{t \to \infty} (\log W_t/W_0)/t \) equals to a positive number. Note by the expression of wealth, \( (\log W_t/W_0)/t \) can be written as the sum of the log growth rate of stock \( \mu - \sigma^2/2 \), the time average of spending ratio \( S_t/W_t \), and the time scaled Brownian motion. Given the ergodic properties of spending ratio, the time average of spending ratio converges to the mean of \( S_t/W_t \) with \( L^2 \). Finally, by Chebyshev’s inequality, we can prove that \( \text{plim}_{t \to \infty} (\log W_t/W_0)/t = Q. \)

\[\text{Definition (1) is not the only possible definition of preserving capital. A stronger definition can be that at every state of nature, the wealth converges to positive infinity as time evolves. In other words, if we measure once a year, e.g., in the end of every year, the wealth converges to positive infinity almost surely. Although with a stronger definition, the proof is similar to the weaker version, but needs to apply a functional limit theorem.}

\[\text{The proof can be generalized to the case that given } \forall K > 0, \lim_{t \to \infty} \Pr (W_t < K) = 0.\]
See the Appendix for the proof.

The condition (13) means that the log growth rate of the risk asset have to be larger than the expected spending rate $E[S_t/W_t] = \exp (\tau + \sigma^2 / (4\kappa))$. Note spending rate $S_t/W_t$ is stationary and lognormally distributed. This results carry quite similar intuitions as that in subsection 2.2, and forms a contrast to the case when capital is not preserved and the spending rate tends to increase over time as wealth converges to zero.

Note the term $\mu - \sigma^2/2$ in the drift of spending is the log growth rate of wealth from investment, and term $-S_t/W_t$ is the reduction in wealth from spending. Recall the spending rule preserving capital in the previous subsection requires that $S_t/W_t \leq \mu - \sigma^2/2$. Hence, the smooth spending rule in (11) demonstrates that if the spending is too high, i.e., $\mu - \sigma^2/2 - S_t/W_t < 0$, then reduction in spending at expected rate of decline in wealth is needed to preserve capital. Moreover, the term $\kappa (\tau - \log (S_t/W_t))$ means that on top of preservation of capital, the spending mean reverts to the constant target level. Besides, the spending rule adjusts for the expected change in wealth instead of the random part. As a result, the spending is still differentiable and smooth.

**Remark 2 (Knife-edge Case):** It is a knife-edge case when the expected log return equals the expected spending rate. Since we never know when the knife-edge case is exactly true, it is not an case of great interest. It is easy to find out that just like standard winner process returns to initial value infinitely many times over an infinite horizon, the wealth reaches the initial wealth infinitely many times in the knife-edge case. If defining preservation of capital by returning to original value of wealth infinitely many times, this knife-edge case can preserve capital. However, this is not a quite reasonable definition of preservation of capital. Because as time evolves to infinity, most of the probability mass concentrates in the tails. As a result, as time $t$ evolves, the probability that wealth is in an interval from $1/t$ times initial wealth to $t$ times initial wealth converges to zero. The wealth can be either above the interval or below the interval. Conventionally, this is not consistent with the spirit of preserving capital. Therefore,
in most senses of preservation of capital, the knife-edge case does not preserve capital over time.

4 General Condition for Preservation of Capital

The moving average spending rule in the previous section assumes continuity of underlying parameters, e.g., constant volatility of stock return and constant return growth rate. In this section, we provide a general condition of preservation of capital which allows stock return growth, volatility, and the spending rate to follow numerous general type of processes.

4.1 General Condition

Suppose a return process on the endowment with local mean $\mu_t$ and local standard deviation $\sigma_t$, where $\mu_t$ and $\sigma_t$ are some general processes, and $Z$ is a standard Wiener process. Then the wealth dynamic follows

$$dW_t = W_t (\mu_t dt + \sigma_t dZ) - S_t dt = (W_t \mu_t - S_t) dt + W_t \sigma_t dZ,$$

which implies that

$$W_t = W_0 \exp \left[ \int_0^t \left( \mu_v - \frac{1}{2} \sigma_v^2 - s_v \right) dv - \int_0^t \sigma_v dZ_v \right].$$

Then the Theorem (3) can be easily generalized to a more general case as the following theorem.

**Theorem 4** Given some general stochastic processes of $\mu_v$, $\sigma_v^2$, and $s_v$, and for $\forall v > 0$,
\( \sigma_v > 0 \) and \( s_v > 0 \), and the following limit exist

\[
\lim_{t \to \infty} \frac{1}{t} \mathbb{E} \left[ \int_{v=0}^{t} \left( \mu_v - \frac{1}{2} \sigma_v^2 - s_v \right) dv - \int_{v=0}^{t} \sigma_v dZ_v \right] = B,
\]

\[
\lim_{t \to \infty} \frac{1}{t^2} \text{Var} \left[ \int_{v=0}^{t} \left( \mu_v - \frac{1}{2} \sigma_v^2 - s_v \right) dv - \int_{v=0}^{t} \sigma_v dZ_v \right] = 0.
\]

then the spending process preserves capital in the sense that

\[
\lim_{t \to \infty} \Pr (W_t < W_0) = 0,
\]

if \( B > 0 \). However, if \( B < 0 \), then the spending rule does not preserves capital, i.e.,

\[
\lim_{t \to \infty} \Pr (W_t < W_0) = 1.
\]

Note the process of \( \mu_v, \sigma_v^2 \), and \( s_v \) do not need to be each stationary and ergodic, as long as the conditions are satisfied. However, these conditions might not be easily utilized by practitioners, since they are not explicit and simple enough. Hence, we provide some simple conditions which are the special cases of the general condition and capture the basic properties growth rate, volatility, and spending rate in the real world, and obtain the following corollary.

**Corollary 2** Assume \( \mu_v \) and \( \sigma_v^2 \) is covariance-stationary, and \( s_v \) is ergodic, then

\[
\mathbb{E} [\mu_v - \sigma_v^2 / 2] > \mathbb{E} [s_v]
\]

ensures preservation of capital.

Moreover, by the general condition, we can study some interesting cases: spending with temporarily negative real risk-free rate and spending with stochastic volatility.
4.2 Preserving Capital with Temporarily Negative Real Risk-Free Rate

These calculations by practitioners are done in real terms (as they should be). An interest rate environment like the current one where inflation exceeds the nominal rate is a special challenge. The endowment never preserve capital if the real risk-free rate is always negative. For example, if investments in real riskless bonds are available but the local expectations hypotheses holds, then given a little regularity, no strategy with non-negative spending will preserve capital if the long-term expected short real interest rate is negative. However, under some condition, capital can be persevered even the risk-free rate is temporarily negative. This subsection models temporarily negative real rate and provides the conditions needed for preserving capital by employing the results of Theorem (4).

