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Many studies investigate the impact of heterogeneous beliefs in the first moment, while very few in the second moment. This is partially due to continuous-time setup which makes it difficult to incorporate heterogeneous beliefs in the second moment. In a two-period exponential–normal model with Bayesian learning, I demonstrate that heterogeneous prior variances give rise to the economic value of option markets. Investors speculate in option market and public information improves allocation efficiency of markets only when there is heterogeneity in prior variances. Heterogeneity in mean is neither a necessary nor a sufficient condition for generating speculations in option markets. With heterogeneous beliefs, options are non-redundant assets which can facilitate side-betting and enable investors to take advantage of the disagreements and the differences in confidence. This fact leads to a higher growth rate in the investors’ certainty equivalents and, thus, a higher equilibrium interest rate. Furthermore, option exhibits a unique feature of enabling signal precision to affect the \textit{ex ante} risk premium of underlying asset, which quadratic derivative and stock do not have.

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1. Introduction

A growing body of research has shown that disagreements in investors’ beliefs on firm fundamentals have significant impact on market characteristics including risk-free rate, risk premium, excess volatility, cross-section return, bond yield, and option pricing. Most of the literature focuses on the impact of heterogeneous beliefs in the first moment, while very few in the second moment. This is partially due to continuous-time setup which makes it difficult to incorporate heterogeneous beliefs in the second moment. Since the Girsanov’s theorem implies that the instantaneous volatility of endowment is identical across investors under both individual perceived dynamics and risk-neutral dynamics. In a two-period model with an exponential–normal specification and Bayesian learning, this paper examines the impact of the heterogeneous prior beliefs in variance on market equilibrium in an incomplete market. The first major contribution in this paper is that heterogeneous prior variances provide economic value to option markets in the sense that the investors speculate in the option market and imperfect public signal improves the allocation efficiency of markets measured by investors’ certainty equivalent only when there is heterogeneity in prior variances. Therefore, in my setting, the option is non-redundant. Whereas heterogeneity in mean is neither a necessary nor a sufficient condition for generating speculations in option markets.

On the other hand, public information or signal such as earnings and dividend announcements, mergers and acquisitions, macroeconomic announcements, accounting reports are long recognized to have substantial impacts on the financial markets. However, also under an exponential–normal specification, both Brennan and Cao (1996) and Christensen and Qin (2013) show that when investors trade in stock and bond, and/or quadratic derivative, signal precision does not affect the \textit{ex ante} risk premium of stock. Naturally, a question arises: Under what condition does public information quality exert influence on risk premium on stock and option? As the second main finding in the paper, I show that option payoff exhibits a unique feature of enabling signal precision to affect the \textit{ex ante} risk premium of underlying asset, while nonlinear payoff of quadratic derivative and linear payoff of stock do not have this characteristic.

The model is a derivative oriented and two-period extension of the classic single-period capital asset pricing model (CAPM) with heterogeneous beliefs ofLintner (1969). Specifically, the investors

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1 Researches modelling heterogeneous beliefs in the first moment includes Detemple and Murthy (1994), Basak (2000), Basak (2005), and Buraschi and Jiltsov (2006), among many others.

2 Recent empirical evidence suggests that macroeconomic announcements and employment figures have pronounced impact on financial markets. Literatures include, e.g., Feltham and Pae (2000); Andersen et al. (2003), and Richardson et al. (2005).

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hold different prior beliefs at $t = 0$ on the normally distributed $t = 2$ dividend, i.e., the prior beliefs of mean and precision (the inverse of variance) differ. These assumptions imply that the investors agree to disagree due to, for instance, difference in their experiences or DNA. The investors update their beliefs according to the Bayes’ rule with a public signal received at $t = 1$ from a simple public information system. The public signal is equal to the $t = 2$ dividend on the risky underlying asset plus independent noise. Moreover, the investors have concordant beliefs (Milgrom and Stokey, 1982) or homogeneous information beliefs (Hakansson et al., 1982) on the normally distributed noise in the signal, i.e., a zero mean and a common signal precision. These assumptions allow to measure the informativeness of public information system by the public signal precision. Furthermore, the investors can trade and speculate in the option markets, the underlying asset markets and the zero-coupon bond markets at $t = 0$ and $t = 1$, and consume at $t = 0$ and $t = 2$. Solving for equilibriums in an exchange economy, I investigate the impact of the heterogeneity in beliefs, the strike price, and the public information quality on risk-free rate, risk premium, and other asset pricing properties in option markets.

