Heterogeneous Beliefs and Information: Cost of Capital, Trading Volume and Investor Welfare.pdf

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Information and Heterogeneous Beliefs: Cost of Capital, Trading Volume, and Investor Welfare

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ABSTRACT: In an incomplete market with heterogeneous prior beliefs, we show that public information can have a substantial impact on the ex ante cost of capital, trading volume, and investor welfare. The Pareto efficient public information system is the system enjoying the maximum ex ante cost of capital and the maximum expected abnormal trading volume. Imperfect public information increases the gains-to-trade based on heterogeneously updated posterior beliefs. In an exchange economy, this leads to higher growth in the investors’ certainty equivalents and, thus, a higher equilibrium interest rate, whereas the ex ante risk premium is unaffected by the informativeness of the public information system. Similar results are obtained in a production economy, but the impact on the ex ante cost of capital is dampened compared to the exchange economy due to welfare-improving reductions in real investments to smooth the investors’ certainty equivalents over time.

Keywords: heterogeneous beliefs; public information; dynamic trading; cost of capital; real effects; investor welfare.

I. INTRODUCTION

Financial markets are not complete, and investors in financial markets are not alike in terms of preferences, wealth, and beliefs. Acknowledging these facts, we develop a simple analytical model with exponential utility investors, who have heterogeneous beliefs over normally distributed dividends, which shows that the public information system plays a key role for the

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investors’ welfare, asset prices, trading volume, and for real investments. We show that the Pareto efficient public information system is the system enjoying the maximum expected abnormal trading volume. In an incomplete market, public information facilitates dynamic trading opportunities based on heterogeneously updated posterior beliefs, which allow the investors to take better advantage of their differences in optimism and confidence.1

In a partial equilibrium analysis, the investors’ expected utility (i.e., welfare) increases if the expected returns on their assets increase for some exogenous reason. In our general equilibrium analysis, the causality is the reverse: increased gains-to-trade due to imperfect public information increase the investors’ expected utility of future consumption, and this increases the equilibrium expected returns due to a reduced demand for additional units of future consumption. In addition, the higher expected utility of future consumption increases the hurdle rate for real investments to be valuable; i.e., the cost of capital, leading to welfare-improving reductions in aggregate investments. Therefore, policy implications based on empirical analyses of the impact of financial reporting on the cost of capital must be interpreted with great care.

The vast majority of prior studies in the accounting and finance literature on the impact of public information system choices—such as financial reporting regulation, on equilibrium asset prices, trading volume, real investments, and investor welfare—recognize differences in preferences and/or wealth, but assume that the investors’ prior beliefs are identical, although their posterior beliefs may vary due to differences in the information they have received. In complete markets, this assumption leads to so-called no-trade theorems (Milgrom and Stokey 1982), implying that the theory cannot explain the significant trading volume in actual financial markets, for example, around earnings announcements as first documented by Beaver 1968), unless some unmodeled noise trading is injected into the price system (Grossman and Stiglitz 1980; Hellwig 1980; Verrecchia 1982; Kyle 1985).

But why should all investors have been born equal (cf. Harsanyi 1968)? Some investors may be more optimistic or more confident in their estimates than others, for example, due to different DNA profiles or past experiences that are completely unrelated to the uncertainty and information in financial markets (Morris [1995] provides a critical discussion of the common prior assumption in economic theory).2 Moreover, despite significant financial innovations over the last four decades, financial markets are probably still incomplete even if we allow for dynamic trading strategies, for example, due to heterogeneous prior beliefs. We develop a simple equilibrium model with heterogeneous prior beliefs and incomplete markets allowing us to study the impact of public information system choices on both equilibrium asset prices, trading volume, real investments, and investor welfare.

A large literature in accounting and finance studies the impact of information on firms’ cost of equity capital both theoretically and empirically.3 The underlying idea in this literature and the conventional wisdom among accounting standard-setters seems to be that more mandated public disclosure of economy-wide information will reduce firms’ cost of equity capital. For example, “Based on our framework, increasing the quality of mandated disclosures should in general move

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1 We use the term “optimism” as pertaining to the expected future dividends, and the term “confidence” as pertaining to the precision (or inverse variance) of future dividends.
2 There is a growing literature on heterogeneous beliefs and asset pricing. Theoretical studies include, for example, Williams (1977), Detemple and Murthy (1994), Morris (1996), Basak (2005), and Bhamra and Uppal (2010). Buraschi and Jiltsov (2006) and David (2008) also provide empirical evidence of the impact of heterogeneous beliefs. However, this literature is silent with respect to the impact of the informativeness of public information on the cost of capital, real investments, or investor welfare.
the cost of capital closer to the risk-free rate for all firms in the economy” (Lambert, Leuz, and Verrecchia 2007, 387–388); and “Numerous academic studies have concluded that more information in the marketplace lowers the cost of capital. Upon reflection, although it is nice to have empirical support, academic studies are not really necessary to reach this conclusion—it is intuitive” (Foster 2003, 1). The argument put forward is simple: A firm’s cost of equity capital (required expected rate of return) is the riskless interest rate plus a risk premium. Releasing more informative public signals reduces the uncertainty about the size and timing of future cash flows and, therefore, also the risk premium.4

This argument, however, pertains only to the cost of capital when measured after the release of information, i.e., the ex post cost of capital. Christensen et al. (2010) show that if the cost of capital is measured before any signals from the information system are realized, i.e., the ex ante cost of capital, then the public information system has no impact on the ex ante cost of capital and, thus, no impact on the ex ante stock prices in competitive exchange economies with homogeneous prior beliefs and both public and private investor information. The public information system only serves to affect the timing of release of information and, thereby, to affect the allocation of the total risk premium for future cash flows over time.

Is a low ex ante cost of equity capital and, thus, high ex ante stock prices, “good” or “bad” for investors? In a partial equilibrium analysis focusing on a single firm and its shareholders, the answer is clearly “good.” This is merely a cousin of the familiar value maximization principle for competitive markets (cf. Debreu 1959). However, financial reporting regulation (and other mandated disclosure requirements) is about choosing information systems for the economy at large. In such settings, a general equilibrium analysis is warranted, and we show that the welfare consequences of policy changes can be very different from what should be expected from partial equilibrium analyses based on the cost of capital.

For example, how is the other component of the cost of equity capital, i.e., the riskless interest rate, affected by changes in the information system in the economy? In competitive exchange economies with homogeneous prior beliefs, time-additive preferences, and public information, the ex ante riskless interest rates are not affected by changes in the information system (see, e.g., Christensen et al. [2010] and the references therein). We show that even for an exchange economy, but with heterogeneous prior beliefs and incomplete markets, the ex ante equilibrium interest rate is affected by the informativeness of the public information system. The ex ante equilibrium interest rate is a linear increasing function of the growth in the investors’ certainty equivalents. More efficient dynamic trading opportunities based on the heterogeneity in prior beliefs and public information increase the growth in certainty equivalents and, thus, the interest rate, while the ex ante risk premium is unaffected by the public information system (Proposition 1). In other words, from a general equilibrium perspective, the preferred public information system is the system enjoying the highest ex ante cost of equity capital and, thus, the lowest ex ante stock prices (Proposition 6).

Our initial analysis focuses on a competitive exchange economy. An important question is whether the higher ex ante cost of capital comes with a negative real effect due to costlier financing

4 If the information pertains to firm-specific risks, it is diversifiable and does not affect expected returns. But if it pertains to economy-wide risk factors, it lowers market and other systematic factor risk premia and, thus, in general, moves expected returns closer to the risk-free rate (see the discussion in Easley and O’Hara [2004], Lambert et al. [2007], and Hughes et al. [2007]). The term “in general” refers to the fact that information may affect the cross-sectional differences in expected returns through posterior betas, but the value-weighted sum of posterior betas for, for example, the market factor must still be equal to 1, and the fact that posterior expected rates of returns are signal-contingent through the impact of the particular signals on the posterior means of future normally distributed dividends. For more general distributions of dividends, dollar risk premia may also be signal-contingent.
of firms’ real investments in a more general production economy. We show that with a standard production technology, a higher ex ante cost of capital is associated with positive real effects on welfare. The higher ex ante cost of capital is a consequence of a higher growth in certainty equivalents due to more efficient dynamic trading opportunities and, thus, the intertemporal trade-off between current and future aggregate consumption changes such that it becomes optimal for investors to reduce real investments and, thus, to consume more now and consume less in the future. Such welfare-improving reductions in real investments decrease the equilibrium ex ante cost of capital, but not all the way back to the level with less efficient dynamic trading opportunities (Proposition 7).

In our model, the impact of public information on the ex ante cost of capital, real investments, and investor welfare is due to more efficient dynamic trading opportunities in an incomplete capital market with heterogeneity in prior investor beliefs. The model is a two-period extension of the classical single-period capital asset pricing model with heterogeneous beliefs of Lintner (1969). For simplicity, we assume there is a single risky asset paying a known dividend at \( t = 0 \) and a normally distributed dividend at \( t = 2 \). In addition, there is a zero-coupon bond available for trade paying one unit of account at \( t = 2 \). The investors have time-additive exponential utility, and we assume, for simplicity, that they have identical time-preference rates and risk-aversion parameters. However, their subjective prior beliefs at \( t = 0 \) for the dividend at \( t = 2 \) can differ with regard to both the mean and the precision.

In settings with heterogeneous beliefs, Pareto efficient allocations require not only an efficient sharing of the risks, but also an efficient side-betting arrangement (see, e.g., Wilson 1968). If the investors’ prior precisions are identical, then Pareto efficient side-betting (or speculative positions) based on their disagreements about the mean can be achieved by trading in the risky asset and the zero-coupon bond at \( t = 0 \). The optimistic (pessimistic) investors hold more (less) than their efficient risk-sharing fraction of the risky asset (Proposition 3). In other words, if the investors have homogeneous prior precisions, the risky asset and the zero-coupon bond constitute an effectively complete market with no need for subsequent information-contingent trading after the initial trading at \( t = 0 \).

If the investors have different prior precisions, trading in the risky asset and the zero-coupon bond at \( t = 0 \) does not facilitate efficient side-betting. An investor with a low (high) prior precision would like to have a payoff at \( t = 2 \), which is a convex (concave) function of the dividend. The key is that investors with low precisions value a convex payoff more than investors with higher precisions.

In this setting, it can be valuable to have public information and another round of trading at the interim date \( t = 1 \). We consider a public information system generating a public signal at \( t = 1 \) equal to the \( t = 2 \) dividend on the risky asset plus independent noise. The investors have homogeneous normally distributed beliefs for the noise in the signal, i.e., a zero mean and a common signal precision, such that the investors’ posterior precisions for the \( t = 2 \) dividend are equal to their heterogeneous prior dividend precisions plus the common signal precision. This

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\(^5\) It is straightforward to extend our analysis to an economy with multiple risky assets. The impact of the public information system on the ex ante cost of capital is through the riskless interest rate (which is common to all firms) and not through the ex ante risk premium even in a multi-asset extension of our model. Hence, no key additional economic insights are obtained by such an extension. Therefore, we follow the literature (see, e.g., Christensen et al. 2010; Bloomfield and Fischer 2011; Lambert et al. 2012) and assume there is only a single risky asset. However, note that in a model with heterogeneous prior beliefs, diversification plays a much smaller role than in homogeneous prior beliefs settings. With heterogeneous prior beliefs, there will be a demand for side-betting opportunities on both economy-wide and firm-specific events (Christensen and Feltham 2003, Section 4.1.3). Furthermore, side-betting opportunities facilitated by firm-specific information may affect the equilibrium interest rate and, thus, the ex ante cost of capital, even if there is no economy-wide information.
specification allows us to measure the informativeness of the public information system by the
gain-to-trade following from an imperfect public signal at \( t = 1 \) translate directly into
higher ex ante certainty equivalents of the investors’ \( t = 2 \) consumption, and this reduces the
demand for the zero-coupon bond at \( t = 0 \) in an exchange economy.\(^6\) Since the zero-coupon bond is
in zero net-supply, the gains-to-trade increase the equilibrium interest rate from \( t = 0 \) to \( t = 2 \).\(^7\) Hence, the equilibrium interest rate is also maximized for the public information system with a signal
precision equal to the investors’ average prior dividend precision. This is also the information
system that has the maximum expected abnormal trading volume at \( t = 1 \) (Proposition 5).

Next, Section II presents the model and derives the equilibrium asset prices and asset demands
in the incomplete exchange economy. Section III establishes the relationship between the
informativeness of the public information system and the equilibrium asset prices, the ex ante
cost of capital, the expected abnormal trading volume, and the investors’ welfare. The preceding results
are extended to a production economy in Section IV, and Section V concludes.

II. THE MODEL

In our incomplete market model, we examine the impact of heterogeneity in prior beliefs and
signal precision on equilibrium asset prices, trading volume, real investments, and investor welfare
for a two-period economy in which investors have identical preferences but differ in their prior
beliefs about the future dividends on a single risky asset. The following two subsections describe
the model and the equilibrium, respectively.

\(^6\) In the production economy, there are two ways of increasing current consumption at the expense of future
consumption: (1) reducing the demand for the zero-coupon bond as in the pure exchange economy, and (2)
reducing investments in the riskless production technology. Therefore, in equilibrium, the impact on the
interest rate will be smaller in the production economy than in an otherwise identical pure exchange
economy.