Let the nominal interest rate \( r_t \) modeled by some diffusion processes. Hence, the stock price follows a diffusion process as

\[
\frac{dP_t}{P_t} = (r_t - \iota + \pi) dt + \sigma dZ_t,
\]

where \( \iota \) is a constant inflation rate and \( \pi \) is a constant risk premium. With a fixed portfolio \( \theta \), the wealth process follows,

\[
dW_t = (r_t - \iota) W_t dt + W_t \theta (\sigma dZ_t + \pi dt) - S_t dt,
\]

\[
= W_t ((r_t - \iota + \theta \pi) dt + \theta \sigma dZ_t) - S_t dt.
\]

Employing the results in Theorem (4), we can obtain the following theorem:

**Theorem 5** With a constant portfolio in stock, the endowment can preserve capital if

\[
E \left[ r_t - \iota + \theta \pi - \frac{\theta^2 \sigma^2}{2} \right] > E [S_t],
\]

(14)
where the spending rate $s$ is covariance-stationary process. If

$$E\left[r_t - \pi + \theta \pi - \frac{\theta^2 \sigma^2}{2}\right] < E[s_t],$$

then capital is not preserved.

By Theorem 5, we can cannot gain a high expected log rate of return after taking return volatility into account, which is different from the implausible implications of the traditional rule. Since the quadratic function with a negative coefficient of the second order term is capped over the choice of portfolio.

Now can provide examples of spending rule with negative real interest rate, both rules preserving capital and rules not preserving capital.

**Example 3 (Successful preservation of capital with negative real rate):** Let the nominal interest rate follows a CIR model, i.e.,

$$dr_t = a_0(b - r_t)dt + \sigma \sqrt{r_t} dZ_t,$$

where $a_0$ is a constant and $b$ is the long-term mean. Let the spending rule be a moving average rule as

$$dS_t = S_t \left(\kappa \left(\tau - \log \left(\frac{S_t}{W_t}\right)\right) + r_t - \pi + \theta \pi - \frac{\theta^2 \sigma^2}{2} - \frac{S_t}{W_t}\right)dt,$$

which, by the results in Theorem (3), implies that $E[s_t] = \exp[\tau + \sigma^2/(4\kappa)].$

Given $\lambda = 4\%$, $b = 4\%$, $\pi = 5\%$, $\sigma = 15\%$, $\theta = 0.8$, $\tau = -3.5$ (with target rate of $s = 0.03$, i.e., $-3.5 = \log(0.03)$), and $\kappa = 1$, then the real interest rate is zero, just quite similar to real rate in the current financial market. However, the spending still can be covered by a high enough risk premium. Consequently, in a long horizon, the capital can be preserved. For instance, suppose at a point of time, the inflation rate is 4% and the real rate is $-4\%$, then given the risk premium is 5% and the endowment cannot cover a positive spending rate with a negative return at this point. However, capital is
still preserved since when during a good time, say, real interest rate is 8% and, thus, the expected return of portfolio is 13%. If the endowment still has the target spending rate, then capital is preserved. To sum up, the point is that preservation of capital is not about a point of time, it is about the whole paths of underlying dynamics. Finally, by applying Theorem (5), it is easy to see (14) is satisfied, since

\[ b - \iota + \theta \pi - \frac{\theta^2 \sigma^2}{2} - \exp \left[ \tau + \frac{\sigma^2}{4K} \right] = 0.0024, \]

hence, capital is preserved.

**Example 4 (Unsuccessful preservation of capital with negative real rate):**
Given \( \iota = 6\% \), \( E[r_t] = 0 \), \( \pi = 5\% \), and \( \sigma = 15\% \), then no choice of a fixed portfolio \( \theta \) can preserve capital locally. Since even the portfolio which maximizes the growth rate of log wealth, i.e., \( \theta = \pi / \sigma^2 \) maximizing \( \theta \pi - \theta^2 \sigma^2 / 2 \), can not preserve capital. Note according to (14) in Theorem (5), we can calculate the expected log turn with highest growth rate:

\[ E[r_t] - \iota + \theta \pi - \frac{\theta^2 \sigma^2}{2} = E[r_t] - \iota + \frac{\pi^2}{2\sigma^2} = -0.004444, \]

which is a negative number. However, expected spending cannot be negative. Hence, (14) is not satisfied, and capital is not preserved due to a too high expected inflation and a too low expected nominal interest rate. There are also good reason not to take on so much leverage. If \( \theta = 0.8 \), and \( \iota = 3.5\% \), then capital is still not preserved, since \( E[r_t] - \iota + \theta \pi - \theta^2 \sigma^2 / 2 = -0.0022. \)

**Example 5 (Preservation of capital by spending rule with stochastic volatility):** Assume the spending rate is given as an affine function of nominal interest rate, i.e.,

\[ s_t = S_0 + S_1 r_t, \]

where \( r_t \) is the nominal interest rate following the CIR model (15), with \( S_0 > 0 \), and
Therefore, the spending rate is covariance-stationary and always positive, and we have

$$E[s_t] = S_0 + S_1 E[r_t].$$

Given $\nu = 4\%$, $b = 4\%$, $\pi = 5\%$, $\sigma = 15\%$, $\theta = 0.8$, $S_0 = 3\%$, and $S_1 = 0.6$, we have capital preserved, since according to (14) in Theorem (5), we have

$$b - \nu + \theta \pi - \frac{\theta^2 \sigma^2}{2} - (S_0 + S_1 E[r_t]) = (1 - S_1) b - \nu + \theta \pi - \frac{\theta^2 \sigma^2}{2} - S_0 = 0.0088.$$

Note it is possible that at some point, the nominal rate reaches zero, meanwhile, the spending rate is positive. However, even this case happens, the endowment can still preserve capital. Since, again, preservation of capital is not about several points of times, it is about an infinitely long horizon. Hence, even the expected log real rate of the assets can be less than the spending rate when the interest rate is temporarily low, however, the turn of asset can well covers the spending when interest rate is high. Consequently, capital is preserved.