Heterogeneity in the prior variance creates the opportunities for speculation in option markets. With homogeneous prior variance, the investors do not trade in the option markets. The intuition is related to the results in Wilson (1968): Pareto efficient allocations in settings with heterogeneous beliefs require not only an efficient sharing of the risks, but also an efficient side-betting arrangement. With homogeneous prior variance, the Pareto efficient side-betting based on their disagreements about the mean can be achieved by trading only in the risky underlying asset and the zero-coupon bond at $t = 0$. The CAPM-like equilibrium price under heterogeneous beliefs is obtained. However, when the investors have different prior precision, trading only in the underlying asset and the zero-coupon bond at $t = 0$ does not facilitate efficient side-betting. The investors tend to speculate in the option markets. Take a European call option market for example, the investor with a low (high) prior precision takes long (short) position in the call option with convex payoff to achieve a terminal payoff which is a convex (concave) function of the dividend. This speculative strategy is the so-called Gamma trading strategy.

Speculations in option markets increase the allocation efficiency of the equilibrium. This result can be detected from the change of asset pricing properties when investors’ speculative behaviors change. First, conditional on identical average prior precision, the higher heterogeneity in beliefs, the more opportunities in speculations, and the more advantage of the disagreements and the differences in confidence among the investors can be taken. This effect leads to a higher efficiency of side-betting and more gains in trading options. The trading gains translate into increased certainty equivalents of the terminal consumption, and result in a higher equilibrium consumption growth, and thus a higher equilibrium interest rate. Second, investors tend to trade in options with an intermediate strike price. Since this type of options carry the most substantial convexity in their payoff, and thus the investors can effectively and actively speculate in the option markets.

Third, the imperfect public signal facilitates speculations. When the investors have heterogeneous prior dividend precision, they update their posterior beliefs differently with imperfect public signal, and this gives the basis for additional trading gains contingent on the imperfect public signal. Another round of trading using Gamma trading strategies at $t = 1$ partly facilitates the efficient side-betting. Eventually, a combination of the option with intermediate strike price and public information system of the intermediate signal precision enables the investors to achieve the highest efficiency of side-betting, reflected by the unique maximum point of the ex ante equilibrium interest rate.

Options have a particular feature to allow public signal precision to affects the ex ante equilibrium risk premium on the risky underlying asset. The underlying mechanism is that the convexity of the option payoff at $t = 1$ varies with the variance of the posterior beliefs which are determined by the public signal precision. Through this relationship, the speculative positions in the underlying asset and the option are affected by the public signal precision. This fact gives rise to a signal-precision-dependent covariance between the marginal utility of consumption and the dividend and, thus, a signal-precision-dependent ex ante equilibrium risk premium. With an intermediate strike price, the impact of the public signal precision on the ex ante equilibrium risk premium is nontrivial (see Fig. 9). In contrast, also under an exponential–normal specification, both Brennan and Cao (1996) and Christensen and Qin (2013) show that when investors trade in stock and bond, and/or quadratic derivative, signal precision does not affect the risk premium. Note Brennan and Cao (1996) inject some unmodeled noise trading into the price system, while the noise trading does not bridge a relationship between the risk premium and the signal precision. On the contrary, speculations in option markets bridge this link.