\(^7\) For simplicity, we assume that there is no consumption at the interim date \( t = 1 \) and, thus, only the equilibrium
interest rate from \( t = 0 \) to \( t = 2 \) has any economic substance (and not how that interest rate is divided between the
two periods). Therefore, even though the interest rate covers two periods, we can still refer to it as a “rate of
return” per consumption period.
Investor Beliefs and Preferences

There are two consumption dates, $t=0$ and $t=2$, and there are $I$ investors who are endowed at $t=0$ with a portfolio of securities, potentially receive public information at $t=1$, and receive terminal normally distributed dividends from their portfolio of securities at $t=2$ (the key notation is summarized in Appendix B). The trading of the marketed securities takes place at $t=0$ and $t=1$ based on heterogeneous prior and posterior beliefs, respectively. There are two securities available for trade at $t=0$ and $t=1$: a zero-coupon bond that pays one unit of consumption at $t=2$ and is in zero net-supply, and the shares of a single risky asset that has a fixed non-zero net-supply $Z$ throughout. The investors are endowed with $\gamma_i$ units of the $t=2$ zero-coupon bond and $z_i$ shares of the risky asset, $i = 1, 2, \ldots, I$. In addition, the investors are endowed with $\kappa_i$ units of a zero-coupon bond, also in zero net-supply, paying one unit of consumption at $t=0$. Let $\gamma_{it}$ and $x_{it}$ represent the units held by investor $i$ of the $t=2$ zero-coupon bond and the risky asset after trading at date $t$, respectively. The market-clearing conditions at date $t$ are:

$$
\sum_{i=1}^{I} \gamma_{it} = 0, \quad \sum_{i=1}^{I} x_{it} = Z, \quad t = 0, 1.
$$

A share of the risky asset pays a dividend $d_0$ at date $t=0$ and a dividend $d$ at date $t=2$. We assume the investors have heterogeneous prior beliefs with respect to the $t=2$ dividend represented by $\phi_i(d) \sim N(m_i, \sigma_i^2)$, $i = 1, \ldots, I$, where $m_i$ is the expected dividend per share and $\sigma_i^2$ is the variance of the dividend per share for investor $i$. In our initial analysis, these dividends are exogenously specified, i.e., we consider an exchange economy, but we extend the model to a production economy in Section IV in which the dividends are endogenous.

At $t=1$, all investors receive a public signal $y$, which is jointly normally distributed with the dividend paid by the risky asset at $t=2$. The public signal is given as the $t=2$ dividend plus noise, i.e., $y = d + \varepsilon$, where $\varepsilon$ and $d$ are independent and $\varepsilon \sim N(0, \sigma^2)$. We refer to $h_i = 1/\sigma_i^2$ as the common signal precision, and we use $h_{(i)} = 1/\sigma_{(i)}^2$ throughout to denote precisions for the associated variances. Hence, while the investors may disagree about the fundamentals in the economy (i.e., the future dividends), we assume the investors have homogeneous beliefs about the noise in the public signal. This is in contrast to the growing “differences-of-opinion” literature in which the investors have homogeneous beliefs about the fundamentals, but disagree on how to interpret common public signals. This literature is targeted toward explaining empirical stylized facts for the relationship between trading volume and stock returns. Our specification of the heterogeneity in beliefs allows us to ask how the informativeness of the public information, i.e., the signal precision $h_{(i)}$, affects the equilibrium asset prices, trading volume, real investments, and the investors’ welfare.

The prior beliefs of investor $i$ for the public signal and the $t=2$ dividend are $\phi(y, d) \sim N(\mu_i, \Sigma_i)$, where:

$$
\mu_i = \left( \begin{array}{c} m_i \\ m_i \end{array} \right), \quad \Sigma_i = \left( \begin{array}{cc} \sigma^2 + \sigma_i^2 & \sigma_i^2 \\ \sigma_i^2 & \sigma^2 \end{array} \right).
$$

Hence, conditional on the public signal, the posterior beliefs of investor $i$ at $t=1$ about the $t=2$ dividend are:

$$
\phi_{it}(y, d) = \frac{1}{\sqrt{2\pi h_{(i)}}} \exp \left( -\frac{(yd - \mu_i)^2}{2h_{(i)}} \right).
$$
dividend are \( \varphi_i(d\mid y) \sim N(m_{i1}, \sigma^2_{i1}) \), where:

\[ m_{i1} = \omega_i y + (1 - \omega_i) m_i, \quad \omega_i = \frac{\sigma_i^2}{\sigma_i^2 + \sigma_e^2}, \]  

\[ \sigma^2_{i1} = \omega_i \sigma_e^2, \quad h_{i1} = h_i + h_e. \]  

(1a)

(1b)

The posterior mean is a linear function of the investors’ signal, while the posterior variance only depends on the informativeness of the public signal and not on the signal realization.

The investors trade in the zero-coupon bond with equilibrium price \( p_0 \) at \( t = 0 \) and \( \beta_1 \) at \( t = 1 \). We assume without loss of generality that \( \beta_1 = 1 \) since there is no consumption at \( t = 1 \). The equilibrium price of the risky asset at \( t = 0 \) is denoted \( p_0 \). The ex post equilibrium price of the risky asset at \( t = 1 \) given the public signal \( y \) is denoted \( p_1(y) \).

Investor \( i \)’s consumption at date \( t = 0 \) and \( t = 2 \) is denoted \( c_0 \) and we assume the investors have time-additive exponential utility. The investors have common period-specific exponential utility functions, i.e., \( \mu_{i0}(c_0) = -\exp[-rc_0]\) and \( \mu_{i2}(c_2) = -\exp[-\delta \exp[-rc_2]] \), where \( r > 0 \) is the investors’ common constant absolute risk aversion, and \( \delta \) is the common utility discount rate for date \( t = 2 \) consumption. Our results are qualitatively unaffected by allowing heterogeneity in risk aversion and utility discount rates.\(^{10}\)

Equilibrium with Heterogeneous Beliefs and Public Information

There are two rounds of trading: one round of trading at \( t = 0 \) prior to the release of information, and a second round of trading subsequent to the release of the public signal at \( t = 1 \). We solve for the equilibrium by first deriving the ex post equilibrium \( t = 1 \), and given this equilibrium, we can subsequently derive the ex ante equilibrium at \( t = 0 \).

Equilibrium Prices and Demand Functions at Date \( t = 1 \)

From the perspective of \( t = 1 \), date \( t = 2 \) consumption for investor \( i \) is \( c_{i2} = x_{i1}d + y_{i1} \), and is thus normally distributed given the public signal \( y \) at \( t = 1 \). Investor \( i \) maximizes his certainty equivalent of \( t = 2 \) consumption subject to his budget constraint, and given period-specific exponential utility, this can be expressed as:

\[
\max_{x_{i1}, y_{i1}} CE_{i2}(x_{i1}, \gamma_{i1}\mid y, \gamma_0, x_0) = \gamma_{i1} + m_{i1} x_{i1} - \frac{1}{2} \rho \sigma^2_{i1} x^2_{i1},
\]

subject to \( \gamma_{i1} + p_1(y)x_{i1} \leq \gamma_0 + p_1(y)x_0 \).

The first-order conditions imply that the optimal portfolio at \( t = 1 \) is given by:

\[
x_{i1}(y) = \rho h_{i1}\left( m_{i1} - p_1(y) \right); \quad \gamma_{i1}(y) = \gamma_0 + p_1(y)x_0 - p_1(y)x_{i1}(y),
\]

where \( \rho \equiv 1/r \) is the investors’ common risk tolerance, and \( h_{i1} = h_i + h_e \) is investor \( i \)’s posterior precision for the terminal dividend. Market clearing at date \( t = 1 \) implies that:

\[
p_1(y) = m^\rho_{i1} - r \sigma^2_{i1} Z / I,
\]

where \( m^\rho_{i1} \) is the precision weighted average of the investors’ posterior means, i.e.:\(^{10}\)

\[ 10 \text{ The difference is that whenever we calculate averages of means and precisions in equilibrium relations, these averages will be risk tolerance weighted averages, and the common utility discount rate will be replaced by a risk tolerance weighted average of the investors’ personal utility discount rates.}
\[\hat{m}_i^t = \frac{1}{I} \sum_{i=1}^I \frac{h_i^t}{h_i} m_{i1}, \quad \hat{h}_i = \frac{1}{I} \sum_{i=1}^I h_i,\]

and \(\bar{\sigma}_i^2\) is the inverse of the average posterior precision, i.e., \(\bar{\sigma}_i^2 = 1/\hat{h}_i\).

Inserting the equilibrium price of the risky asset into investor \(i\)'s demand function in (2), yields the equilibrium demand functions:

\[x_i^t(y) = \rho h_i (m_{i1} - [\hat{m}_i^t - r\bar{\sigma}_i^2 Z/L]).\]

The posterior mean and precision, i.e., \(m_{i1}\) and \(h_{i1}\), are functions of the priors and the signal precision. Hence, the equilibrium price of the risky asset and the equilibrium demand functions at date \(t = 1\) are affected by both the priors and the signal precision. Moreover, the equilibrium demand functions are linear functions of the public signal (through the posterior mean, \(m_{i1} = \omega_3 y + (1 - \omega_3) m_i\)), which implies that, in general, there is non-trivial trading at \(t = 1\), in equilibrium.

**Equilibrium Prices and Demand Functions at Date \(t = 0\)**

We now determine the equilibrium \(ex\) \(ante\) prices and demand functions at \(t = 0\), taking the equilibrium at \(t = 1\) characterized by Equations (3) and (4) as given. From the perspective of \(t = 0\), investor \(i\)'s date \(t = 2\) consumption is \(c_{i2} = [d - p_1(y)] x_i^t(y) + p_1(y) x_0 + \gamma_{i0}\), and investor \(i\)'s date \(t = 0\) consumption is \(c_{i0} = [p_0 + d_0 \bar{\tilde{z}}_i + \beta_0 \bar{\tilde{y}}_i + \kappa_i - p_0 x_0 - \beta_0 \gamma_{i0}].\) Conditional on the public signal at \(t = 1\), investor \(i\)'s \(t = 1\) certainty equivalent of \(t = 2\) consumption is:

\[C_{E2} \left( x_{i0}, \tilde{\gamma}_{i0}, x_i^t(y) | y \right) = \gamma_{i0} + p_1(y) x_0 + [m_{i1} - p_1(y)] x_i^t(y) - \frac{1}{2} r \bar{\sigma}_i^2 (x_i^t(y))^2.\]

From the perspective of \(t = 0\), the second term in \(C_{E2} \left( x_{i0}, \tilde{\gamma}_{i0}, x_i^t(y) | y \right)\) is a normally distributed variable, while the last two terms contain products of normally distributed variables if the \(t = 1\) equilibrium demand function \(x_i^t(y)\) varies with the public signal at \(t = 1\). Substituting in the equilibrium demand functions and the equilibrium price of the risky asset at \(t = 1\), i.e., Equations (4) and (3), allows us to calculate investor \(i\)'s \(t = 0\) certainty equivalent of \(t = 2\) consumption \(C_{E2} \left( x_{i0}, \tilde{\gamma}_{i0} \right)\) as a function of the portfolio \(\left( x_{i0}, \tilde{\gamma}_{i0} \right)\) chosen at \(t = 0\) (see Lemma A.1 in Appendix A).

With the investors’ \(t = 0\) certainty equivalent of their \(t = 2\) consumption determined, investor \(i\)'s decision problem at \(t = 0\) can be stated as follows:

\[
\max_{\gamma_{i0}, x_{i0}} \left( - r C_{E0} (x_{i0}, \gamma_{i0}) - \exp(-\delta) \exp \left( - r C_{E2} (x_{i0}, \gamma_{i0}) \right) \right),
\]

where \(C_{E0} (x_{i0}, \gamma_{i0}) = \left[ p_0 + d_0 \bar{\tilde{z}}_i + \beta_0 \bar{\tilde{y}}_i + \kappa_i - p_0 x_0 - \beta_0 \gamma_{i0} \right]\) is the \(t = 0\) certainty equivalent. The first-order condition for investments in the zero-coupon bond implies that:

\[t = \delta + r \left( C_{E2} (x_{i0}, \gamma_{i0}) - C_{E0} (x_{i0}, \gamma_{i0}) \right),\]

where \(t = -\ln \beta_0\) is the interest rate from \(t = 0\) to \(t = 2\) of the zero-coupon bond. The first-order condition for investments in the risky asset then implies that:

\[p_0 = \beta_0 \frac{\partial C_{E2} (x_{i0}, \gamma_{i0})}{\partial x_{i0}}.\]

Invoking the market-clearing conditions at \(t = 0\), the following proposition yields the \(t = 0\) equilibrium price of the risky asset (all proofs are relegated to Appendix A).
Proposition 1: The ex ante equilibrium price of the risky asset at \( t = 0 \) is equal to the equilibrium riskless discount factor times the risk-adjusted expected dividend, i.e.:\(^{11}\)

\[
p_0 = \beta_0 E^Q[d]. \tag{8}\]

The risk-adjusted expected dividend is independent of the signal precision \( h_e \), and it can be expressed as a function of the prior means and variances, i.e.:

\[
E^Q[d] = \bar{m}^h - r\sigma^2 Z/I, \tag{9a}\]

where:

\[
\bar{m}^h = \frac{1}{I} \sum_{i=1}^{I} h_i m_i, \quad \bar{h} = \frac{1}{I} \sum_{i=1}^{I} h_i, \quad \sigma^2 = \frac{1}{\bar{h}}. \tag{9b}\]

Hence, given the priors, the risk-adjusted expected dividend is independent of the informativeness of the public signal at \( t = 1 \) (\( h \)) and, in particular, it is determined entirely by the prior beliefs as if there would be no second round of trading at \( t = 1 \). In other words, the informativeness of the public signal at \( t = 1 \) affects the ex ante equilibrium asset price only through the impact on the equilibrium interest rate.