### 5 Optimization Models

We have been emphasizing preservation of capital as a constraint facing by the universities. The traditional practice by endowments postulates a candidate portfolio strategy and spending rule, followed by a check of what parameter values, e.g., spending rate target and portfolio weights, are consistent with preservation of capital. Alternatively, we can impose preservation of capital as a constraint in an optimization problem. Unfortunately, the traditional rules we have been studying is not up to this task. We investigate this in the following Problem 1.

**Problem 1** Given the initial wealth $W_0$, choose adapted portfolio process $\{\theta_t\}_{t=0}^\infty$, adapted spending process $\{S_t\}_{t=0}^\infty$ and wealth process $\{W_t\}_{t=0}^\infty$ to maximize the expected
utility,

$$\sup_{\theta, S} \left[ \int_{t=0}^{\infty} D_t u (t, S_t) \, dt \right]$$

s.t. \( dW_t = rW_t \, dt + \theta_t (\mu - r) \, dt + \sigma dZ_t = S_t \, dt, \)

\( \forall t, W_t \geq 0. \)

\( p \lim_{t \to \infty} W_t = \infty. \) \hspace{1cm} (16)

where \( u (t, S_t) \) is supposed to be concave increasing in its second argument and measurable in its first. It is assumed that \( \mu - r, \sigma, \) and \( r \) are all positive and the discount factor \( D_t \geq 0, \) and

$$0 < \int_{s=0}^{\infty} D_s \, ds < \infty.$$

The optimization model reveals the weakness of the constraint since the constraint only concerns the infinite limit, without setting any restrictions on the wealth process on the intermediate dates. To obtain a positive amount of wealth at infinity, one can employ strategies that guarantee a nonnegative value at any finite date \( T, \) and from that date, invests part of the wealth into the riskless asset. Theoretically, this amount of wealth converges to infinity over time exponentially with riskless rate of return. Consequently, the definition of preservation of capital we used in this paper so far is not really adequate for putting into an optimization model as a constraint. We summarize the results in the following proposition:

**Theorem 6** Let \( \theta_t^k \) and \( S_t^k \) be feasible spending and investment policies for Problem 1 without the constraint (16). The supremum in Problem 1 is at least the value of following this strategy. Specifically, there exists a sequence \( (\theta_t^k, S_t^k) \) of feasible policies such that

$$\lim_{k \to \infty} \mathbb{E} \left[ \int_{t=0}^{\infty} D_t u (t, S_t^k) \, dt \right] \geq \mathbb{E} \left[ \int_{t=0}^{\infty} D_t u (t, S_t^*) \, dt \right].$$

Proof. See Appendix.
5.1 Discussion on Optimization Models for Endowments

Preservation of capital and the smoothed spending are two key important ingredients in an optimization models for endowments. To make the wealth constraint more effective in preservation of capital, we can impose the drawdown constraint that wealth should never fall below a certain percentage of previous maximum of wealth, i.e.,

\[ W_t \geq b\bar{W}_t = b \sup_{s \leq t} W_s, \quad \forall t, \text{ and } b \in (0, 1). \]

Drawdown constraint carries strong sense of preservation of capital. Since it adds requirements on intermediate cash flows, contrasting to the implications of traditional definition of preserving capital that the wealth converges to infinity approximately for sure, which sounds pretty conservative but actually is not. Elie and Touzi (2006) treat a optimization problem with drawdown constraint thoroughly, while Rogers (2013) gives a very concise summarization of the main results in Elie and Touzi (2006). The solution is given in dual and is analytical up to some constants determined numerically. To apply their model to endowment management, we need to additionally consider smoothing the spending stream. However, considering both drawdown constraint and smoothing simultaneously can give rise to a model with three state variables, i.e., spending, wealth, and the previous maximum wealth. With property of homogeneity of power utility function, we can reduce the number of state variables to two, but we still face a PDE which are hard to solve and interpret.

Actually, a typical problem with the feature of smoothed spending and without drawdown constraint could already be quite intractable and needs to be solved numerically. A natural way to model the incentive to smooth spending is to incorporate the adjustment cost of changing spending. For example, an assumption choice could be adding a term of adjustment cost of changing spending into the dynamics of wealth as a function of change in spending. Moreover, a quadratic cost term can capture the idea that a larger change in spending leads to a higher adjustment cost. However, this conventional methods of modeling adjustment cost does not yield analytical solutions
in an extended Merton model, which is illustrated as follows:

Consider the portfolio problem faced by the endowment which can possible invest in a riskless asset and a single risky asset (a stock) whose price process evolves according to

$$
\frac{dP_t}{P_t} = \mu dt + \sigma dZ_t.
$$

The instantaneous riskless rate is $r$. To simplify interpretation later, we assume without loss of generality that $\mu > r$, so that the risky asset is an attractive investment. Assume the endowment has incentive to smooth spending, the problem of the endowment can be described as follows.

**Problem 2** Given the initial wealth $W_0$ and initial spending $S_0$, choose adapted portfolio process $\{\theta_t\}_{t=0}^\infty$ and adapted marginal spending process $\{\delta_t = S_t^\prime\}_{t=0}^\infty$ to maximize the expected utility,

$$
\max_{\theta, \delta} \mathbb{E} \left[ \int_{t=0}^{\infty} e^{-pt} \frac{S_t^{1-R}}{1-R} dt \right]
$$

s.t. $dW_t = rW_t dt + \theta_t ((\mu - r) dt + \sigma dZ_t) - S_t dt - k \frac{\delta_t^2}{S_t}$,

$$
dS_t = \delta_t dt.
$$

where $\rho$ is the pure rate of time preference, and $R$ is the constant relative risk aversion. It is assumed that $\mu - r, \rho, \sigma, k,$ and $r$ are all positive.