1.1. Review of the literature

Some studies on the impact of information system and heterogeneity in beliefs on asset pricing are closely related to this work. Back (1993) shows that when investors receive asymmetric information about the future price of an underlying asset, an introduction of option can cause a stochastic volatility of the underlying asset and make the option non-redundant. And Li (2012) assumes that investors believe the growth rate of the dividend to be a constant and known perfectly. This assumption enables closed-form solutions for vanilla European option prices and closed-form approximations for barrier options. His model offers a rationale for observed implied volatility patterns in an equilibrium setting and is also easy to implement in practice. However, in both models, the role of the option to facilitate side-betting is not explored, and both model are silent with respect to the influence of information system. Those issues are investigated in this paper.

In a similar effort, Buraschi and Jiltsov (2006) employ a model, based on the work by Detemple and Murthy (1994), to investigate the option markets with heterogeneous beliefs. They show that the heterogeneity in beliefs has significant pricing implications by plotting equilibrium asset pricing properties such as stock price and stock volatility as functions of the difference in beliefs. Their results indicate that the heterogeneous beliefs are strongly related to optimal portfolio holdings, stock volatility, equity premium, stock prices, option prices and skewness in equity returns.

My model differs with Buraschi and Jiltsov (2006) in several aspects. First, their studies focus on the impact of heterogeneity in means, while this paper mainly investigates the role of heterogeneity in prior variances. Second, Cuoco and He (1994) demonstrate

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3 Note this paper considers the economy-wide impacts of public information and, thus, the public information should be interpreted as, for instance, macroeconomic reports of aggregate consumption. An information system is a set of potential signals that present conditional (or signal dependent) probabilities that each state at the terminal date occurs.

4 See more about Gamma trading strategies in e.g., Hull, 2009, Chapter 17.

5 Recent contributions of asset pricing in economies with incomplete information include, e.g., David (1997), Brennan (1998), Veronesi (1999) Veronesi (1999), Veronesi (2000), and Brennan and Xia (2001). Efforts to establish the so-called differences-of-opinion models in financial markets include, e.g., Cao and Ou-Yang (2009), and Banerjee and Kremer (2010). These literature assume the investors have homogeneous beliefs about the fundamentals in the economy, but disagree on how to interpret common public signals.
the equilibrium with a stochastic weight in the representative agent utility is in general not Pareto efficient. Hence markets in Buraschi and Jiltsov (2006) are essentially incomplete, and they do not illustrate how much the option can help to improve the efficiency of side-betting. This paper demonstrates to what degree the option can enhance the market allocation efficiency. Third, the approach to study the impacts of heterogeneity in beliefs is different. They plot the asset pricing properties as functions of the difference in the updated beliefs scaled by the signal volatility, hence heterogeneous beliefs in their model carry the effect of the information system. In fact, they do not study the impact of the information quality. However, I plot the asset pricing properties as functions of the different priors, thus the analysis in this paper probes the impact of the heterogeneity in beliefs and the public information quality separately.

Also note that Buraschi and Jiltsov (2006) and David (2008) solve for the equilibrium by constructing a representative agent utility through taking weight of the two different individual utilities, and the weight is stochastic and endogenized in the equilibrium as a function of the difference in beliefs. Hence the change of heterogeneity in beliefs results in the change of the value of the weight. However, when plotting the asset pricing properties as functions of the heterogeneity in beliefs, they fix the value of the weight at a certain level. Hence, their approach to analyze the impacts of the heterogeneous beliefs can be considered to be conditional on a certain value of the weight. While models of unconditional analyses are provided in this paper: Exponential-normal specification gets rid of the need of the stochastic weighted representative agent. Thus, I do not need to fix any kind of quantities which are functions of heterogeneity and/or signal precision when varying the level of heterogeneity and/or signal precision.

Christensen and Qin (2013) study a benchmark model in which the investors speculate in a risky asset and a zero-coupon bond. They show that the public information system is able to facilitate side-betting and enable the investors to take advantage of the disagreements in the prior variance. A more efficient market gives rise to a higher equilibrium interest rate.