Remark 1: In equilibrium, investor \( i \)'s \( t = 0 \) equilibrium demand function for the risky asset is given by:

\[
x_i^0 = \rho h_i [m_i - E^Q[d]]. \tag{10}\]

Note that the equilibrium demand for the risky asset is the same as in an otherwise identical economy in which there is no public information at \( t = 1 \). In other words, the investors’ equilibrium demands are myopic, independently of the informativeness of the forthcoming public signal. The equilibrium demand is increasing in the investors’ prior mean and in the prior dividend precision. In other words, the more optimistic (i.e., higher prior mean) and the more confident (i.e., higher prior precision) investors invest more in the risky asset than the more pessimistic (i.e., lower prior mean) and less confident (i.e., lower prior precision) investors. This result is a consequence of the investors’ incentive to take speculative positions based on their heterogeneous prior beliefs and, hence, the equilibrium entails “side-betting.” On the other hand, with homogeneous priors, all investors hold the same efficient risk sharing equilibrium positions in the risky asset, i.e.: \( x_{i0}^* = Z/I \).

Substituting the equilibrium demand for the risky asset into the ex ante certainty equivalents of \( t = 0 \) and \( t = 2 \) consumption (see Lemma A.1), respectively, we obtain the result.

Remark 2: In equilibrium, investor \( i \)'s ex ante certainty equivalents of \( t = 0 \) and \( t = 2 \) consumption, respectively, can be expressed as:

\[
CE_{i0}^* = d_0 z_i + p_0 [z_i - x_{i0}^*] + \beta_0 [\gamma_i - \gamma_{i0}] + \kappa_i, \tag{11a}\]

\(^{11}\) This means that we can define the risk-adjusted probability measure \( Q \) explicitly such that under \( Q \), the terminal dividend is normally distributed as \( d \sim N(\bar{m}^h - r\sigma^2 Z/I, \sigma^2) \), and the noise in the public signal \( \varepsilon \) is normally distributed as \( \varepsilon \sim N(0, \sigma^2) \). Note that while the expected dividend under \( Q \) is uniquely determined in equilibrium, the variance of the dividend under \( Q \) is not uniquely determined due to the market incompleteness and, thus, we may just take it to be \( \sigma^2 \). The lack of the uniqueness of the variance has no consequences in the subsequent analysis.
\[ CE_{i2}^* = \gamma_{i0}^s + U_{1i} + \frac{1}{2} \rho h_i m_i^2 - (E_Q^0[d])^2, \]  

(11b)

where:

\[ U_{1i} = \frac{1}{2} \rho \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i (\bar{h} + h_i)^2} \right]. \]

(11c)

The \textit{ex ante} certainty equivalent of \( t = 2 \) consumption depends only on the signal precision \( h_i \) through \( U_{1i} \) and \( \gamma_{i0}^s \). As a function of \( h_i \), \( U_{1i} \) is bell-shaped with respect to the signal precision \( h_i \) and attains its maximum for \( h_i = \bar{h} \).\(^{12}\) If investor \( i \)'s prior dividend precision is equal to the investors' average prior dividend precision, i.e., \( h_i = \bar{h} \), then \( U_{1i} = 0 \) independently of the signal precision \( h_i \).

It follows from Equations (11b) and (11c) that the investors’ equilibrium \textit{ex ante} certainty equivalents of \( t = 2 \) consumption are maximized (holding the investment in the zero-coupon bond fixed) when the public signal is imperfect with a signal precision equal to the investors’ average prior dividend precision, and that investors with extreme prior dividend precisions benefit the most. However, the equilibrium investment in the zero-coupon bond \( \gamma_{i0}^s \) also depends on the signal precision. If the signal precision is such that the investors have a high (low) value of \( U_{1i} \), then they also have an incentive to reduce (increase) the investment in the zero-coupon bond to smooth consumption over the two consumption dates. In equilibrium, however, the interest rate must reflect these incentives such that the net-demand for the zero-coupon bond is equal to the net-supply of zero. Changes in the equilibrium interest rate affect the \textit{ex ante} price of the risky asset (see (8)) and, thus, the value of the investors’ endowments, which also affect the investors’ equilibrium investment in the zero-coupon bond. We investigate the sources and implications of these dependencies of the signal precision on asset prices, trading volume, and investor welfare in Section III.

Substituting the equilibrium certainty equivalents in expressions (11a) and (11b) into the expression for the interest rate (6) yields:

\[ i = \delta + r(CE_{i2}^* - CE_{i0}^*). \]

(12)

Using the market-clearing conditions for the riskless and risky assets, and simplifying yield the equilibrium interest rate.

**Proposition 2:** The equilibrium interest rate is given by:

\[ i = \delta + r \bar{U}_1 + \Phi(\{m_i, \sigma_i^2\}_{i=1,\ldots,I}), \]

(13a)

where:

\[ \bar{U}_1 = \frac{1}{I} \sum_{i=1}^I U_{1i} = \frac{1}{2} \rho \hat{\sigma}^2 \frac{1}{I} \sum_{i=1}^I \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i (\bar{h} + h_i)^2} \right], \]

(13b)

and \( \Phi(\cdot) \) is a function of the priors but independent of the signal precision:

\[ \Phi(\{m_i, \sigma_i^2\}_{i=1,\ldots,I}) = r[\bar{m}^h - d_0]Z/I - \frac{1}{2} r^2 \sigma^2(Z/I)^2 + \frac{1}{2} \frac{1}{I} \sum_{i=1}^I h_i m_i^2 - \frac{1}{2} (\bar{m}^h)^2 \bar{h}. \]

(13c)

\(^{12}\) By “bell-shaped” we mean a function that is first increasing, obtains a unique maximum, and is then decreasing.
If the investors have homogeneous prior expected dividends, i.e., $m_i = m$, then:

$$\Phi(\{m, \sigma_i^2\}_{i=1,...,I}) = r[m - d_0]Z/I - \frac{1}{2} \sigma^2(Z/I)^2. \quad (13d)$$

If the investors have homogeneous prior dividend precisions, the equilibrium interest rate is independent of the signal precision.

The equilibrium interest rate is equal to the utility discount rate plus a function of the signal precision and the priors. The function $\Phi(\cdot)$ is a function of the priors only and is thus independent of the signal precision. Hence, the signal precision only affects the equilibrium interest rate and, thereby, the equilibrium price of the risky asset (since $E^Q[d]$ is independent of the signal precision $h_i$ by Proposition (1)) through the logarithmic terms $\{U_i\}_{i=1,...,I}$.

If the investors have homogenous prior precisions (i.e., $h_i = h$ for all $i$), the logarithmic terms are all equal to 0. In this case the signal precision does not affect the equilibrium prices at $t = 0$. Moreover, with homogeneous prior beliefs, i.e., $m_i = m$, and $h_i = h$, for $i = 1, 2, .. I$, the equilibrium interest rate can be expressed as:

$$t = \delta + r(m - d_0)Z/I - \frac{1}{2} \sigma^2(Z/I)^2.$$

Hence, in a benchmark setting with homogeneous prior beliefs, the equilibrium interest rate is given as the utility discount rate plus a risk-adjusted expected dividend growth minus a risk premium for the uncertainty in the dividend growth. This is the standard expression for the equilibrium interest rate in effectively complete markets with time-additive HARA utilities and homogeneous prior beliefs. On the other hand, if the investors have homogeneous prior expected dividends, but heterogeneous prior dividend precisions, then there is an additional component to the equilibrium interest rate, i.e.:

$$t = \delta + r\bar{U}_1 + r[m - d_0]Z/I - \frac{1}{2} \sigma^2(Z/I)^2.$$

This additional component $r\bar{U}_1$ depends on the signal precision, and it plays a key role in the following analysis (as also indicated by Remark 2).

**III. THE IMPACT OF SIGNAL PRECISION**

We are interested in how the informativeness of the public signal, i.e., the signal precision, affects the *ex ante* equilibrium prices, the trading volume, and the investors’ *ex ante* expected utilities at $t = 0$ when the investors have heterogeneous beliefs including heterogeneous prior means and/or heterogeneous prior dividend precisions.

**Ex Ante Equilibrium Prices and Trading Volume**

Proposition 1 establishes that the equilibrium asset prices at $t = 0$ are only affected by the signal precision through the equilibrium interest rate. Furthermore, Proposition 2 establishes that the equilibrium interest rate is also independent of the signal precision if the investors have homogeneous prior dividend precisions. This is due to the fact that in this case there is no equilibrium trading at $t = 1$ based on the public signal.

**Proposition 3:** When the investors have identical prior dividend precisions, i.e., $h_i = h$, $i = 1, \ldots, I$, the date $t = 1$ equilibrium portfolios are independent of both the signal precision and the realized public signal, and they are equal to the date $t = 0$ equilibrium portfolios, i.e.:
If the investors have homogeneous prior dividend precisions, i.e., $h_i = h$ for all $i$, the impact of the public signal on the posterior mean, i.e., $m_1 = \omega y + (1 - \omega) m_i$, is the same for all investors, since $\omega_i = \omega$ for all $i$ (see Equation (1a)). Consequently, the impact of the public signal on the investors’ demand functions for the risky asset, i.e., $x_i(y) = \rho h_i (m_1 - p_1(y))$, is also the same for all investors (see Equation (2)). Market clearing then dictates that the dollar risk premium $m_1 - p_1(y)$ cannot depend on the public signal $y$ (see Equation (4)) and, thus, there can be no equilibrium information-contingent trading at $t = 1$.

However, with heterogeneous prior dividend precisions, the public signal affects the investors’ posterior means differently and, thus, there can be equilibrium information-contingent trading at $t = 1$. This further implies that the signal precision plays a key role in determining the equilibrium interest rate and, thereby, the equilibrium price of the risky asset at $t = 0$. As noted above, the impact of the signal precision on the equilibrium interest rate is only through the logarithmic terms in Equation (13a). The following proposition characterizes the equilibrium interest rate as a function of the signal precision.

**Proposition 4:** Assume the investors have heterogeneous prior dividend precisions. The equilibrium interest rate is bell-shaped with respect to the signal precision $h_e$. The unique maximum for the equilibrium interest rate is attained when $h_e = \bar{h}$, and its minimum is attained for uninformative information ($h_e = 0$) and for perfect information ($h_e \to \infty$).

The intuition for the result in Proposition 4 can be obtained from Equation (12), in which the interest rate is expressed as a linear increasing function of the growth in the investors’ certainty equivalents, $CE_i = CE_{i2} - CE_{i0}$. In equilibrium, all investors have the same growth in certainty equivalents. For the two extreme values of the signal precision ($h_e = 0$ and $h_e \to \infty$) there is no trading at $t = 1$ based on the public signal: (a) for $h_e = 0$, no new information is released at $t = 1$ and, thus, the equilibrium portfolios after trading at $t = 0$ remain equilibrium portfolios; and (b) when the signal precision increases, the investors’ posterior beliefs converge and the risk premium in the equilibrium price of the risky asset at $t = 1$ decreases, and in the limit for $h_e \to \infty$ all uncertainty is resolved at $t = 1$ and, thus, there is no basis for additional trading. On the other hand, for intermediate values of the signal precision ($h_e \in (0, \infty)$) there is non-trivial trading based on the public signal at $t = 1$ if the investors have heterogeneous prior dividend precisions. The source of this trading is that the investors can achieve improved side-betting based on their heterogeneously updated posterior beliefs. These gains-to-trade translate directly into increased certainty equivalents of $t = 2$ consumption (by $r U_1$) and, thus, a higher growth in their certainty equivalents (cf. Remark 2), *ceteris paribus*. A highly informative or an almost uninformative public signal at $t = 1$ yields only limited side-betting benefits and, thus, the highest growth in certainty equivalents is obtained for a unique interior signal precision $h_e = \hat{h}$. The equilibrium price of the risky asset is the product of the equilibrium riskless discount factor and the risk-adjusted expected dividend (which is independent of the signal precision by Proposition 1) and, thus, the equilibrium price of the risky asset is an inverted bell-shaped function of the signal precision $h_e$ with a minimum point at $h_e = \hat{h}$.

**Ex Ante Cost of Capital**

In settings with homogeneous beliefs, the *ex ante* cost of capital, i.e., the required expected rate of return on investments, is equal to the equilibrium expected rate of return on the risky asset. However, the expected rate of return on the risky asset, i.e., $\exp(\mu_i) = m_i/p_0$, is an investor-specific concept in settings in which the investors have heterogeneous prior means $m_i$ for the dividend on
the risky asset—the investors agree on the equilibrium *ex ante* price, but they disagree on the expected future dividend. This raises the question as to how to define the *ex ante* cost of capital in such settings.