Denote the value function of the endowment as $V$. The HJB equation is given by

$$
u (S_t) - \rho V + V_W \left( rW_t + \theta_t (\mu - r) - S_t - k \frac{\delta_t^2}{S_t} \right) + \delta_t V_S + \frac{\sigma^2 \theta^2}{2} V_{WW} = 0.
$$

By the first-order condition, the optimal choice of change of spending is given as

$$
\delta_t = \frac{S_t V_S}{2k V_W}.
$$
Substitute the optimal change in spending into the HJB equation, we have

\[ u(S_t) - \rho V + V_W(rW + \theta(\mu - r) - S_t) + \frac{S_t V^2_S}{4kV_W} + \frac{\sigma^2 \theta^2}{2} V_{WW} = 0. \]

We can simplify it by let \( x \equiv W/S \), and \( \Theta \equiv \theta/S \), and conjecture \( V(S,W) = S^{1-R}v(x) \). As a result, we have

\[ V_W(W, S) = S^{-R}v_x, \quad V_{WW}(W, S) = S^{-R-1}v_{xx}, \quad \text{and} \quad V_S = (1 - R) S^{-R}v(x) - S^{-R}xv_x. \]

The HJB equation is thus simplified and transferred into

\[ \frac{\sigma^2 \Theta^2}{2} v_{xx} + \frac{((1 - R)v - xv_x)^2}{4kv_x} + v_x (rx + \Theta(\mu - r) - 1) - \rho v + \frac{1}{1 - R} = 0. \quad (17) \]

Again by first-order condition, we have the optimal scaled portfolio in stock is given as

\[ \Theta = -\frac{v_x(\mu - r)}{\sigma^2 v_{xx}}, \]

and substitute it into (17) we have,

\[ -\frac{v^2_x \rho^2}{2v_{xx}} + \frac{((1 - R)v - xv_x)^2}{4kv_x} + (rx - 1)v_x - \rho v + \frac{1}{1 - R} = 0. \]

This ODE needs to be solved numerically, even via dual approach. Besides, assuming linear adjustment cost with wealth dynamics following

\[ dW_t = rW_t dt + \theta_t ((\mu - r) dt + \sigma dZ_t) - S_t dt - k|\delta_t|, \]

or assuming the utility function incorporating a term of cost of adjustment

\[ \max_{S, \theta} E \left[ \int_{t=0}^{\infty} e^{-\rho t} \left( \frac{S_{t}^{1-R}}{1-R} - \frac{a}{1-R} |d(S_{t}^{1-R})| \right) dt \right], \]

will not yields desirable properties of smoothed spending for endowment, e.g., the
spending processes are continuous but not differentiable everywhere, although solutions of the optimization problem could be analytical and easy to interpret.

Therefore, unfortunately, even without drawdown constraint, the desirable problem for endowment needs to be solved numerically. No mention a useful model for endowment with drawdown constraint could be really hard to solve and interpret. Alternatively, one could also consider a wealth constraint as a lower bound on capital which is a function of time, starting at a positive number and increasing over time. We leave the realistic problem with both effective wealth constraint and smooth spending for future research.

6 Conclusion

Two commonly used rules of thumb used for managing endowments that are supposed to preserve capital actually do not preserve capital. Having a spending rate less than the expected return on assets is not strong enough and is based on a fallacious application of the law of large numbers. A correct analogous rule would take logs. We can think of an approximate rule (correct for a lognormal world) that the spending rate has to be less than the mean return on the portfolio minus half the variance.

The second rule of thumb that has problems is the use of a moving average rule to smooth spending. This type of rule never preserves capital in a model where returns are random and i.i.d. We provide alternative rules that smooths spending but in a way that preserves capital for appropriate choice of parameter values.

Although optimization method is a standard approach to decision making on investment and spending in academia of finance, it is less useful for practitioner than we think. Because most of the problems practitioners facing are more complicated than what we know how to solve. The optimization methods work well on finding optimal solutions, while they are not good at identifying weakness in the assumption that one is making. Consequently, to make our results to be readily utilized by practitioners, we stick to the simple and non-optimazation models.
We hope our results will help universities to do a better job managing their endowments.

References


A Appendix: Proofs and Algorithms

A.1 Proof of Proposition 1

Proof. We can rewrite (8) and (9) as

$$d\begin{pmatrix} W_t \\ S_t \end{pmatrix} = A \begin{pmatrix} W_t \\ S_t \end{pmatrix} dt,$$

where

$$A = \begin{pmatrix} r & -1 \\ \kappa \tau & -\kappa \end{pmatrix}.$$

The above ODE can be solved by using an eigenvalue-eigenvector decomposition of $A$. The solution is given as

$$\begin{pmatrix} W_t \\ S_t \end{pmatrix} = K_1 e^{\lambda_1 t} \phi_1 + K_2 e^{\lambda_2 t} \phi_2,$$

where $\lambda_2 < 0 < \lambda_1$ given by

$$\lambda = \frac{r - \kappa \pm \sqrt{(\kappa - r)^2 - 4\kappa (\tau - r)}}{2} = \frac{r - \kappa \pm \sqrt{(\kappa + r)^2 - 4\kappa \tau}}{2}.$$
are the two roots of the eigenvalue equation $\det(A - \lambda I) = 0$, and $\phi_i = (1, r - \lambda_i)^T$. Note that $0 < r - \lambda_1 < r - \lambda_2$, so that if $S_0/W_0 > r - \lambda_2$ (say after an unanticipated negative shock to wealth), then $K_2 > W_0$ and $K_1 = W_0 - K_2 < 0$, so wealth goes to zero in finite time and, thus, capital is not preserved in this case.

Moreover, since
\[
W_t = \frac{W_0 (r - \lambda_2) - S_0}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{W_0 (\lambda_1 - r) + S_0}{\lambda_1 - \lambda_2} e^{\lambda_2 t},
\]
\[
S_t = \frac{W_0 (r - \lambda_2) - S_0}{\lambda_1 - \lambda_2} (r - \lambda_1) e^{\lambda_1 t} + \frac{W_0 (\lambda_1 - r) + S_0}{\lambda_1 - \lambda_2} (r - R_2) e^{\lambda_2 t},
\]
hence, the limit of spending rate is given as $\lim_{t \to \infty} S_t/W_t = r - \lambda_1 > 0$.