Based on the idea of Wilson (1968), Christensen and Qin (2013) introduce a dividend derivative which pays off the square of dividend in the terminal date to facilitate side-betting and achieve a Pareto efficient equilibrium. This extreme case gets rid of the need for dynamic trading based on the public signals. Note that the dividend derivative in their model resembles a “smooth” straddle. This idea gives a motivation to employ a true straddle, i.e., long positions in both a call and a put option with the same strike price, as a replacement of the dividend derivative and explore the ability of the options in facilitating side-betting. The straddle may not be able to effectively complete the market. However, the investors still can employ the Gamma strategy by trading straddles to achieve a more satisfactory convexity of their payoff. This paper investigates this intermediate case to address issues such as the role of options to help establish allocation efficient markets and the role of the public information system in the option markets.

This paper is organized as follows. The primitives of the economy and the learning mechanism are established in Section 2. Section 3 establishes a single-period model in which investors receive no signal or perfect signal, and investigate the effects of the heterogeneous beliefs and the strike price on asset pricing properties such as the equilibrium interest rate and the equilibrium risk premium with numerical examples. Section 4 investigates the equilibrium with imperfect public signal and discusses the effects of the public signal precision on asset pricing properties. Section 5 concludes the paper and discusses possible extensions. Proofs and algorithms to solve for the equilibriums are presented in Appendix A. Appendix B provides the derivations of models when investors speculate with straddles.

2. The model

I examine the impacts of the heterogeneity in priors, the strike price and the public signal precision on option pricing properties in an exchange economy in which many types of agents have identical preferences but differ in their prior beliefs about the distribution of forthcoming dividend.

2.1. The investors’ beliefs and preferences

There are two consumption dates, \( t = 0 \) and \( t = 2 \), and there are \( I \) investors who are endowed at \( t = 0 \) with a portfolio of underlying assets, potentially receive public information at \( t = 1 \), and receive terminal normally-distributed dividends from their portfolio of underlying asset at \( t = 2 \). There are three marketed securities: a zero-coupon bond which pays one unit of consumption at \( t = 2 \) and is in zero net supply, the shares of a single risky firm which have net supplies \( Z \) at \( t = 0 \) and \( t = 1 \), and a European call option with a strike price at \( K \) which has zero net supplies at \( t = 0 \) and \( t = 1 \), and the underlying asset is the share of the firm. The net supplies are fixed: the \( I \) investors are endowed with firm shares \( z_i \); \( i = 1, 2, \ldots, I \). In addition, the investors have endowments \( h_0 \) units of the European call option in zero net supply at \( t = 0 \), and the investors are endowed with \( y_i \) units of zero-coupon bond in zero net supply also. The trading of the marketed securities takes place at \( t = 0 \) and \( t = 1 \). Let \( h_0, x_0 \) and \( y_0 \) present the investor’s portfolio of the option, the share and the units had of the zero-coupon bond after trading date \( t \), respectively. Hence, the market clearing conditions at date \( t \) are

\[
\sum_{i=1}^{I} h_i = 0, \quad \sum_{i=1}^{I} y_i = 0, \quad \sum_{i=1}^{I} x_i = Z = \sum_{i=1}^{I} y_i, \quad t = 0, 1.
\]

The dividend paid by the firm’s shares at date \( t = 0 \) is denoted \( d_0 \) and at date \( t = 2 \) is denoted \( d_2 \). The investors have heterogeneous prior beliefs on the distribution of the dividend, and the individual perceived distribution is represented by \( N(m_i, \sigma_i^2) \), where \( m_i \) is the expected dividend per share and \( \sigma_i^2 \) is the variance of firm dividends per share for the investor \( i \). Note that since the investors have common and not asymmetric information, they are aware of each other’s different inferences, arising from their different priors. Under this heterogeneous beliefs formulation, the investors agree to disagree. Morris (1995) proposes a method to endogenize the difference in beliefs and formulations. He shows that it is fully consistent with rationality to have heterogeneous priors.