We define the *ex ante* cost of capital based on the *value maximization principle*, i.e., whether a specific real investment will increase the cum-dividend market value of the firm (see also Section IV). Assessing the impact of the investment on the market value of the firm requires that the beliefs implicit in the equilibrium market value of the firm are used for the future dividend, i.e., it must be assumed that \( d \sim N(\tilde{m}^h, \tilde{\sigma}^2) \). Hence, the required expected rate of return on investments using these beliefs and, thus, the continuously compounded *ex ante* cost of capital \( \bar{l}^h \) is defined by \( \exp(\bar{l}^h) = \tilde{m}^h/p_0 \). Inserting the *ex ante* equilibrium price of the risky asset (8), and using Proposition 1, we obtain that:

\[
\bar{l}^h = i + \bar{\sigma}^h,
\]

where the (rate of return) risk premium \( \bar{\sigma}^h \) is given by:

\[
\bar{\sigma}^h = \ln \left( \frac{\tilde{m}^h}{\tilde{m}^h - r \tilde{\sigma}^2 Z^1} \right).
\]

Hence, the *ex ante* cost of capital for the risky asset \( \bar{l}^h \) is equal to the equilibrium interest rate \( i \) plus a risk premium \( \bar{\sigma}^h \), which is independent of the informativeness of the public signal. Propositions 2 and 4 then imply that the *ex ante* cost of capital is minimized for no public information (\( h_e = 0 \)) and for perfect public information (\( h_e \to \infty \)), while it is maximized for a unique interior signal precision \( h_e = \tilde{h} \) if the investors have heterogeneous prior precisions for the future dividend. The signal precision \( h_e = \tilde{h} \) is the precision for which the *ex ante* value of the trading gains \( \bar{U}_1 \) is maximized and, hence, the precision for which the growth in certainty equivalents and, therefore, also the interest rate are maximized.

To illustrate our results, we use the following three-investor example throughout with the parameters given in Table 1. Figure 1 illustrates the equilibrium interest rate, the risk premium, and the *ex ante* cost of capital as functions of the signal precision for the parameters in Table 1. Note that the equilibrium interest rate, the risk premium, and the *ex ante* cost of capital are independent of the investors’ individual endowments and, thus, even though the investors have heterogeneous beliefs, the equilibrium admits aggregation as with HARA utilities in effectively complete markets with homogeneous beliefs.

**Trading Volume**

The source of the increased growth in the investors’ certainty equivalents is the gains-to-trade based on the investors’ heterogeneously updated posterior beliefs at \( t = 1 \). In this section we demonstrate that the signal precision, which maximizes the gains-to-trade and, thus, the equilibrium interest rate, also maximizes the expected abnormal trading volume.

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Note that this is also the case for the *ex post* cost of capital in noisy rational expectations models such as Easley and O’Hara (2004), Hughes et al. (2007), Christensen et al. (2010), and Lambert et al. (2012). It is only when an additional expectation of the *ex post* cost of capital based on the investors’ homogeneous prior beliefs is taken that the informed and uniformed investors agree on the *ex post* cost of capital—a somewhat strange concept (see, e.g., the discussion in Christensen et al. [2010]).

We assume that the parameters are such that \( \tilde{m}^h/p_0 > 0 \).

Note that the investor-specific expected rate of return \( \mu_i \) on the risky asset can also be expressed as \( \mu_i = i + \sigma_i \), where \( \sigma_i = \ln(m_i/\tilde{m}^h - r \tilde{\sigma}^2 Z^1) \). Hence, the investor-specific risk premia \( \sigma_i \) are also independent of the informativeness of the public signal.
Investor $i$’s equilibrium net-trade in the risky asset at $t = 1$ is $\tau^i_t(y) = x^i_t(y) - x^i_0$, where the equilibrium demands for the risky asset at $t = 1$ and $t = 0$ are given in (4) and (10), respectively. Inserting the definitions of the posterior means and precisions in (4), and simplifying yield the following result.

**Remark 3:** Investor $i$’s equilibrium net-trade in the risky asset at $t = 1$ is given by:

$$\tau^i_t(y) = \frac{h_e(h - h_i)}{h + h_e}[y - EQ[h]],$$

and the risk-adjusted expected net-trade is equal to 0, i.e., $EQ[\tau^i_t(y)] = 0$.

Hence, the sensitivity of the investor’s equilibrium net-trade increases with the difference between the investor’s prior dividend precision $h_i$ and the average prior dividend precision $h$. Using that $y = d + \varepsilon$, investor $i$’s $t = 2$ equilibrium consumption can be expressed as:

$$c_{i2} = [d - p_1(y)]\tau^i_t(y) + [d - p_1(y)]x^i_0 + p_1(y)x^i_0 + \gamma^i_0 = \rho \frac{h_e(h - h_i)}{h + h_e}d^2 + L(d, \varepsilon),$$

<table>
<thead>
<tr>
<th>TABLE 1</th>
</tr>
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<tbody>
<tr>
<td>Investor and Risky Asset Parameters of the Running Example</td>
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<tr>
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<tr>
<td>Risk aversion ($\rho$)</td>
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<tr>
<td>Utility discount rate ($\delta$)</td>
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<tr>
<td>Prior mean ($m_i$)</td>
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<tr>
<td>Prior variance ($\sigma^2_i$)</td>
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<tr>
<td>Initial dividend ($d_0$)</td>
</tr>
<tr>
<td>Supply ($Z$)</td>
</tr>
</tbody>
</table>

**FIGURE 1**

Equilibrium Interest Rate, Risk Premium, and *Ex Ante* Cost of Capital as Functions of the Signal Precision $h_e$ given the Parameters in Table 1

The scale on the horizontal axis is $x = \ln (1 + h_e \cdot 1.5E + 07)$. 

<table>
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where \( L(d,e) \) is a linear function of the \( d \) and \( e \). This implies that investor \( i \)'s equilibrium consumption is a convex (concave) function of the \( t = 2 \) dividend if, and only if, \( h_i < \bar{h} (h_i > \bar{h}) \). This relationship between the public signal and the \( t = 2 \) equilibrium consumption is the key source of the improved side-betting opportunities (i.e., gains-to-trade) following from the fact that investors with low prior dividend precisions value convex payoffs more than investors with high prior dividend precisions.

The risk-adjusted expected net-trade is equal to zero, but the investors’ expected net-trade is not equal to zero, and it depends on their subjective prior dividend beliefs. Therefore, to investigate the impact of the signal precision on the expected trading volume, we define the abnormal net-trade of investor \( i \) as the difference between the net-trade and the expected net-trade conditional on the \( t = 2 \) dividend, i.e.,

\[
\alpha_{t_i}^*(y) \equiv \tau_i^*(y) - E[\tau_i^*(y)|d] = \frac{h_i(h - \bar{h})}{\bar{h} + h_e} e.
\]

Since the investors have homogeneous beliefs about the noise in the public signal \( e \), they have homogeneous beliefs about their abnormal net-trades, and the abnormal net-trades are normally distributed with a zero mean. Recognizing that some investors are selling while others are buying, the abnormal trading volume per investor is defined as:

\[
T^* = \frac{1}{2} \sum_{i=1}^{I} |\alpha_{t_i}^*(y)|.
\]

**Proposition 5:** The expected abnormal trading volume is:

\[
E[T^*] = \frac{\sqrt{h_e}}{h + h_e \sqrt{2\pi}} \frac{\rho}{\bar{h}} \sum_{i=1}^{I} |h - h_i|.
\]

Assume the investors have heterogeneous prior dividend precisions. The expected abnormal trading volume is bell-shaped with respect to the signal precision \( h_e \). The unique maximum for the expected abnormal trading volume is attained when \( h_e = \bar{h} \), and its minimum is attained for uninformative information (\( h_e = 0 \)) and for perfect information (\( h_e \to \infty \)).

The proposition establishes that the expected abnormal trading volume has the same comparative statics as the equilibrium interest rate with respect to the signal precision (cf. Proposition 4). The key empirical implication is that there is a direct positive relationship between the empirically unobservable growth in certainty equivalents (i.e., investor welfare) and the (empirically observable) expected (average) abnormal trading volume.

**Ex Ante Expected Utilities**

In this subsection, we investigate the impact of the public information system on ex ante investor welfare. Our concept of ex ante investor welfare is the standard concept of Pareto efficiency. A change in the public information system, i.e., a change in the signal precision, is said to increase investor welfare if it strictly increases at least some investors’ ex ante expected utilities, while it does not reduce any other investors’ ex ante expected utilities. Note that the investors’ ex ante expected utilities are calculated using each investor’s subjective prior beliefs.
and, thus, we do not assume that there is some “true” or “objective” prior beliefs (see, e.g., Savage 1954; Wilson 1968; Christensen and Feltham 2003, Section 4.1.3). The investors’ ex ante expected utilities are affected in two ways by changes in the signal precision. First, changes in the signal precision affect the gains-to-trade based on heterogeneously updated posterior beliefs and, thus, the growth in the certainty equivalents as illustrated in the preceding subsection. Second, the signal precision affects the ex ante equilibrium asset prices through the equilibrium interest rate and, thus, affects the value of the investors’ individual endowments. The latter may affect the investors in different ways depending on their individual endowments relative to their equilibrium portfolio at \( t = 0 \). The following lemma (partly) characterizes the impact of changing the signal precision on the investors’ ex ante expected utilities.

**Lemma 1:** The derivative of the investors’ ex ante expected utilities with respect to the signal precision \( h_e \) is given by:

\[
\frac{\partial}{\partial h_e} \frac{EU^*_i}{C_0} = r \exp(-rCE^*_i) \left\{ \beta_0 \frac{\partial}{\partial h_e} U_{1i} + [\gamma_{i0}^* - \gamma_i + E^2[x_i^* - z_i]] \frac{\partial}{\partial h_e} \beta_0 \right\},
\]

where:

\[
\frac{\partial}{\partial h_e} \beta_0 = -r \beta_0 \frac{\partial}{\partial h_e} \bar{U}_1.
\]

All the investors’ ex ante expected utilities have a stationary point at \( h_e = \bar{h} \).

The gains-to-trade based on the public signal are reflected in the first term in the braces of (18), \( \frac{\partial}{\partial h_e} U_{1i} \), where \( U_{1i} \) is investor \( i \)'s ex ante value of the trading gains at \( t = 1 \) and is given in (11c). The investors can always refuse to engage in any second-round trading at \( t = 1 \) and, thus, the ex ante value of trading gains is always non-negative, and it is maximized for the signal precision \( h_e = \bar{h} \). Note that an investor with a prior dividend precision \( h_i \) equal to the average dividend precision \( \bar{h} \) has an ex ante value of the trading gains equal to zero, i.e., \( U_{1i} = 0 \). This is due to the fact that an investor does not engage in signal-contingent trading at \( t = 1 \) if \( h_i = \bar{h} \) (see Equation (16)). All “more confident” and “less confident” investors than the average have a strictly positive ex ante value of trading gains, and these ex ante values are maximized for all investors exactly at the signal precision at which the equilibrium interest rate is maximized, \( h_e = \bar{h} \) (see Remark 2 and Proposition 4).

The positive ex ante values of trading gains (which will be realized when consuming at \( t = 2 \)) shift the investors’ incentive to consume more at \( t = 0 \) (to smooth consumption over time) and, thus, increase the equilibrium interest rate. This reduces the value of the investors’ (positive) endowments of the \( t = 2 \) zero-coupon bond and the risky asset through a reduction in the zero-coupon bond price, \( \beta_0 \). This reduction is maximized when the average ex ante value of the trading gains \( \bar{U}_1 \) is maximized at the signal precision \( h_e = \bar{h} \). However, the reduction in the asset prices also makes it cheaper to buy these assets. Hence, the impact of the changed equilibrium prices on the

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16 This is in contrast to the behavioral economics and finance literature, which typically assumes some form of irrational investor behavior, for example, that investors may make mistakes in applying Bayes’ rule (such as being “overconfident”). In a setting of heterogeneous subjective prior beliefs there is no such thing as a true probability distribution. In settings in which econometricians get to observe infinitely many independent draws from the same distribution (e.g., by flipping a coin), there is clearly an objective probability distribution. However, this does not seem to be a descriptive assumption in many economic settings (see, e.g., the discussion in Morris [1995]).
investors’ consumption possibilities at $t = 0$ depends on whether the investor wants to increase or decrease the holdings of the assets, i.e., whether $(\bar{c}_i - \bar{c}_i^0)$ and $(x_i^0 - z_i)$ are positive or negative. These effects are reflected in the second term in the braces of (18). Hence, in general, even though the signal precision has a unique impact on the \textit{ex ante} value of trading gains and on the equilibrium interest rate, the investors’ \textit{ex ante} expected utilities may be affected in different directions by changes in the signal precision. In other words, changes in the information system may not result in Pareto improvements—some investors may be better off, while others may be worse off depending on their endowments relative to their equilibrium asset holdings at $t = 0$.