### A.2 Proof of Proposition 2
Assume the dynamic of spending and wealth are given as
\[
\begin{align*}
    dS_t &= \kappa (\tau W_t - S_t) dt, \\
    dW_t &= W_t (\mu dt + \sigma dZ) - S_t dt = (W_t \mu - S_t) dt + W_t \sigma dZ,
\end{align*}
\]
i.e.,
\[
\begin{align*}
    \frac{dW_t}{W_t} &= \frac{W_t \mu - S_t}{W_t} dt + \sigma dZ, \\
    \frac{dS_t}{S_t} &= \frac{\kappa (\tau W_t - S_t)}{S_t} dt.
\end{align*}
\]
Let
\[
    a = \frac{W_t \mu - S_t}{W_t}, \quad b = \sigma \quad \text{and} \quad f = \frac{\kappa (\tau W_t - S_t)}{S_t}, \quad g = 0,
\]
let $U = \frac{W}{S}$, then by Itô’s lemma
\[
\frac{dU}{U} = (a - f + g^2) dt + bdZ.
\]
\[
= \left(\frac{W_t \mu - S_t}{W_t} - \frac{\kappa (\tau W_t - S_t)}{S_t}\right) dt + \sigma dZ
\]
\[
= \left(\mu + \kappa - \frac{S_t}{W_t} - \kappa \tau \frac{W_t}{S_t}\right) dt + \sigma dZ,
\]
i.e.,
\[ d \left( \frac{W_t}{S_t} \right) = W_t \left( \mu + \kappa - \frac{S_t}{W_t} - \kappa \tau \frac{W_t}{S_t} \right) dt + \frac{W_t}{S_t} \sigma dZ. \]

Let \( F \) be a function of the ratio \( \frac{W_t}{S_t} \),

\[ dF \left( \frac{W_t}{S_t} \right) = F' \left( \frac{W_t}{S_t} \right) \left[ W_t \left( \mu + \kappa - \frac{S_t}{W_t} - \kappa \tau \frac{W_t}{S_t} \right) dt + \frac{W_t}{S_t} \sigma dZ \right] + \frac{1}{2} F'' \left( \frac{W_t}{S_t} \right) \left( \frac{W_t}{S_t} \right)^2 dt. \]

To make the drift \( F \) equal to zero, \( F \) has to satisfy that

\[ F' \left( \frac{W_t}{S_t} \right) \left( \mu + \kappa \right) \frac{W_t}{S_t} - 1 - \kappa \tau \left( \frac{W_t}{S_t} \right)^2 = -\frac{1}{2} F'' \left( \frac{W_t}{S_t} \right) \left( \sigma \frac{W_t}{S_t} \right)^2 \]

\[ F' \left( \frac{W_t}{S_t} \right) \left( \mu + \kappa \right) \frac{W_t}{S_t} - 1 - \kappa \tau \left( \frac{W_t}{S_t} \right)^2 = -\frac{1}{2} F'' \left( \frac{W_t}{S_t} \right) \left( \sigma \frac{W_t}{S_t} \right)^2 \]

\[ \frac{F'' \left( \frac{W_t}{S_t} \right)}{F' \left( \frac{W_t}{S_t} \right)} = -\frac{2 \left( \left( \mu + \kappa \right) \frac{W_t}{S_t} - 1 - \kappa \tau \left( \frac{W_t}{S_t} \right)^2 \right)}{\left( \sigma \frac{W_t}{S_t} \right)^2} \]

\[ \left[ \ln F' \left( \frac{W_t}{S_t} \right) \right]' = -\frac{2 \left( \left( \mu + \kappa \right) \frac{W_t}{S_t} - 1 - \kappa \tau \left( \frac{W_t}{S_t} \right)^2 \right)}{\left( \sigma \frac{W_t}{S_t} \right)^2} \]

\[ \left[ \ln F' \left( \frac{W_t}{S_t} \right) \right]' = -\frac{2 \sigma^2}{\sigma^2} \left[ \left( \mu + \kappa \right) \frac{S_t}{W_t} - \left( \frac{S_t}{W_t} \right)^2 - \kappa \right], \]
hence

\[
\ln F' \left( \frac{W_t}{S_t} \right) = -\frac{2}{\sigma^2} \left( (\mu + \kappa) \ln \frac{W_t}{S_t} + \frac{S_t}{W_t} - \kappa \tau \frac{W_t}{S_t} \right) + S_0
\]

\[
F' \left( \frac{W_t}{S_t} \right) = \exp \left[ -\frac{2}{\sigma^2} \left( (\mu + \kappa) \ln \frac{W_t}{S_t} + \frac{S_t}{W_t} - \kappa \tau \frac{W_t}{S_t} \right) \right]
\]

\[
= \exp S_0 \exp \left[ -\frac{2}{\sigma^2} \left( (\mu + \kappa) \ln \frac{W_t}{S_t} + \frac{S_t}{W_t} - \kappa \tau \frac{W_t}{S_t} \right) \right]
\]

\[
= S_1 \exp \left[ -\frac{2}{\sigma^2} \left( (\mu + \kappa) \ln \frac{W_t}{S_t} + \frac{S_t}{W_t} - \kappa \tau \frac{W_t}{S_t} \right) \right], \text{ where } S_1 = \exp S_0,
\]

and

\[
F \left( \frac{W_t}{S_t} \right) = S_1 \int \exp \left[ -\frac{2}{\sigma^2} \left( (\mu + \kappa) \ln \frac{W_t}{S_t} + \frac{S_t}{W_t} - \kappa \tau \frac{W_t}{S_t} \right) \right] \frac{dW_t}{S_t} + S_2.
\]

Let \( x = \frac{W_t}{S_t} \),

\[
F (x) = S_1 \int \exp \left[ -\frac{2}{\sigma^2} \left( (\mu + \kappa) \ln x + \frac{1}{x} - \kappa \tau x \right) \right] dx + S_2
\]

\[
= S_1 \int \exp \left( -\frac{2}{\sigma^2} (\mu + \kappa) \ln x - \frac{2}{\sigma^2} \frac{1}{x} + \frac{2}{\sigma^2} \kappa \tau x \right) dx + S_2
\]

\[
= S_1 \int \exp \left( \ln x^{\frac{2}{\sigma^2}(\mu + \kappa)} - \frac{2}{\sigma^2} \frac{1}{x} + \frac{2\kappa \tau}{\sigma^2} x \right) dx + S_2
\]

\[
= S_1 \int \exp \left( \ln x^{\frac{2}{\sigma^2}(\mu + \kappa)} \right) \exp \left( -\frac{2}{\sigma^2} \frac{1}{x} + \frac{2\kappa \tau}{\sigma^2} x \right) dx + S_2
\]

\[
= S_1 \int x^{-\frac{2}{\sigma^2}(\mu + \kappa)} \exp \left( -\frac{2}{\sigma^2} \frac{1}{x} + \frac{2\kappa \tau}{\sigma^2} x \right) dx + S_2.
\]

Note \( F (\infty) \) is explosive, i.e., \( F (\infty) = +\infty \) is the upper bound of the value of \( F \), \( F (0) \) converges to some finite number and it is the lower bound of \( F \). Note \( F \) is a martingale and it is a increasing function of \( x \).