At \( t = 1 \), all investors receive a public signal \( y \), which has impacts on asset prices. The investors can trade in the riskless asset in zero net supply with price \( b_0 \) at \( t = 0 \) and price \( b_1 \) at \( t = 1 \). The underlying asset price and the call option price at \( t = 0 \) are denoted \( p_0 \) and \( p_0 \), respectively. The underlying asset price and call option price at \( t = 1 \) given the public signals \( y \) are represented by \( p(y) \) and \( \pi(y) \), respectively, which reflect the fact that the \( \text{ex post} \) asset prices may be affected by the public signals available to the investors at \( t = 1 \).

The investor’s consumption at date \( t \) is denoted \( c_t \) and they have time-additive utility. The common period-specific utility is negative exponential utility with respect to consumption, i.e., \( u_0(c_0) = -\exp(-r_0c_0) \) and \( u_2(c_2) = -\exp[-\delta \cdot \exp(-r_2c_2)] \), where \( r > 0 \) is the investors’ common constant absolute risk aversion parameter. Moreover, the investors have common utility discount rate, \( \delta \), for date \( t = 2 \) consumption.

2.2. Learning mechanism

Bayesian learning provides a core concept of information processing in financial markets. I assume that the investors update their beliefs in a Bayesian fashion.
Specifically, the public signal $y$ is generated by $y = d + \varepsilon$, where $\varepsilon \sim N(0, \sigma^{2}_y)$, $d \equiv 1/\sigma^{2}_y$ is the signal precision, which is common knowledge of all the investors. In other words, the investors have concordant beliefs (Milgrom and Stokey, 1982) or homogeneous information beliefs (Hakansson et al., 1982). I use $h_i \equiv 1/\sigma^{2}_i$ throughout to denote the precisions for the associated variances. When $h_i \to \infty$, the public signal is the realization of the dividend. Moreover, the terminal dividend $d$ and the noise $\varepsilon$ are independent and jointly normally distributed. Thus, the joint distribution of the public signal and dividend from the perspective of investor $i$ is $\phi(y,d) \sim N(\mu_i, \Sigma_i)$, where

$$
\mu_i = \left( \begin{array}{c} m_i \\ 0 \end{array} \right), \quad \Sigma_i = \left( \begin{array}{cc} \sigma^{2}_i + \sigma^{2}_y & \sigma^{2}_y \\ \sigma^{2}_y & \sigma^{2}_y \end{array} \right),
$$

I represent investor $i$'s posterior beliefs of the terminal dividend $d$ given his signal as $N(m_i, \sigma^{2}_i)$, hence the posterior of the investor $i$ at $t=1$ is $\phi(x_i(d,y)) \sim N(m_i, \sigma^{2}_i)$, where

$$
\begin{align*}
& m_i = m + \sigma^{2}_i (\sigma^{2}_y + \sigma^{2}_i)^{-1} (y - m), \\
& \sigma^{2}_i = \sigma^{2}_y - \sigma^{2}_i (\sigma^{2}_y + \sigma^{2}_i)^{-1} \sigma^{2}_y.
\end{align*}
$$

Therefore, the posterior mean, $m_i$, is a linear function of the investor's public signal, while the posterior variance, $\sigma^{2}_i$, is independent of the specific signal.

Moreover, when the variance of the disturbance term $\sigma^{2}_y \to 0$, the posterior mean $m_i \to y$, hence the posterior mean tends to be independent of the priors as public signal precision increases. When $\sigma^{2}_y \to 0$, the posterior variance $\sigma^{2}_i \to 0$, this indicates that with higher public signal precision, investors are more confident on their inferences. When $\sigma^{2}_y \to \infty$, the public signal becomes completely uninformative, thus the posterior beliefs equal to the prior beliefs. From the perspective of the investor, the signal $y$ is normally distributed with $N(m, \sigma^{2}_y + \sigma^{2}_i)$, hence the posterior mean $m_i \sim N(m, \sigma^{2}_y (\sigma^{2}_y + \sigma^{2}_i)^{-1})$. For further information on Bayesian learning model, see e.g., Raiffa and Schlaifer (1961) and DeGroot (1970).