To illustrate these effects of changes in the signal precision on the \textit{ex ante} expected utilities, we define investor $i$’s equilibrium \textit{ex ante} certainty equivalent $C_{\text{Exa}}^i(h_e)$ as:

$$C_{\text{Exa}}^i(h_e) = -\exp\left(-rC_{\text{Exa}}^i(h_e)\right) = -\exp(-rCE_{\text{Ex}}^i(h_e)) - \exp(-\delta)\exp(-rCE_{\text{Ex}}^i(h_e)).$$

The change in the equilibrium \textit{ex ante} certainty equivalent relative to the no public information case $C_{\text{Exa}}^i(h_e = 0)$, i.e., $\Delta C_{\text{Exa}}^i(h_e) = CE_{\text{Exa}}^i(h_e) - CE_{\text{Exa}}^i(h_e = 0)$, is a measure of investor $i$’s increased \textit{ex ante} expected utility from changing the informativeness of the public signal from being uninformative to having signal precision $h_e$. Figure 2 illustrates these changes in investor welfare for the three-investor example given in Table 1 assuming that the investors have efficient risk sharing endowments of the risky asset and zero endowments of the $t = 2$ zero-coupon bond, i.e., $z_i = Z/3$ and $\bar{c}_i = 0, i = 1, 2, 3$, while the endowments of the $t = 0$ zero-coupon bond are $\bar{k}_1 = \bar{k}_3 =$ 5,000 and $\bar{k}_2 = -10,000$.

All three investors in Figure 2 have a stationary point for their equilibrium \textit{ex ante} certainty equivalent at the signal precision $h_e = \bar{h}$ (see Lemma 1). However, while the equilibrium \textit{ex ante} certainty equivalents of investors 1 and 3 are both maximized at $h_e = \bar{h}$, investor 2’s equilibrium \textit{ex ante} certainty equivalent is minimized at that point. Hence, investors 1 and 3 are both better off with an interior signal precision, whereas investor 2 is better off with no public information ($h_e = 0$) or perfect information ($h_e \to \infty$). The parameters in Table 1 are such that investor 2 has a prior dividend precision equal to the average prior dividend precision, i.e., $h_2 = \bar{h}$. As noted above, this implies that investor 2 does not engage in signal-contingent

\begin{figure}
\centering
\caption{Changes in Equilibrium \textit{Ex Ante} Certainty Equivalents $\Delta CE_{\text{Exa}}^i(h_e)$ as Functions of the Signal Precision $h_e$ given the Parameters in Table 1 and Endowments $z_i = Z/3, \bar{c}_i = 0, i = 1,2,3,$ and $\bar{k}_1 = \bar{k}_3 = 5,000, \bar{k}_2 = -10,000$}
\includegraphics[width=\textwidth]{figure2.png}
\end{figure}

The scale on the horizontal axis is $x = \ln(1 + h_e \cdot 1.5E + 07)$. 

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trading at \( t = 1 \) and, thus, his \textit{ex ante} value of trading gains \( U_{12} \) is equal to 0. Therefore, his equilibrium \textit{ex ante} certainty equivalent is only affected by the changes in the \textit{ex ante} equilibrium asset prices. With efficient risk sharing endowments of the risky asset and negative endowment of the \( t = 0 \) zero-coupon bond, investor 2 is a “net seller” of assets (for all levels of \( h \)) at \( t = 0 \) (i.e., \((\gamma_{0} - \hat{\gamma}) + E[ d ](x_{0}^{*} - z_{0}) < 0 \)) and, thus, he is hurt by the lower equilibrium asset prices with interior signal precisions. On the other hand, both investor 1 and investor 3 have strictly positive \textit{ex ante} asset prices with \textit{ex ante} certainty equivalents as with the original endowments. Lemma 1 then yields that:

\[
\frac{\partial}{\partial h_{i}} EU_{i0} = r \exp(-rCE_{0}^{*}) \left\{ \beta_{0} \frac{\partial}{\partial h_{i}} U_{1i} + \frac{1}{2} \exp(-rCE_{0}^{*}) \beta_{0} \frac{\partial}{\partial h_{i}} \ln \left[ 1 + \frac{(\hat{h} - h_{i})^{2}}{h_{i} (\hat{h} + h_{i})^{2}} \right] \right\}.
\]

Since the common term for all investors \( h_{i}/(\hat{h} + h_{i})^{2} \) is a concave function of the signal precision, which is maximized for \( h_{i} = \hat{h} \), all investors are weakly better off by marginally increasing (decreasing) the signal precision for \( h_{i} < \hat{h} (h_{i} > \hat{h}) \). Hence, for any \( h_{i} \neq \hat{h} \) and heterogeneous prior dividend precisions, there is an allocation of the endowments such that there exists a Pareto superior equilibrium with a marginal change in the signal precision. If \( h_{i} = \hat{h} \), no such Pareto improvements can be obtained, since \( \frac{\partial}{\partial h_{i}} U_{1i} = 0 \) for all investors in this case. These arguments establish the following result.

It is well known that even though improved gains-to-trade can be achieved by changing the public information system, the change of information system may leave some investors better off and others worse off if the implementation of the equilibrium consumption plans requires trading of assets at equilibrium prices that depend on the information system (see Christensen and Feltham [2003, Chapter 7]). The example in Figure 2 illustrates such a setting. To investigate whether there exists a Pareto efficient information system, consider a setting in which the investors have equilibrium endowments relative to the signal precision and, thus, the \textit{ex ante} cost of capital, is maximized, i.e., for \( h_{i} = \hat{h} \). Hence, this level of signal precision is the obvious candidate for a Pareto efficient information system, and this is indeed the case.

To see why, consider the investors’ equilibrium portfolios at \( t = 0 \). It follows from (10) that the investors’ equilibrium demand for the risky asset at \( t = 0 \), \( x_{0}^{*} = ph_{i}[m_{i} - E[ d ] d] \), depends on the prior beliefs but is independent of the informativeness of the public signal at \( t = 1 \). However, the investors’ equilibrium demand for the \( t = 2 \) zero-coupon bond at \( t = 0 \) varies with the signal precision. Consider any given signal precision different from the average prior dividend precision, i.e., \( h_{i} \neq \hat{h} \), and the associated equilibrium certainty equivalents, \( \{CE_{0}^{*}, CE_{0}^{*}\}_{i=1,...,t} \), and equilibrium prices, \( \beta_{0}, p_{0} \). We want to show that this system cannot be a Pareto efficient information system. The equilibrium demand for the \( t = 2 \) zero-coupon bond \( \gamma_{0}^{*} \) is determined by Equation (11a). As noted above, the equilibrium prices are independent of the investors’ individual endowments. Hence, a re-allocation of the endowments of the three assets defined by:

\[
\hat{z}_{i} = x_{0}^{*}, \quad \hat{\gamma}_{i} = \gamma_{0}^{*}, \quad \hat{h}_{i} = CE_{0}^{*} - d_{0}x_{0}^{*}, \tag{19}
\]

implies that the investors do not trade at \( t = 0 \) given these endowments. That is, the investors have equilibrium endowments relative to the signal precision \( h_{i} \), and they achieve the same certainty equivalents as with the original endowments. Lemma 1 then yields that:

\[
\frac{\partial}{\partial h_{i}} EU_{i0} = r \exp(-rCE_{0}^{*}) \left\{ \beta_{0} \frac{\partial}{\partial h_{i}} U_{1i} + \frac{1}{2} \exp(-rCE_{0}^{*}) \beta_{0} \frac{\partial}{\partial h_{i}} \ln \left[ 1 + \frac{(\hat{h} - h_{i})^{2}}{h_{i} (\hat{h} + h_{i})^{2}} \right] \right\}.
\]

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Proposition 6: Assume the investors have heterogeneous prior dividend precisions.

(a) For equilibrium endowments, the information system with signal precision \( h_e = \bar{h} \) is the unique Pareto efficient public information system, and it enjoys the maximum equilibrium \( ex \ ante \) cost of capital and the maximum expected abnormal trading volume.

(b) For fixed endowments, some investors may be worse off with information system \( h_e = \bar{h} \) than with \( h_e \neq \bar{h} \), but at least one investor is better off with information system \( h_e = \bar{h} \) than with \( h_e \neq \bar{h} \).

Table 2 illustrates a setting in which the endowments for the example in Figure 2 are re-allocated as in (19) to achieve equilibrium endowments relative to signal precision \( h_e = 0 \).

With the endowments in Table 2, all three investors do not trade at \( t = 0 \) if \( h_e = 0 \), and they achieve the same certainty equivalents as with the endowments in Figure 2. As the signal precision is increased, the investors continue not to trade at \( t = 0 \) in the risky asset (since \( x^*_i \) does not depend on \( h_e \) but investor 2, who has a prior dividend precision \( h_2 \) equal to the average prior dividend precision \( \bar{h} \), starts to increase his holdings of the \( t = 2 \) zero-coupon bond due to its lower equilibrium price. The other two investors reduce their holdings of the \( t = 2 \) zero-coupon bond, i.e., sell units of the bond, even though its equilibrium price decreases. This is due to the fact that increasing the signal precision increases their future trading gains (which are realized at \( t = 2 \)) and, thus, they also want to consume more at \( t = 0 \) to smooth consumption across the two consumption dates. Hence, all three investors gain from a higher signal precision (up to \( h_e = \bar{h} \)) as illustrated in Figure 3 (although the gains to investor 2 are hardly visible in the graph). A prior round of trading based on the belief that the signal precision will be \( h_e = 0 \), such that the investors have equilibrium endowments at \( t = 0 \) relative to \( h_e = 0 \), ensures that the investors unanimously support a public information system change to a system with \( h_e = \bar{h} \).

IV. REAL EFFECTS

The preceding analysis has demonstrated that the investors’ \( ex \ ante \) welfare is maximized with respect to the signal precision when the expected abnormal trading volume and the \( ex \ ante \) cost of capital are the highest. However, this analysis assumes that real investment decisions are not affected by the signal precision and the resulting \( ex \ ante \) cost of capital. In this section, we extend the analysis to a setting in which real investment decisions may be affected by the impact of the signal precision on the \( ex \ ante \) cost of capital. A higher \( ex \ ante \) cost of capital will reduce real investments, which may seem to indicate that the increase in \( ex \ ante \) welfare due to additional trading gains with an imperfect public signal comes with a cost of negative real effects. We show that allowing for endogenous real investment decisions, in fact, improves the \( ex \ ante \) welfare relative to a setting in which real investments are fixed. This is due to a preferred allocation of aggregate consumption possibilities over
time. On the other hand, the impact of the signal precision on the equilibrium ex ante cost of capital is “dampened” relative to the setting with fixed investments.

We assume that the dividends on the risky asset are endogenously determined by real investments $q$ in a riskless production technology characterized by an increasing strictly concave twice differentiable production function $f(q)$ such that the $t = 0$ and $t = 2$ dividends are given by $d_0 = \hat{d}_0 - q$ and $d = \hat{d} + f(q)$, where $\hat{d}_0$ and $\hat{d}$ are the fixed component of the dividends at $t = 0$ and $t = 2$, respectively, which are independent of the real investment. The investors’ prior means of the $t = 2$ dividend now depend on the real investment, i.e., $m_i = m_i(q) = \hat{m}_i + f(q)$, where $\hat{m}_i$ is investor $i$’s prior mean of $\hat{d}$, whereas their prior precisions are independent of the real investment. The ex-dividend ex ante equilibrium price for the risky asset is given by (see Proposition 1):

$$p_0(q) = \exp(-i)[\hat{m}^h(q) - r\sigma^2Z/I], \quad \text{where } \hat{m}^h(q) = \frac{1}{I} \sum_{i=1}^{I} h_i \hat{m}_i + f(q).$$

Taking the interest rate $i$ and the market price of risk $r\sigma^2Z/I$ as given, the first-order condition for a cum-dividend value-maximizing real investment is:

$$\exp(-i) \frac{\partial \hat{m}^h(q)}{\partial q} = 1, \quad \text{where } \frac{\partial \hat{m}^h(q)}{\partial q} = f'(q).$$

Rearranging yields the first-order condition for a value-maximizing real investment $q^*$:

$$f'(q^*) = \exp(i) = \exp\left(\delta + r(C_{E_2}^* - C_{E_0}^*)\right). \quad (20)$$

17 It is straightforward to extend the analysis to a risky production technology in which the real investment at $t = 0$ has a multiplicative impact on the $t = 2$ dividend, i.e., $d = f(q)\hat{d}$. However, since the key in our analysis is to facilitate a preferred consumption smoothing over the two consumption dates, the simpler riskless production technology serves our purpose.
That is, if an increase in the signal precision increases the growth in certainty equivalents between \( t = 0 \) and \( t = 2 \) and, thus, increases the equilibrium interest rate for a fixed real investment (as in the preceding analysis), this leaves the investors with a desire to reduce real investments at \( t = 0 \), i.e., shift consumption from \( t = 2 \) to \( t = 0 \), to smooth the trading gains realized in the second period over the two consumption dates. As the real investments are reduced, the equilibrium interest rate decreases and, in equilibrium, the interest rate is equal to the marginal rate of transformation in the production technology. The risks associated with the future dividend are not affected by the reduction in the riskless real investments and, hence, the dollar risk premium in the equilibrium price of the risky asset (i.e., \( r\hat{\sigma}^2Z/I \)) is also unaffected. On the other hand, the reduction in the riskless real investments reduces the precision weighted average of the prior expected mean of the future dividend (i.e., \( \hat{m}^h \)) and, thus, reduces the risk-adjusted expected dividend (i.e., \( \hat{m}^h - r\hat{\sigma}^2Z/I \)).

In other words, the reduction in the real investments increases the dollar risk premium per dollar invested, implying that the rate of return risk premium \( \hat{\sigma}^h \) increases (see Equation (15b)). Since the real investments do not affect the risks associated with the future dividend, this impact on the rate of return risk premium is merely a “leverage” effect.\(^{18}\) The following proposition formalizes these arguments.