As time converges to \(+\infty\), let the probability of converging to upper bound of \(+\infty\)
be \( \varphi_u \), let the probability of converging to lower bound of \( F(0) \) be \( \varphi_l \), and we have

\[
\begin{align*}
\varphi_l F(0) + \varphi_u F(\infty) &= F \left( \frac{W_0}{S_0} \right), \\
\varphi_l + \varphi_u + \varphi_0 &= 1, \\
\varphi_0 &= 0.
\end{align*}
\]

where \( \varphi_0 \) denote the probability converge to any finite number which is larger than \( F(0) \), and it is possible to proof that as time evolves to infinity, \( \varphi_0 = 0 \), since the volatility of the process is positive. Hence, we have

\[
\varphi_l F(0) + F(\infty) - \varphi_l F(\infty) = F \left( \frac{W_0}{S_0} \right) \rightarrow \varphi_l = \frac{F \left( \frac{W_0}{S_0} \right) - F(\infty)}{F(0) - F(\infty)},
\]

and \( \varphi_u \to 0 \) and \( \varphi_l \to 1 \) as \( t \to \infty \).

### A.3 Proof of Theorem 3

Note the dynamic of spending and wealth are given as

\[
\begin{align*}
dS_t &= h \left( \frac{S_t}{W_t} \right) dt, \text{ where } h \left( \frac{S_t}{W_t} \right) = S_t \left( \kappa \left( \tau - \log \left( \frac{S_t}{W_t} \right) \right) - \frac{S_t}{W_t} + \mu - \frac{\sigma^2}{2} \right), \\
dW_t &= W_t (\mu dt + \sigma dZ) - S_t dt = (W_t \mu - S_t) dt + W_t \sigma dZ,
\end{align*}
\]

i.e.,

\[
\begin{align*}
&\begin{cases}
\frac{dS_t}{S_t} = h \left( \frac{S_t}{W_t} \right) dt, \\
\frac{dW_t}{W_t} = \frac{W_t \mu - S_t}{W_t} dt + \sigma dZ.
\end{cases}
\end{align*}
\]

Let

\[
a = h \left( \frac{S_t}{W_t} \right), \quad b = 0 \quad \text{and} \quad f = \frac{W_t \mu - S_t}{W_t}, \quad g = \sigma,
\]

let \( U_t = S_t/W_t \), then by Itô’s lemma

\[
d\frac{U_t}{U_t} = (a - f + g^2) dt - gdZ_t = \left( h \left( \frac{S_t}{W_t} \right) - \frac{W_t \mu - S_t}{W_t} + \sigma^2 \right) dt - \sigma dZ_t.
\]
Therefore,

\[
d\log \left( \frac{S_t}{W_t} \right) = \left( \frac{h \left( \frac{S_t}{W_t} \right)}{S_t} - \frac{W_t \mu - S_t}{W_t} + \sigma^2 \right) dt - \sigma dZ_t - \frac{1}{2} \sigma^2 dt
\]

\[
= \left( \frac{h \left( \frac{S_t}{W_t} \right)}{S_t} - \frac{W_t \mu - S_t}{W_t} + \frac{1}{2} \sigma^2 \right) dt - \sigma dZ_t.
\]

Simplifying the above equation, we obtain an Ornstein-Uhlenbeck process as

\[
d\log \left( \frac{S_t}{W_t} \right) = \kappa \left( \tau - \log \left( \frac{S_t}{W_t} \right) \right) dt - \sigma dZ_t,
\]

and

\[
\log \left( \frac{S_t}{W_t} \right) = \log \left( \frac{S_0}{W_0} \right) e^{-\kappa t} + \tau (1 - e^{-\kappa t}) - \sigma e^{-\kappa t} \int_{v=0}^{t} e^{\kappa v} dZ_v.
\]

Note \( \log (S_t/W_t) \) is a Gaussian process, in particular, conditional on \( \log (S_0/W_0) \). Assuming \( \log (S_0/W_0) \) has finite variance, then the mean and variance of \( \log (S_t/W_t) \) follow

\[
\mathbb{E} \left[ \log \left( \frac{S_t}{W_t} \right) \right] = \left( \mathbb{E} \left[ \log \left( \frac{S_0}{W_0} \right) \right] \right) e^{-\kappa t} + \tau \left( 1 - e^{-\kappa t} \right),
\]

\[
\text{Cov} \left[ \log \left( \frac{S_v}{W_v} \right), \log \left( \frac{S_t}{W_t} \right) \right] = \frac{\sigma^2}{2\kappa} e^{-\kappa v} (e^{\kappa t} - e^{-\kappa t}) + e^{-\kappa (v+t)} \text{Var} \left( \log \left( \frac{S_0}{W_0} \right) \right).
\]

If assume

\[
\log \left( \frac{S_0}{W_0} \right) = \tau - \sigma \int_{-\infty}^{0} e^{\kappa t} dZ_v,
\]

hence, the process of \( \log (S_t/W_t) \) is stationary with constant mean, variance, and au-
t covariance as

$$E \left[ \log \left( \frac{S_t}{W_t} \right) \right] = \tau,$$

$$\text{Var} \left[ \log \left( \frac{S_t}{W_t} \right) \right] = \frac{\sigma^2}{2\kappa},$$

$$\text{Cov} \left[ \log \left( \frac{S_v}{W_v} \right), \log \left( \frac{S_t}{W_t} \right) \right] = \frac{\sigma^2}{2\kappa} e^{-\kappa|t-v|}.$$  

As a result, $S_t/W_t$ is log-normal distributed with mean, variance, and autocorrelation as

$$E \left[ \frac{S_t}{W_t} \right] = \exp \left( \tau + \frac{\sigma^2}{4\kappa} \right),$$

$$\text{Var} \left[ \frac{S_t}{W_t} \right] = \left( \exp \left( \frac{\sigma^2}{2\kappa} \right) - 1 \right) \exp \left( 2\tau + \frac{\sigma^2}{2\kappa} \right),$$

$$\text{Cov} \left[ \frac{S_v}{W_v}, \frac{S_t}{W_t} \right] = \left( \exp \left( \frac{\sigma^2}{2\kappa} e^{-\kappa|t-v|} \right) - 1 \right) \exp \left( 2\tau + \frac{\sigma^2}{2\kappa} \right).$$

Note that the autocovariance depends only on the lag $|t - v|$ and not on time $t$. Therefore, $S_t/W_t$ is also stationary.