3. Benchmark equilibrium when investors receive no or perfect public signal

In this section, I derive equilibriums in the economy populated with investors with heterogeneous beliefs. The investors can speculate in European call option markets. In order to see the effect of the heterogeneous beliefs and the strike price of European call options clearly without interruptions from the impact of the public information system, I first investigate a benchmark case in which the investors receive no or perfect public signal at $t=1$, then there is no basis for trading at $t=1$ based on posterior beliefs. Hence, the model is essentially equivalent to a single period model.

3.1. Equilibrium in a single period economy

The payoff of a European call option with strike price $K$ at the date $t=2$ is max $d - K, 0$. Hence, from the perspective of $t=0$, the date $t=2$ consumption for the investor $i$ is

$$
c_{i2} = \theta_i \max(d - K, 0) + x_{i0}d + \gamma_{i0}. 
$$

Given the period-specific negative exponential utility, the investor $i$’s date $t=0$ certainty equivalent of date $t=2$ consumption, receiving no or perfect public information at $t=1$, $CE_{i2}(\theta_{i0}, x_{i0}, \gamma_{i0})$, can be calculated according to Lemma 1.

**Lemma 1.** Assume the investors receive no or perfect public information at $t=1$, given the portfolios in the underlying asset markets, the call option markets, and the zero-coupon bond markets at $t=0$, the investor $i$’s $t=0$ certainty equivalent of $t=2$ consumption is

$$
CE_{i2}(\theta_{i0}, x_{i0}, \gamma_{i0}) = \gamma_{i0} + m_{i0}x_{i0} + \frac{1}{2} r\sigma^{2}_x x_{i0} + f_i(x_{i0}, \theta_{i0}),
$$

where

$$
f_i(x_{i0}, \theta_{i0}) = -\frac{1}{\theta_{i0}} \ln \left( \phi\left( \frac{K - (m_{i0} - r\sigma^{2}_x x_{i0})}{\sigma_{\theta_{i0}}} \right) \right) + \exp \left[ -\theta_{i0} \left( -K + m_{i0} + \frac{1}{2} r\sigma^{2}_x x_{i0} \right) \right] \times \left( 1 - \phi\left( \frac{K - (m_{i0} - r\sigma^{2}_x (\theta_{i0} + x_{i0}))}{\sigma_{\theta_{i0}}} \right) \right).
$$

and $\phi(x)$ is the cumulative distribution function of the standard normal distribution.

**Proof.** See Appendix A.1. \Box

Given prices and portfolios at $t=0$, the $t=0$ consumption is a constant. Note the certainty equivalent of a constant is the constant itself, hence, the investor $i$’s $t=0$ certainty equivalent equals to the date $t=0$ consumption, i.e.,

$$
CE_{i0} = c_{i0} = d_i + (\theta_i - \theta_{i0}) z_{i0} + (\gamma_i - \gamma_{i0}) \theta_i + (z_i - x_{i0}) \theta_{i0},
$$

the investor $i$’s decision problem at $t=0$ can be stated as follows

$$
\max U_i(\theta_{i0}, \gamma_i; x_{i0}), \quad \text{where } U_i(\theta_{i0}, \gamma_i; x_{i0}) = -\exp(-rCE_{i0}) - \exp(-\delta \exp(-rCE_{i0})).
$$

The market equilibrium is characterized by the first-order conditions of the investor $i$’s optimal portfolio choice problem, i.e.,

$$
\frac{\partial U_i}{\partial \theta_{i0}} = 0, \quad \frac{\partial U_i}{\partial \gamma_i} = 0, \quad \text{and } \frac{\partial U_i}{\partial x_{i0}} = 0,
$$

and the market clearing condition for each asset, i.e.,

$$
\sum_{i=1}^{N} \theta_{i0} = 0, \quad \sum_{i=1}^{N} \gamma_i = 0, \quad \text{and } \sum_{i=1}^{N} x_{i0} = Z. 
$$

The equilibrium portfolios and the prices are implicit solutions of the system of equations which arise from the first-order conditions for the portfolio of each asset, and the market clearing conditions. More details of equilibrium equations and the calculation of the first derivatives in the first-order condition are provided in Appendix A.2.1. The methods to further simplify the equations to nonlinear equations with only three unknowns and the algorithms to solve for the equilibrium numerically are also described in Appendix A.4.