**Proposition 7:** Assume the investors have heterogeneous prior dividend precisions.

(a) The equilibrium value-maximizing real investment is inverted bell-shaped with respect to the signal precision \( h_e \), while the equilibrium interest rate and the rate of return risk premium are both bell-shaped with respect to the signal precision \( h_e \). The unique minimum (maximum) for the equilibrium real investment (interest rate and rate of return risk premium) is attained when \( h_e = \hat{h} \), and its maximum (minimum) is attained for uninformative information \( (h_e = 0) \) and for perfect information \( (h_e \to \infty) \).

(b) If the production technology is such that \( q^e(h_e = 0) = 0 \), then the equilibrium interest rate is strictly lower than in an otherwise identical exchange economy in which \( q(h_e) = 0 \) for all \( h_e \) (except in the extreme cases with \( h_e = 0 \) and \( h_e \to \infty \)).

Figure 4 illustrates the results in the proposition with a production function of the form \( f(q) = a(q + b)^\eta \), \( \eta < 1 \). We calibrate the parameters such that the value-maximizing real investment for uninformative information \( (h_e = 0) \) is equal to zero, and we choose the fixed components of the prior means \( \hat{m}_t \) such that the prior means \( m_t(q) = \hat{m}_t + f(q) \) are the same as in our running example with a zero real investment (cf. Table 1). This allows us to directly compare the equilibrium interest rates with the equilibrium interest rates in the exchange economy examined in the preceding sections.

Although the shape of the equilibrium interest rate as a function of the signal precision \( h_e \) is the same as in the exchange economy, the equilibrium interest rate with an endogenous real investment is at a strictly lower level than in the exchange economy, except in the extreme cases \( (h_e = 0 \text{ and } h_e \to \infty) \) where they are equal. This is due to the disinvestment for intermediate levels of \( h_e \) as the investors increase current consumption at the expense of future consumption to smooth the trading gains from the imperfect public signal at \( t = 1 \) over the two consumption dates, i.e., \( t = 0 \) and \( t = 2 \). The reduction in the interest rate is the highest when the disinvestment is the largest (at \( h_e = \hat{h} \)). In our numerical example, the reduction in the interest rate for intermediate levels of \( h_e \) is so large that the equilibrium interest rate is almost the same for all levels of \( h_e \). The bell-shape of the risk

\(^{18}\) We term this a “leverage” effect, since the future dividend \( d = f(q)\hat{d} \) can be considered the payoff on a portfolio consisting of a riskless asset with payoff \( f(q) \) and a risky asset with a normally distributed payoff \( \hat{d} \), in which only the former depends on the real investment \( q \). See Lambert et al. (2007) for a similar “leverage” effect in their analysis of real effects in a single-consumption-date economy.
premium is also hardly visible in the figure, since the relatively small disinvestments (compared to the prior means of the future dividend with zero investment) only yield a small “leverage” effect.

In the preceding analysis of real effects, it is assumed that the real investment is determined by the value-maximizing real investment taking the interest rate and the market price of risk as given. However, is the value-maximizing real investment also unanimously supported by the investors? It is straightforward to show that this is indeed the case if investors have equilibrium endowments (details are available from the authors upon request).

V. CONCLUDING REMARKS

We develop a simple analytical model of public information and heterogeneous prior beliefs that allow us to study the relationship between the informativeness of the public information system and the investors’ welfare in an incomplete market setting. The source of the welfare improvements due to public information is the gains-to-trade following from the investors’ speculative positions based on the differences in the precision of their prior beliefs.

The model provides a direct positive relationship between welfare improvements and the expected abnormal trading volume and, thus, it provides an empirical measure for the relationship between public information systems and investor welfare. This result is in contrast to the extant literature on the impact of public information on trading volume based on noisy rational expectations models (see, e.g., Kim and Verrecchia 1991; McNichols and Trueman 1994; Demski and Feltham 1994). Due to the unmodeled “noise traders, these models do not allow welfare comparisons of market structures with different public information systems. Similarly, the
differences-of-opinion literature is also silent about the relationship between trading volume and investor welfare, mainly because trading volume in these models is generated based on unmodeled heterogeneous beliefs about payoff-irrelevant events as opposed to about the fundamentals of the economy.

Our analysis shows that in an incomplete market setting with heterogeneous prior beliefs early, although imperfect, interim information relative to final cash flows can be valuable. No early information rules out valuable dynamic trading strategies to take advantage of the differences in prior dividend precisions, and so does perfect early information. In this sense, our results are related to the “information-risk problem” (see, e.g., Hirshleifer 1971), i.e., too much information being released before trading can occur, and to the literature on dynamically completing markets by trading long-lived securities (see, e.g., Ohlson and Buckman 1981; Duffie and Huang 1985; Christensen and Feltham 2003, Chapter 7). Hence, the model provides an argument for interim reporting through, for example, earnings forecasts and quarterly earnings announcements, instead of withholding information until the annual audited earnings announcements (see Gigler, Kanodia, Sapra, and Venugopalan [2013] for an opposing impact of higher reporting frequency due to myopic investment behavior induced by asymmetric information between managers and outside investors).

Our model provides a direct positive relationship between welfare improvements and the *ex ante* cost of capital, i.e., the Pareto efficient public information system is the system enjoying the maximum *ex ante* cost of capital and, thus, the lowest equilibrium *ex ante* price of the risky asset. This (maybe at first counterintuitive) result shows the importance of using a general equilibrium analysis in the evaluation of public information systems, which have consequences for the economy at large. In our model, the impact of the public information system on investor welfare is reflected through the equilibrium interest rate.

The *ex ante* risk premium, however, is not affected by the public information system. The lack of an impact on the *ex ante* risk premium is quite likely specific to our particular model with exponential utilities and normally distributed dividends. On the other hand, riskless interest rates may be much more “robust” empirical proxies for the impact of economy-wide information on investor welfare, since, independent of specific assumptions about time-additive preferences, dividend distributions, and market incompleteness, equilibrium zero-coupon bond prices are by the investors’ first-order conditions determined as the ratio between individual expected marginal utility of future consumption and marginal utility of current consumption (compare to Equation (6)).

REFERENCES


**APPENDIX A**

**Additional Results and Proofs**

**Lemma A.1**

Investor $i$’s $t = 0$ certainty equivalent of $t = 2$ consumption is given by:

$$CE_{t=2}(x_0, y_0) = y_{i0} + U_{1i} + U_{2i} + M_i x_0 - \frac{1}{2} r V_i x_0^2,$$

where:

$$U_{1i} = \frac{1}{2} \rho \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_e}{(\bar{h} + h_e)^2} \right],$$

(A.2a)

$$U_{2i} = \frac{1}{2} \rho \frac{h_i [m_i \bar{m} - \bar{m}h + r Z / I]^2}{\bar{h}^2 + \bar{h} h_e},$$

(A.2b)

$$M_i = \frac{h_i m_i + \bar{h}^2 \bar{m} h - r \bar{h} Z / I}{\bar{h}^2 + \bar{h} h_i},$$

(A.2c)
Proof of Lemma A.1

Substituting the $t = 1$ demand functions (2) into investor $i$’s certainty equivalent (5) yields that:

$$CE_{i2}(x_0, y_0, x_1(y)) = \gamma_0 + x_0 \rho p_1(y) + \frac{1}{2} \rho h_{1i} \left( m_{i1} - p_1(y) \right)^2,$$

where $p_1(y)$ is given by (3). Investor $i$’s posterior mean in (1a) for the $t = 2$ dividend is a weighted average of the prior mean and the public signal, i.e., $m_{i1} = \omega_1 y + (1 - \omega_1)$, where $\omega_1 = \sigma^2_i / (\sigma^2_i + \sigma^2_h)$. Thus, the precision weighted average of the investors’ posterior means is:

$$m^h = \frac{1}{T} \sum_{i=1}^{T} \frac{h_i}{h^*} m_{i1}, \quad h^* = \frac{1}{T} \sum_{i=1}^{T} h_i, \quad \bar{\sigma}^2 = \frac{1}{h}.$$

and, consequently, the $t = 1$ equilibrium price of the risky asset is $p_1(y) = \sigma^2_i [y_h + \bar{\sigma}^2 m^h - rZ/I]$. Inserting in the above expression for the certainty equivalent and simplifying yield:

$$CE_{i2}(x_0, y_0, x_1(y)) = \gamma_0 + x_0 \sigma^2_i h_{y} + x_0 \sigma^2_i [h^* m^h - rZ/I]$$

$$+ \frac{1}{2} \rho h_{1i} \left( [\sigma^2_i - \bar{\sigma}^2_i] h_{y} + \sigma^2_i h_{m} - \bar{\sigma}^2_i [h^* m^h - rZ/I] \right)^2.$$

For notational simplicity, let:

$$E_{1i} \equiv [\sigma^2_i - \bar{\sigma}^2_i] h_{y}; \quad E_{2i} \equiv \bar{\sigma}^2_i h_{y}; \quad E_{3i} \equiv \bar{\sigma}^2_i h^* m^h - rZ/I; \quad E_{4i} \equiv \sigma^2_i h_{m} - E_{3i},$$

and substituting into the certainty equivalent yields:

$$CE_{i2}(x_0, y_0, x_1(y)) = \gamma_0 + x_0 E_{3i} + \frac{1}{2} \rho h_{1i} E_{2i}^2 + [x_0 E_{2i} + \rho h_{1i} E_{4i}] y + \frac{1}{2} \rho h_{1i} E_{4i}^2 y^2.$$
w = y; \bar{q} = m_i; \bar{a} = (\sigma_i^2 + \sigma_e^2)^{-1}; a = \gamma_{i0} + x_{0i}E_{3i} + \frac{1}{2}\rho h_{1i}E_{4i}; \bar{h} = x_{0i}E_{2i} + \rho h_{1i}E_{4i}; \bar{c} = \rho h_{1i}E_{1i}^2,

i.e., investor i’s t = 0 certainty equivalent of t = 2 consumption is:

\[ CE_{i2}(x_{0i}, \gamma_{i0}) = \gamma_{i0} + \frac{1}{2}\rho \ln[1 + h_{1i}E_{1i}^2[\sigma_i^2 + \sigma_e^2]] + \frac{1}{2}\rho h_{1i}E_{4i}^2 + \frac{1}{2}\rho \frac{m_i^2}{\sigma_i^2 + \sigma_e^2} + x_{0i}E_{3i} \]

\[ - \frac{1}{2}\rho \left( \frac{\sigma_i^2 + \sigma_e^2}{h_{1i}E_{1i}\left(\sigma_i^2 + \sigma_e^2\right)} \right)^2. \]

Collecting terms yields that \( CE_{i2} = \gamma_{i0} + U_{i1} + U_{i2} + M_i x_{i0} - \frac{1}{2} V_i x_{i0}^2 \), where:

\[ U_{i1} = \frac{1}{2}\rho \ln[1 + h_{1i}E_{1i}^2[\sigma_i^2 + \sigma_e^2]], \]

\[ U_{i2} = \frac{1}{2}\rho h_{1i}E_{4i}^2 + \frac{1}{2}\rho \frac{m_i^2}{\sigma_i^2 + \sigma_e^2} - \frac{1}{2}\rho \left( \frac{\sigma_i^2 + \sigma_e^2}{h_{1i}E_{1i}\left(\sigma_i^2 + \sigma_e^2\right)} \right)^2, \]

\[ M_i = E_{3i} - \frac{(\sigma_i^2 + \sigma_e^2)E_{2i}}{1 + h_{1i}E_{1i}^2(\sigma_i^2 + \sigma_e^2)}, \]

\[ V_i = \frac{(\sigma_i^2 + \sigma_e^2)^2}{1 + h_{1i}E_{1i}^2(\sigma_i^2 + \sigma_e^2)}. \]

Using the definition of \( E_{1i} = [\sigma_i^2 - \sigma_i^2]h_e \), we obtain:

\[ E_{1i} = \left[ \frac{1}{h_i + h_e} - \frac{1}{h + h_e} \right] h_e = h_e \frac{\bar{h} - h_i}{(h_e + h_i)(\bar{h} + h_e)}. \]

This implies that:

\[ A_i = 1 + h_{1i}E_{1i}^2[\sigma_i^2 + \sigma_e^2] = 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_e}{(\bar{h} + h_e)^2}, \]

and thus:

\[ U_{i1} = \frac{1}{2}\rho \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_e}{(\bar{h} + h_e)^2} \right], \]

which establishes (A.2a).