We now prove it is a mean-square ergodic process. Note the integral time scale of the stationary random process $S_t/W_t$ is given as

$$\Upsilon_{\text{int}} = \frac{1}{\left( \exp \left( \frac{\sigma^2}{2\kappa} \right) - 1 \right) \exp \left( 2\tau + \frac{\sigma^2}{2\kappa} \right) \int_0^\infty \left( \exp \left( \frac{\sigma^2}{2\kappa} e^{-\kappa\varphi} \right) - 1 \right) \exp \left( 2\tau + \frac{\sigma^2}{2\kappa} \right) d\varphi}$$

$$= \frac{1}{\exp \left( \frac{\sigma^2}{2\kappa} \right) - 1} \int_0^\infty \left( \exp \left( \frac{\sigma^2}{2\kappa} e^{-\kappa\varphi} \right) - 1 \right) d\varphi.$$  

Let

$$u = \frac{\sigma^2}{2\kappa} e^{-\kappa\varphi} \Rightarrow \frac{2\kappa}{\sigma^2} u = e^{-\kappa\varphi} \Rightarrow -\kappa\varphi = \log \left( \frac{2\kappa}{\sigma^2} u \right) \Rightarrow -\kappa d\varphi = d \log \left( \frac{2\kappa}{\sigma^2} u \right)$$

$$\Rightarrow -\kappa d\varphi = \frac{2\kappa}{\sigma^2} \frac{\sigma^2}{2\kappa u} du \Rightarrow -\kappa d\varphi = \frac{1}{u} du \Rightarrow d\varphi = \frac{1}{-\kappa u} du,$$
hence, we have

$$\int_0^\infty \left( \exp \left( \frac{\sigma^2}{2\kappa} e^{-\kappa \varphi} \right) - 1 \right) d\varphi = -\int_0^0 e^u - 1 \frac{du}{\kappa u} = \frac{1}{\kappa} \int_0^{\frac{\sigma^2}{2\kappa}} e^u - 1 \frac{du}{u}.$$  

Note

$$\lim_{u \to 0} \frac{e^u - 1}{u} = \lim_{u \to 0} \frac{e^u}{u} = 1,$$

and \((e^u - 1)/u\) strictly increases in \(u\), hence,

$$1 \leq \frac{e^u - 1}{u} \leq \frac{2\kappa}{\sigma^2} \left( e^{\frac{\sigma^2}{2\kappa}} - 1 \right), \text{ where } 0 \leq u \leq \frac{\sigma^2}{2\kappa}. $$

Therefore,

$$\int_0^{\frac{\sigma^2}{2\kappa}} \frac{e^u - 1}{u} du < \infty \implies \gamma_{int} < \infty.$$  

Hence, based on the Mean-Square Ergodic Theorem (Finite Autocovariance Time),\(^{10}\) we have that the process \(S_t/W_t\) is mean-square ergodic in the first moment, i.e.,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{S_v}{W_v} dv = \exp \left[ \tau + \frac{\sigma^2}{4\kappa} \right],$$

converges in squared mean. According to the properties of mean-square ergodic convergence, we have

$$\lim_{t \to \infty} \mathbb{E} \left[ \frac{1}{t} \int_0^t \frac{S_v}{W_v} dv \right] = \exp \left[ \tau + \frac{\sigma^2}{4\kappa} \right], \quad (18)$$

$$\lim_{t \to \infty} \text{Var} \left[ \frac{1}{t} \int_0^t \frac{S_v}{W_v} dv \right] = 0. \quad (19)$$

By the definition of preservation of capital, to prove the spending rule preserves

\(^{10}\)Original proof of ergodic theorem was in Neumann (1932). It is based on the spectral decomposition of unitary operators. Later a number of other proofs were published. The simplest is due to F. Riesz, see Halmos (1956).
capital, we only need to prove

\[ p \lim_{t \to \infty} \log \frac{W_t}{W_0} = \infty. \]

Note

\[ W_t = W_0 \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) t - \sigma Z_t - \int_{v=0}^{t} \frac{S_v}{W_v} dv \right], \]

hence, we have

\[ \log \frac{W_t}{W_0} = \left( \mu - \frac{\sigma^2}{2} - \frac{1}{t} \int_{v=0}^{t} \frac{S_v}{W_v} dv \right) t - \sigma Z_t, \]

\[ \implies \frac{1}{t} \log \frac{W_t}{W_0} = \mu - \frac{\sigma^2}{2} - \frac{1}{t} \int_{v=0}^{t} \frac{S_v}{W_v} dv - \frac{\sigma}{t} Z_t. \]

According to the Chebyshev’s inequality, we have for \( \forall \epsilon > 0, \)

\[ \Pr \left( \left| \frac{1}{t} \log \frac{W_t}{W_0} - \mathbb{E} \left( \frac{1}{t} \log \frac{W_t}{W_0} \right) \right| \geq \epsilon \right) \leq \frac{\Var \left( \frac{1}{t} \log \frac{W_t}{W_0} \right)}{\epsilon^2}. \] (20)

Moreover, note

\[ Z_t \sim N \left( 0, t \right), \quad \text{and} \quad -\frac{\sigma}{t} Z_t \sim N \left( 0, \frac{\sigma^2}{t} \right), \]

and

\[ \frac{1}{t} \int_{v=0}^{t} \frac{S_v}{W_v} dv \overset{t \to \infty}{\to} \exp \left[ \tau + \frac{\sigma^2}{4\kappa} \right], \]

hence, based on the results of (18) and (19), we have as \( t \to \infty, \)

\[ \mathbb{E} \left( \frac{1}{t} \log \frac{W_t}{W_0} \right) = \mu - \frac{\sigma^2}{2} - \exp \left[ \tau + \frac{\sigma^2}{4\kappa} \right], \quad \text{and} \quad \Var \left( \frac{1}{t} \log \frac{W_t}{W_0} \right) = \frac{\sigma^2}{t}. \]

Then according (21), we have as \( t \to \infty, \)

\[ \Pr \left( \left| \frac{1}{t} \log \frac{W_t}{W_0} - \left( \mu - \frac{\sigma^2}{2} - \exp \left[ \tau + \frac{\sigma^2}{4\kappa} \right] \right) \right| \geq \epsilon \right) \leq 0. \]
Since probability cannot be negative, hence, we have as $t \to \infty$, for $\forall \epsilon > 0$

$$\Pr \left( \left| \frac{1}{t} \log \frac{W_t}{W_0} - \left( \mu - \frac{\sigma^2}{2} - \exp \left[ \frac{\tau + \sigma^2}{4\kappa} \right] \right) \right| \geq \epsilon \right) = 0.$$ 