3.2. The impact of heterogeneous beliefs and strike price on asset pricing properties

In this section, I demonstrate how the heterogeneity in beliefs and the strike price of option affect the equilibrium prices at $t=0$ when the investors hold heterogeneous beliefs including heterogeneous prior means and/or heterogeneous prior variances.

Solving for the equilibrium, I find that with homogeneous prior variance, the investors do not trade in the option markets. This is because the Pareto efficient side-betting based on their disagreements about the mean can be achieved by trading only in the risky underlying asset and the zero-coupon bond at $t=0$. The model reduces to a single period benchmark case in which the investors only trade in the stock and the zero-coupon bond. Therefore, heterogeneity in mean is neither a necessary nor a sufficient condition for generating speculations in option markets. Hence in this...
The zero-coupon bond is given as

\[ \text{strike price} \times \text{prior variances} \]

The investor's decision problem, i.e., Eq. (6), and the investor's certainty equivalents at \( t = 0 \) and \( t = 2 \), i.e., Eqs. (5) and (4), the first-order condition with respect to the portfolio in the zero-coupon bond is given as

\[ -r \exp(-rCE_0)\theta_0 + r \exp(-\delta) \exp(-rCE_2) = 0. \]

Solve for the price of zero-coupon bond, I obtain

\[ \theta_0 = \exp(-\delta + rCE_2 - CE_0)). \]

With the definition of the equilibrium interest rate, I achieve the following proposition.

**Proposition 1.** Assume the investors with heterogeneous beliefs can potentially trade in option markets. The equilibrium interest rate is given as the time discount rate plus the risk-adjusted growth in certainty equivalents, i.e.,

\[ \tau = \delta + r(CE_2 - CE_0). \]  

(9)

In Eq. (9), the equilibrium interest rate is expressed as an increasing linear function of the growth in the investors' certainty equivalents. In equilibrium, due to the assumptions of common constant absolute risk aversion parameter \( r \) and common utility discount rate \( \delta \), it directly follows from the first-order condition that all the investors have the same growth in certainty equivalents. Note an identical relationship between the equilibrium interest rate and growth in certainty equivalents can be found in the benchmark model in Christensen and Qin (2013).

The equilibrium interest rate is endogenized in the equilibrium as a function of the prior precision and the strike price. To see the impact of those two pricing factors, and I plot the equilibrium interest rate as a function of heterogeneous prior precision at a fixed level of the strike price, and as a function of the strike price at a given level of the heterogeneity in beliefs in Fig. 1 for the parameters in Table 1.

Note the parameters are selected to guarantee: First, significant variations in both the level of equilibrium interest rate (from 3% to 6% in Fig. 1) and equilibrium risk premium; Second, the scale of both the equilibrium interest rate and the equilibrium risk premium should be reasonable, e.g., the interest rate should be around 5%, and the risk premium should be around 10%; Third, considerably high accuracy of the numerical solutions of nonlinear equations within an acceptable computing time. Also note the parameters of mean and dividend is scale free. Hence, the magnitude of the parameters does not change the intuition of the results, and the general equilibrium properties which are analyzed in this paper maintain effective regardless the change of the parameters, given the parameters meet basic requirements, for instance, the risk aversion \( r > 0 \), and the prior variance \( \sigma_j^2 > 0 \).

In each panel, when I plot the asset pricing properties as functions of the heterogeneity in beliefs, a diagonal line indicates that the two investors agree with each other, and depart further from the diagonal line indicates higher heterogeneity in beliefs. As we can see from the left panel of Fig. 1, the equilibrium interest rate increases with the heterogeneity in prior precision. Since conditional on identical average prior precision, the higher heterogeneity in beliefs, the more opportunity in speculation in option market, and the more advantage of the disagreements and the differences in confidence among the investors can be taken. This effect leads to a higher efficiency of side-betting and more gains in trading. The trading gains translate into increased certainty equivalents of the \( t = 2 \) consumption, and result in a higher equilibrium consumption growth, and thus a higher equilibrium interest rate. Note the increase of the equilibrium interest rate can be viewed as an analog of the side-betting trading gains. This intuition can be gained directly from Eq. (9).