Turning to \( V_i \) and using the definitions of \( E_{2i} = \sigma_i^2 h_e \) and of \( A_i = 1 + h_{1i}E_{1i}^2[\sigma_i^2 + \sigma_e^2] \), we obtain:

\[ V_i = \frac{(\sigma_i^2 + \sigma_e^2)(\sigma_e^2 h_e)^2}{A_i} = \frac{(h_e + h_i)h_e}{h_i(h_e + h_i)^2 + (\bar{h} - h_i)^2 h_e} = \frac{h_e}{\bar{h} + h_e h_i}, \]

which establishes (A.2d).
Turning to $M_i$ and using the definition of $V_i$, we can express $M_i$ as:

$$M_i = E_3i - \left( \frac{\sigma_i^2 + \sigma_e^2}{1 + h_i E_1^2(\sigma_i^2 + \sigma_e^2)} \right) V_i \left[ \frac{E_3i}{V_i} - h_i \frac{E_4i}{E_2i} \frac{m_i h_i}{E_2i(h_c + h_l)} \right].$$

Inserting the expressions for $E_{ji}$, $j = 1, 2, 3, 4$, and simplifying yield:

$$M_i = V_i \left[ h_i m_i + \left( \frac{1}{V_i} + (h_l - h_i) \right) \sigma_i^2 \left[ \hat{m} h_Z - r Z/\bar{I} \right] \right].$$

Substituting the expression for $V_i$ back in and simplifying establish (A.2c):

$$M_i = \frac{h_i h_l m_i + \hat{h} \hat{m} h_l - r \hat{h} Z/\bar{I}}{\bar{h}^2 + h_l h_i}.$$ 

Finally, turning to $U/2$ we obtain:

$$2rU_2 = h_i \frac{E_4i + m_i E_3i}{1 + h_i E_1^2(\sigma_i^2 + \sigma_e^2)} = h_i \frac{(E_4i + m_i E_3i)^2}{1 + h_i E_1^2(\sigma_i^2 + \sigma_e^2)}.$$ 

Substituting the expressions for $E_1i$ and $E_4i$ in and simplifying, we obtain:

$$U_2 = \frac{\rho h_i [m_i \hat{h} - \hat{m} \hat{h} Z - r Z/\bar{I}]}{2 \bar{h}^2 + h_l h_i},$$

which establishes (A.2b).  

**Proof of Proposition 1**

Lemma A.1 implies that the first-order condition (7) can be written as $p_0 = \beta_0 [M_i - r V(x_0)]$ and, thus, the demand for the risky asset at $t = 0$ can be expressed as:

$$x_{i0} = \rho \frac{M_i - R_0 p_0}{V_i},$$

(A.3)

where $R_0 = 1/\beta_0$. Hence, the market-clearing condition for the risky asset implies that its equilibrium price at $t = 0$ is:

$$p_0 = \beta_0 [m^0 - r \hat{V} Z/\bar{I}],$$

(A.4)

where:

$$v_i = V_i^{-1}, \quad \hat{v} = \frac{1}{I} \sum_{i=1}^{I} v_i, \quad \hat{m}^0 = \frac{1}{I} \sum_{i=1}^{I} \frac{v_i}{\hat{v}} M_i, \quad \hat{V} \equiv \hat{v}^{-1}.$$ 

In other words, the equilibrium price of the risky asset is equal to its discounted “risk-adjusted expected dividend,” where the latter is defined as $E^Q[d] = \hat{m}^0 - r \hat{V} Z/\bar{I}$. First we calculate:

$$\hat{v} = \frac{1}{I} \sum_{i=1}^{I} v_i = \frac{1}{I} \sum_{i=1}^{I} \left( \frac{\hat{h}^2 + h_l h_i}{h_c + h_l} \right) = \frac{\hat{h}_c^2 + \hat{h}_l}{h_c + \hat{h}_l},$$

and:
\[ m^\nu = \frac{1}{I} \sum_{i=1}^{I} \gamma_i M_i = \frac{h_i \tilde{m}^h + \tilde{h}^2 \tilde{m}^h - \tilde{h}rZ/I}{\tilde{h}^2 + h_i \tilde{h}}. \]

Hence, the risk-adjusted expected dividend is:

\[ E^Q[d] = m^\nu - r\tilde{V}Z/I = \frac{(h_i + \tilde{h})\tilde{m}^h - (\tilde{h} + h_i)rZ/I}{(\tilde{h} + h_i)\tilde{h}} = m^\nu - r\tilde{a}^2Z/I, \]

which establishes (8) and (9a) and proves the claim that the risk-adjusted expected dividend is independent of the signal precision. ■

**Proof of Remark 1**

Substituting the ex ante equilibrium price of the risky asset (8) into the demand functions (A.3), we obtain the investors’ equilibrium demand functions for the risky asset:

\[ x^*_{i0} = \rho V_i^{-1} [M_i - E^Q[d]]. \] (A.5)

Substituting the expressions for \( M_i \) and \( V_i \), i.e., (A.2c) and (A.2d), into (A.5) and simplifying yield:

\[ x^*_{i0} = \rho \frac{1}{h_i} [h_i h_i m_i + \tilde{h}[\tilde{m}^h - rZ/I] - (\tilde{h}^2 + h_i h_i)E^Q[d]]. \]

Using the expression for \( E^Q[d] \), i.e., (9a), yields:

\[ x^*_{i0} = \rho \frac{1}{h_i} [h_i h_i m_i + \tilde{h}^2 E^Q[d] - (\tilde{h}^2 + h_i h_i)E^Q[d]] = \rho h_i \{m_i - E^Q[d]\}. \]

■

**Proof of Remark 2**

Using (10), the last three terms in (A.1) can be expressed as:

\[ U_{2i} + M_i x^*_{i0} - \frac{1}{2} rV_i (x^*_{i0})^2 = U_{2i} + \frac{1}{2} \rho \frac{M_i^2 - (E^Q[d])^2}{V_i}. \]

Substituting in the definitions of \( U_{2i} \), \( M_i \) and \( V_i \) given in (A.2) and simplifying yield that:

\[ 2r \left[ U_{2i} + \frac{1}{2} \rho \frac{M_i^2 - (E^Q[d])^2}{V_i} \right] = h_i \left[ m_i^2 - \frac{[\tilde{h}m^h - rZ/I]^2}{\tilde{h}^2} \right]. \]

Thus, we obtain that:

\[ U_{2i} + M_i x^*_{i0} - \frac{1}{2} rV_i (x^*_{i0})^2 = \frac{1}{2} \rho h_i \left[ m_i^2 - \frac{[\tilde{h}m^h - rZ/I]^2}{\tilde{h}^2} \right] = \frac{1}{2} \rho h_i [m_i^2 - (E^Q[d])]^2, \]

which proves (11b).

Using the definition of \( U_{1i} \) in (11c), it follows immediately that \( U_{1i} = 0 \), when \( h_i = \tilde{h} \) irrespectively of the signal precision \( h_i \), and when \( h_i = 0 \) or \( h_i \rightarrow \infty \) irrespective of the prior dividend precision \( h_i \). Furthermore, it follows that:

\[ \frac{\partial}{\partial h_i} U_{1i} = \frac{1}{2} \rho h_i \frac{(\tilde{h} - h_i)^2}{h_i (\tilde{h} + h_i)^3 + h_i (\tilde{h} - h_i)^2 (\tilde{h} + h_i)} (\tilde{h} - h_i). \]
Hence, \( U_{1i} \) is an increasing (decreasing) function of the signal precision for \( h_e \leq \bar{h}(h_e \leq \bar{h}) \), and it has a unique maximum at \( h_e = \bar{h} \). ■

**Proof of Proposition 2**

Using Remark 2, summing (12) across investors, and using the market-clearing conditions yield:

\[
\bar{h} = \delta + r \frac{1}{I} \sum_{i=1}^{I} (CE_{1i}^n - CE_{1i}^m) = \delta + r \bar{U}_1 + \Phi(\{m_i, \sigma_i^2\}_{i=1, \ldots, I}),
\]

where:

\[
\Phi(\{m_i, \sigma_i^2\}_{i=1, \ldots, I}) = \frac{1}{2} \bar{h} \sum_{i=1}^{I} m_i^2 - (E^0[\sigma^2]) - r d_0 Z/I.
\]

Inserting the expression for \( E^0[\sigma^2] \) in (9a) yields:

\[
\Phi(\{m_i, \sigma_i^2\}_{i=1, \ldots, I}) = \frac{1}{2} \bar{h} \sum_{i=1}^{I} m_i^2 - \frac{1}{2} (\bar{m}^b h_i)^2 h_i.
\]

If investors have homogeneous prior dividend means, i.e., \( m_i = m \), then:

\[
\Phi(\{m, \sigma_i^2\}_{i=1, \ldots, I}) = \frac{1}{2} \bar{h} \sum_{i=1}^{I} m^2 - \frac{1}{2} (\bar{m}^b)^2 h_i.
\]

If the investors have homogeneous prior dividend precisions, i.e., \( h_i = \bar{h} \), then \( \bar{U}_1 = 0 \) and \( \bar{h} = \delta + \Phi(\{m, \sigma^2\}_{i=1, \ldots, I}) \). Hence, the equilibrium interest rate is independent of the signal precision, since \( \Phi(\cdot) \) only depends on the prior beliefs. ■

**Proof of Proposition 3**

With identical prior dividend precisions, let \( h_i = \bar{h} \), for \( i = 1, \ldots, I \). Hence:

\[
\bar{h} = \bar{h}, \quad \bar{m}^b = \frac{1}{I} \sum_{i=1}^{I} m_i, \text{ and } m_{1i} = \frac{h_e}{h + h_e} y + \left( \frac{h}{h + h_e} \right) m_i = \frac{1}{h + h_e} (h_e y + hm_i).
\]

This implies that:

\[
x_{1i}^*(y) = \rho h_{1i} (m_{1i} - [\bar{m}^b - \sigma^2 Z/I]) = \rho h_i (m_i - [\bar{m}^b - \sigma^2 Z/I]).
\]

Using the expression for \( E^0[\sigma^2] \), i.e., (9a), we have that \( E^0[\sigma^2] = \bar{m}^b - \sigma^2 Z/I \), and Remark 1 yields that \( x_{1i}^*(y) = \rho h_i E^0[m_i - E^0[\sigma^2]] = x_{1i}^*(y) \). Therefore, the demand for riskless asset is also time- and signal-invariant, since \( \gamma_{1i}^* = \gamma_{0i}^* + p_1(y) x_{0i} - p_1(y) x_{1i}^*(y) = \gamma_{0i}^* \). ■

**Proof of Proposition 4**

By Proposition 2 the equilibrium interest rate attains its maximum value when the logarithmic term in (13a):

\[
r \frac{1}{I} \sum_{i=1}^{I} \frac{1}{2} \frac{1}{I} \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_e}{(h + h_e)^2} \right],
\]
is maximized with respect to $h_i$. This term is maximized whenever the common term for all investors $h_i/(\hat{h} + h_i)^2$ is maximized, and the result follows from Remark 2.

**Proof of Remark 3**

Using that $m_i^h = \sigma_i^2[\hat{h} + \hat{m}^h] = [\hat{h} + \hat{m}^h]/(\hat{h} + h_i)$ (see the proof of Lemma A.1), (4) can be re-written as:

$$x_{i1}(y) = \rho h_i(m_i - [m_i^h - \hat{m}^h Z/I]) = \rho \left( \frac{h_i(\hat{h} - h_i)}{h + h_e} y + h_i m_i - \frac{\hat{h}(h_i + h_e)}{h + h_e} E^Q[d] \right).$$

It then follows from (10) that:

$$\tau_i^*(y) = \rho \left( \frac{h_i(\hat{h} - h_i)}{h + h_e} y - \left[ \frac{h h_e - h_i h_i}{h + h_e} \right] E^Q[d] \right) = \rho \frac{h_i(\hat{h} - h_i)}{h + h_e} (y - E^Q[d]).$$

The risk-adjusted expected equilibrium net-trade is (see footnote 11 for the definition of the risk-adjusted probability measure $Q$):

$$E^Q[\tau_i^*(y)] = \rho \frac{h_i(\hat{h} - h_i)}{h + h_e} (E^Q[d + \tilde{e}] - E^Q[d]) = \rho \frac{h_i(\hat{h} - h_i)}{h + h_e} E^Q[\tilde{e}] = 0.$$

**Proof of Proposition 5**

Using that the expected value of the absolute value of a zero-mean normally distributed variable $X$ is $E[|X|] = \sqrt{2/\pi} \sqrt{\text{Var}[X]}$, the expected abnormal trading volume is:

$$E[T^*] = \frac{1}{2} \sum_{i=1}^{I} E[|\alpha \tau_i^*(y)|] = \frac{\sqrt{h_e}}{h + h_e} \sqrt{\frac{\rho}{2\pi I}} \sum_{i=1}^{I} |\hat{h} - h_i|.$$

As a function of the signal precision, the expected abnormal trading volume can be expressed as $E[T^*] = \alpha \sqrt{h_e/(\hat{h} + h_e)^2}$ with $\alpha$ being a positive constant. The comparative statics stated in the proposition then follows from the proof of Proposition 4, and the fact that the square-root function is a strictly increasing function.