Therefore, according to the definition of convergence in probability, we have

$$p \lim_{t \to \infty} \left( \frac{1}{t} \log \frac{W_t}{W_0} \right) = \mu - \frac{\sigma^2}{2} - \exp \left[ \frac{\tau + \sigma^2}{4\kappa} \right].$$

By the condition (13)

$$\mu - \frac{\sigma^2}{2} - \exp \left[ \frac{\tau + \sigma^2}{4\kappa} \right] > 0,$$

hence, we have

$$p \lim_{t \to \infty} \left( \log \frac{W_t}{W_0} \right) = \infty \implies \lim_{t \to \infty} \Pr (W_t < W_0) = 0.$$ 

Given

$$\mu - \frac{\sigma^2}{2} - \exp \left[ \frac{\tau + \sigma^2}{4\kappa} \right] < 0,$$

we have

$$p \lim_{t \to \infty} \left( \log \frac{W_t}{W_0} \right) = -\infty \implies \lim_{t \to \infty} \Pr (W_t < W_0) = 1,$$

which completes the proof. ■

**A.4 Proof of Theorem 4**

By the definition of preservation of capital, to prove the spending rule preserves capital, we only need to prove

$$p \lim_{t \to \infty} \log \frac{W_t}{W_0} = \infty.$$ 

Note

$$dW_t = W_t (\mu_t dt + \sigma_t dZ) - S_t dt = (W_t \mu_t - S_t) dt + W_t \sigma_t dZ,$$
implies that

\[ W_t = W_0 \exp \left[ \int_{v=0}^{t} \left( \mu_v - \frac{1}{2} \sigma_v^2 - s_v \right) dv - \int_{v=0}^{t} \sigma_v dZ_v \right]. \]

hence, we have

\[
\log \frac{W_t}{W_0} = \int_{v=0}^{t} \left( \mu_v - \frac{1}{2} \sigma_v^2 - s_v \right) dv - \int_{v=0}^{t} \sigma_v dZ_v \\
\implies \frac{1}{t} \log \frac{W_t}{W_0} = \frac{1}{t} \int_{v=0}^{t} \left( \mu_v - \frac{1}{2} \sigma_v^2 - s_v \right) dv - \frac{1}{t} \int_{v=0}^{t} \sigma_v dZ_v.
\]

According to the Chebyshev’s inequality, we have for \( \forall \epsilon > 0 \),

\[
\Pr \left( \left| \frac{1}{t} \log \frac{W_t}{W_0} - \mathbb{E} \left( \frac{1}{t} \log \frac{W_t}{W_0} \right) \right| \geq \epsilon \right) \leq \frac{\text{Var} \left( \frac{1}{t} \log \frac{W_t}{W_0} \right)}{\epsilon^2}. \tag{21}
\]

and based on the assumption about the expectation that, as \( t \to \infty \),

\[
\mathbb{E} \left( \frac{1}{t} \log \frac{W_t}{W_0} \right) = \mathbb{E} \left[ \frac{1}{t} \int_{v=0}^{t} \left( \mu_v - \frac{1}{2} \sigma_v^2 - s_v \right) dv - \frac{1}{t} \int_{v=0}^{t} \sigma_v dZ_v \right] \\
= \mathbb{E} \left[ \frac{1}{t} \int_{v=0}^{t} \left( \mu_v - \frac{1}{2} \sigma_v^2 - s_v \right) dv \right] \to B > 0,
\]

since as \( t \to \infty \),

\[
\mathbb{E} \left[ \int_{v=0}^{t} \sigma_v dZ_v \right] = 0.
\]

Moreover, we have the assumption about the variance that, as \( t \to \infty \),

\[
\text{Var} \left[ \frac{1}{t} \log \frac{W_t}{W_0} \right] = \frac{1}{t^2} \text{Var} \left[ \int_{v=0}^{t} \left( \mu_v - \frac{1}{2} \sigma_v^2 - s_v \right) dv - \int_{v=0}^{t} \sigma_v dZ_v \right] \to 0.
\]

Therefore, we have as \( t \to \infty \),

\[
\Pr \left( \left| \frac{1}{t} \log \frac{W_t}{W_0} - B \right| \geq \epsilon \right) \leq 0.
\]
Since probability cannot be negative, hence, we have as $t \to \infty$, for $\forall \epsilon > 0$

$$\Pr \left( \left| \frac{1}{t} \log \frac{W_t}{W_0} - B \right| \geq \epsilon \right) = 0.$$ 

Therefore, according to the definition of convergence in probability, we have

$$p \lim_{t \to \infty} \left( \frac{1}{t} \log \frac{W_t}{W_0} \right) = B.$$

By the condition (13), if $B > 0$, hence, we have

$$p \lim_{t \to \infty} \left( \log \frac{W_t}{W_0} \right) = \infty \implies \lim_{t \to \infty} \Pr (W_t < W_0) = 0.$$ 

Given $B < 0$, we have

$$p \lim_{t \to \infty} \left( \log \frac{W_t}{W_0} \right) = -\infty \implies \lim_{t \to \infty} \Pr (W_t < W_0) = 1,$$

which completes the proof. ■

### A.5 Proof of Theorem 6

Let $S_t^*, \theta_t^*$, and $W_t^*$ be the optimal spending, optimal portfolio weight in risky asset, and wealth process, respectively, for the standard optimization problem without the constraint $p \lim_{t \to \infty} W_t = \infty$. Let the sequences of investment strategy, spending strategy and wealth process be:

$$\theta_t = \left( 1 - \frac{1}{1+k} \right) \theta_t^*, \quad S_t = \left( 1 - \frac{1}{1+k} \right) S_t^*, \quad W_t = \left( 1 - \frac{1}{1+k} \right) W_t^* + \frac{1}{1+k} W_0 e^{rt},$$

then we have

$$\lim_{k \to \infty} \theta_t = \theta_t^*, \quad \lim_{k \to \infty} S_t = S_t^*, \quad \lim_{k \to \infty} W_t = W_t^*.$$
Moreover, since the utility function is unbounded and satisfies the Lipschitz condition, the expected utility converges uniformly:

$$
\lim_{k \to \infty} E \left[ \int_{t=0}^{\infty} D_t u \left( t, S^k_t \right) dt \right] = E \left[ \int_{t=0}^{\infty} D_t u \left( t, S^*_t \right) dt \right].
$$

(Incomplete proof, to be revised.) ■