**Proof of Lemma 1**

Using (6), the equilibrium expected utility of investor $i$ can be expressed as:

$$EU^*_i = -\exp(-r CE^*_i)[1 + \exp(-\delta)\exp(-r \{ CE^*_i - CE^*_i \})] = -\exp(-r CE^*_i) \cdot [1 + \beta_0].$$

Hence:

$$\frac{\partial}{\partial h_e} EU^*_i = r \exp(-r CE^*_i) \left\{ [1 + \beta_0] \frac{\partial}{\partial h_e} CE^*_i - \rho \frac{\partial}{\partial h_e} \beta_0 \right\}.$$

Investor $i$’s $t = 0$ certainty equivalent is given by $CE^*_i = H(\beta_0) - \beta_0 \gamma^*_i$, where $H(\beta_0) = [p_0 + d_0] \hat{z}_i + \beta_0 \hat{g}_i + \hat{r}_i - \rho \hat{x}^*_i$ is the value of the endowments minus the investment in the risky asset. The equilibrium investment in the zero-coupon bond is given by its first-order condition (6), and using Remark 2 we obtain:

$$t = \delta + r \left( \gamma^*_i + U_1 + \frac{1}{2} \rho h_i [m_i^2 - (E^Q[d])^2] - H(\beta_0) + \beta_0 \gamma^*_i \right).$$
Solving for $\gamma^*_0$ and using Proposition 2 yield that:

$$\gamma^*_0 = \left\{ \bar{U}_1 - U_{11} + \rho \Phi(\{ m_i, \sigma_i^2 \}_{i=1,...,J}) - \frac{1}{2} \rho h \sigma_i^2 - (E^Q(d))^2 \right\} / (1 + \beta_0).$$

Note by Remark 2 that all except for the first two and the last term in the numerator are independent of the signal precision. Hence:

$$\frac{\partial \gamma^*_0}{\partial h_e} = \frac{\frac{\partial}{\partial h_e} \bar{U}_1 - \frac{\partial}{\partial h_e} U_{11} + [H'(\beta_0) - \gamma^*_0] \frac{\partial}{\partial h_e} \beta_0}{1 + \beta_0}.$$ 

This implies that:

$$[1 + \beta_0] \frac{\partial}{\partial h_e} CE^*_0 = [H'(\beta_0) - \gamma^*_0] \frac{\partial}{\partial h_e} \beta_0 - \beta_0 \left[ \frac{\partial}{\partial h_e} \bar{U}_1 - \frac{\partial}{\partial h_e} U_{11} \right].$$

Furthermore, it follows from (6) and Proposition 2 that:

$$\frac{\partial}{\partial h_e} \beta_0 = \frac{\partial}{\partial h_e} \exp \left( - \left( \delta + r(C E^*_2 - C E^*_0) \right) \right) = -r \beta_0 \frac{\partial}{\partial h_e} \bar{U}_1.$$

Hence:

$$\frac{\partial}{\partial h_e} E U^*_0 = r \exp(-r C E^*_0) \beta_0 \left\{ \frac{\partial}{\partial h_e} U_{11} + r [\gamma^*_0 - H'(\beta_0)] \frac{\partial}{\partial h_e} \bar{U}_1 \right\}.$$ 

Since both $U_{11}$ and $\bar{U}_1$ have a unique maximum for $h_e = \hat{h}$, all investors’ expected utilities have a stationary point for $h_e = \hat{h}$. Using the fact that the risk-adjusted expected dividend of the risky asset is independent of $h_e$, it follows that $H'(\beta_0) = E^Q[d][\hat{x}_i - x^*_0] + \gamma_i$. Hence:

$$\frac{\partial}{\partial h_e} E U^*_0 = r \exp(-r C E^*_0) \beta_0 \left\{ \frac{\partial}{\partial h_e} U_{11} + [\gamma^*_0 - \gamma_i] + E^Q[d](x^*_0 - \hat{x}_i) \right\} \frac{\partial}{\partial h_e} \beta_0 \right\},$$

which is (18). \(\blacksquare\)

**Proof of Proposition 7**

Using Proposition 2, the first-order condition (20) can be expressed as:

$$q^*(h_e) = g \left( \exp \left( \delta + r \bar{U}_1 + \Phi(\{ m_i(q^*_0), \sigma_i^2 \}_{i=1,...,J}) \right) \right),$$

where $g \equiv (f')^{-1}$. Using the definition of $\Phi$ in (13c), we obtain:

$$\frac{\partial \Phi}{\partial h_e} = \left( f'(q^*_0) + 1 \right) \frac{rZ \partial q^*_0}{T \partial h_e}.$$ 

Hence:

$$\frac{\partial q^*_0}{\partial h_e} = g' \left( f'(q^*_0) \right) f'(q^*_0) \times \left[ \frac{\partial \bar{U}_1}{\partial h_e} + \left( f'(q^*_0) + 1 \right) \frac{rZ \partial q^*_0}{T \partial h_e} \right].$$

Rearranging yields that:

$$\frac{\partial q^*_0}{\partial h_e} = \frac{g' \left( f'(q^*_0) \right) f'(q^*_0) \left( f'(q^*_0) + 1 \right) r \partial \bar{U}_1 / \partial h_e}{1 - g' \left( f'(q^*_0) \right) f'(q^*_0) \left( f'(q^*_0) + 1 \right) rZ / T \partial h_e}.$$
Note that \( g'(f'(q^*)) = 1/f''(f'(q^*)) < 0 \) by the concavity of the production function \( f \). Since \( f' > 0 \), the sign of \( \frac{\partial q}{\partial h_e} \) is opposite to the sign of \( \frac{\partial f}{\partial h_e} \) and, thus, the value-maximizing real investment \( q^*(h_e) \) is an inverted bell-shaped function of the signal precision \( h_e \) by Remark 2 and Proposition 4. Furthermore:

\[
\frac{\hat{t}}{\partial h_e} = r \frac{\partial C_1}{\partial h_e} + r \left( f'(q^*) + 1 \right) \frac{Z \partial q^*}{I} \frac{\partial U_1}{\partial h_e} = r \frac{\partial C_1}{\partial h_e} \left[ \frac{1}{1 - g'(f'(q^*))f'(q^*)} \right] \frac{rZ}{I}.
\]

Since \( g'(f') < 0 \) and \( f' > 0 \), the sign of \( \frac{\partial q}{\partial h_e} \) is the same as the sign of \( \frac{\partial f}{\partial h_e} \), i.e., the equilibrium interest rate is a bell-shaped function of the signal precision \( h_e \). It is “dampeden” relative to the exchange economy (in which \( \frac{\hat{t}}{\partial h_0} = r \frac{\partial C_1}{\partial h_0} \)), since the term in the brackets is less than 1. If the production technology is such that \( q^*(h_e = 0) = 0 \), the equilibrium interest rate is strictly less in the real investment setting than in an otherwise identical exchange economy with \( q(h_e) = 0 \) for all \( h_e \) (except for \( h_e = 0 \) and \( h_e \to \infty \) in which cases they are equal). Finally, since \( \frac{\partial q}{\partial h_e} = f'q^* \) and \( f' > 0 \), it follows immediately from the definition of the rate of return risk premium (15b) that it is a bell-shaped function of \( h_e \).

**APPENDIX B**

**Key Notation**

**Preference and Endowments**

\( I \) = number of investors;
\( Z \) = a fixed non-zero net-supply of a single risky asset;
\( \bar{z}_i, \bar{\gamma}_i, \bar{\kappa}_i \) = units of investor \( i \)'s endowment of the risky asset, the \( t = 2 \) zero-coupon bond paying one unit of consumption at \( t = 2 \), and the \( t = 0 \) zero-coupon bond paying one unit of consumption at \( t = 0 \), respectively, \( i = 1, 2, \ldots, I \);
\( \gamma_{it}, x_{it} \) = units held by investor \( i \) of the \( t = 2 \) zero-coupon bond and the risky asset after trading at date \( t = 0 \) and \( t = 1 \), respectively, \( i = 1, 2, \ldots, I \);
\( d_0, d \) = a dividend paid by a share of the risky asset at date \( t = 0 \) and \( t = 2 \), respectively;
\( d_0, \bar{d} \) = fixed component of the dividends at \( t = 0 \) and \( t = 2 \), respectively, in production economy;
\( r, \rho, \delta \) = investors’ common constant absolute risk aversion, risk tolerance, \( \rho = 1/r \), and utility discount rate for date \( t = 2 \) consumption, respectively;
\( c_{it} \) = investor \( i \)'s consumption at date \( t = 0 \) and \( t = 2 \), respectively, \( i = 1, \ldots, I \);
\( u_{it}(c_{it}) \) = investor \( i \)'s common period-specific exponential utility functions at date \( t = 0 \) and \( t = 2 \), respectively, i.e., \( u_{it}(c_{it}) = -\exp[-rc_{it}] \), and \( u_{it}(c_{2i}) = -\exp[-\delta \exp[-rc_{2i}], i = 1, \ldots, I] \);
\( q \) = real investments in production economy; and
\( f(q) \) = production function of real investments.

Throughout the article, a quantity with a * superscript denotes a corresponding equilibrium quantity, e.g., equilibrium demand for the risky asset at \( t = 0 \) is denoted \( x^*_0 \).

**Information System and Beliefs**

\( y \) = a public signal that all investors receive at \( t = 1 \);
\( \varepsilon \) = noise that plus the dividend equals the public signal, i.e., \( y = d + \varepsilon \);
\( h_e, \sigma^2_e \) = signal precision, and variance of the noise in public signal, \( h_e = 1/\sigma^2_e \);
\( m_i(m_{it}) \) = prior (posterior) mean of the dividend per share for investor \( i, i = 1, \ldots, I \);
\( \sigma^2_i(\sigma^2_{it}) \) = prior (posterior) variance of the dividend per share for investor \( i, i = 1, \ldots, I \);
\( \hat{h}(\bar{h}_1) = \) average of investors’ prior (posterior) dividend precision, \( \bar{h} \equiv \frac{1}{I} \sum_{i=1}^{I} h_i \), and \( \bar{h}_1 \equiv \frac{1}{I} \sum_{i=1}^{I} h_{i1} \);
\( \bar{m}^h(\bar{m}_1^h) = \) precision weighted average of investors’ prior (posterior) means, \( \bar{m}^h \equiv \frac{1}{I} \sum_{i=1}^{I} \frac{h_i}{\bar{h}_i} m_i \), and \( \bar{m}_1^h \equiv \frac{1}{I} \sum_{i=1}^{I} \frac{h_{i1}}{\bar{h}_{i1}} m_{i1} \);
\( \bar{\sigma}^2(\bar{\sigma}_1^2) = \) inverse of the average prior (posterior) precision, \( \bar{\sigma}^2 \equiv 1/\bar{h} \), and \( \bar{\sigma}_1^2 \equiv 1/\bar{h}_1 \);
\( \bar{m}_i = \) investor \( i \)'s prior mean of the fixed component of the dividend \( \hat{d}_i \), \( i = 1, ..., I \) in production economy;
\( \varphi_i(d) = \) investor \( i \)'s heterogeneous prior beliefs with respect to the \( t = 2 \) dividend, \( \varphi_i(d) \sim N(m_i, \sigma_i^2), i = 1, ..., I \);
\( \varphi(x) = \) investors’ homogeneous information beliefs for the noise, \( \varphi(x) \sim N(0, \sigma_x^2) \);
\( \varphi_i(y, d) = \) prior beliefs of investor \( i \) for the public signal and the dividend, \( \varphi_i(y, d) \sim N(\mu_i, \Sigma_i), i = 1, ..., I \); and
\( \varphi_{i1}(d | y) = \) posterior beliefs of investor \( i \) at \( t = 1 \) for the \( t = 2 \) dividend conditional on the public signal, \( \varphi_{i1}(d | y) \sim N(m_{i1}, \sigma_{i1}^2), i = 1, ..., I \).

**Equilibrium Quantities**

\( \beta_i = \) equilibrium price of \( t = 2 \) zero-coupon bond at \( t = 0 \) and \( t = 1 \), respectively;
\( \bar{t} = \) zero-coupon interest rate from \( t = 0 \) to \( t = 2 \), \( \bar{t} = -\ln \beta_0 \);
\( p_0 = \) ex ante equilibrium price of the risky asset at \( t = 0 \);
\( p_1(y) = \) ex post equilibrium price of the risky asset at \( t = 1 \) given the public signal \( y \);
\( \bar{\mu}^h = \) ex ante cost of capital defined as the (continuously compounded) expected rate of return, i.e., \( \exp(\bar{\mu}^h) = \bar{m}^h/p_0 \);
\( \vartheta^h = \) ex ante risk premium given as the ex ante cost of capital minus the interest rate;
\( \tau^*_i(y) = \) investor \( i \)'s equilibrium net-trade in the risky asset at \( t = 1 \), \( \tau^*_i(y) = x^*_i(y) - x^0_i, i = 1, ..., I \);
\( a\tau^*_i(y) = \) investor \( i \)'s abnormal net-trade defined as the difference between the net-trade and the expected net-trade conditional on the \( t = 2 \) dividend, \( a\tau^*_i(y) = \tau^*_i(y) - E(\tau^*_i(y)|d), i = 1, ..., I \);
\( T^* = \) abnormal trading volume per investor, \( T^* = \frac{1}{I} \sum_{i=1}^{I} |a\tau^*_i(y)| \);
\( U_{i1}(\bar{U}_1) = \) investor \( i \)'s (investors’ average) ex ante value of the trading gains following from signal-contingent trading at \( t = 1 \):

\[
U_{i1} = \frac{1}{2} \rho \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i (\bar{h} + h_i)^2} \right], \bar{U}_1 \equiv \frac{1}{I} \sum_{i=1}^{I} U_{i1};
\]

\( CE^*_i = \) investor \( i \)'s ex ante certainty equivalents of \( t = 0 \) and \( t = 2 \) consumption, respectively:

\( CE^*_{i0} = d_0 \bar{z}_i + p_0 [\bar{z}_i - x^*_i] + \beta_0 [\bar{\zeta}_i - \gamma^*_i] + \bar{\kappa}_i \), and

\( CE^*_{i2} = \gamma^*_i + U_{i1} + \frac{1}{2} \rho h_i |m_i^2 - (E^Q[d])^2| \); and

\( EU^*_i = \) equilibrium ex ante expected utility of investor \( i, i = 1, ..., J \).