Furthermore, the diagonal line in the left panel shows that when investors hold homogeneous prior precision, the equilibrium interest rate increases with the prior precision. It is a standard
result that with homogeneous belief, the equilibrium interest rate is given as the time discount factors plus a risk-adjusted aggregate consumption growth minus a variance of risk-adjusted aggregate consumption. With homogeneous beliefs, the investors do not trade in the option markets, and the variance of risk-adjusted aggregate consumption decreases with the prior precision.

The right panel of Fig. 1 shows the impact of the strike price on the equilibrium interest rate. When the strike price is very high, i.e., the option is very deep out of the money, the value of option is almost zero. When the strike price is much lower than the mean, i.e., the option is very deep in the money, the option resembles a stock paying a dividend with a very high mean but with identical prior variance as that of the underlying asset. As a result, from the right panel of Fig. 1, we can see, with very high or very low strike price, the equilibrium interest rate converges to that in the benchmark case in which the investors only trade in a stock and a zero-coupon bond as depicted by the bottom line. This result illustrates the lowest efficiency of side-betting and benefit from the disagreement in the prior variance. When the option is slightly in the money with an intermediate strike price, it is substantially different from nothing or a stock. Thus the investors can effectively and actively speculate in the option markets. This kind of call option can more effectively facilitate side-betting, and thus enhance the growth in certainty equivalents and the equilibrium interest rate. With the used parameters, the equilibrium interest rate reaches the maximum point of around 0.062 at an intermediate strike price of around 0.01.

Note that Buraschi and Jiltsov (2006) and David (2008) also plot asset pricing properties as functions of heterogeneity in beliefs. However, they solve for the equilibrium by constructing a representation agent utility through taking weight of the two different individual utilities, and the weight is stochastic and endogenized in the equilibrium as a function of the difference in beliefs. Hence the change of heterogeneity in beliefs results in the change of the value of the weight. However, when plotting the asset pricing properties as functions of the heterogeneity in beliefs, they fix the value of the weight at a certain level. Hence their approach to analyze the impacts of the heterogeneous beliefs can be considered to be conditional. Models in this paper study the asset pricing properties as functions of the heterogeneous beliefs, the strike price, and the public information quality unconditionally.

### 3.2.2. Equilibrium risk premium

Christensen and Qin (2013) define the (continuously compounded) expected rate of return $\mu^\text{eq}$ using the beliefs implicit in the unambiguous ex ante equilibrium price of the risky underlying asset, i.e., $\varphi^j(d) \sim N(m^j, \sigma^2)$

\[
\exp(\mu^\text{eq}) = \frac{m^j}{p_0},
\]

where

\[
m^j \equiv \frac{1}{T} \sum_{t=1}^{T} h_t h_n, \quad h_t \equiv \frac{1}{T} \sum_{t=1}^{T} h_t, \quad \sigma^2 \equiv \frac{1}{h}.
\]

I define the expected rate of return in the same way, and obtain the equilibrium risk premium on the risky underlying asset $\sigma^\text{eq} = \mu^\text{eq} - r$. Since the equilibrium interest rate is defined as $r = -\ln p_0$ and the ex ante price of the underlying asset is a product of the equilibrium riskless discount factor and the risk-adjusted expected dividend, i.e., $p_0 = \beta_0 E^d[\delta]$, hence the equilibrium risk premium $\sigma^\text{eq} = \ln m^j - \ln E^d$. The following subsection provides the expression for the risk-adjusted dividend $\ln E^d\delta$, and the definition of $Q$ measure.

In the benchmark model without the option, the equilibrium risk premium on the risky asset is only affected by the prior mean, the prior precision, the risk aversion and the net supplies of the risky asset. However, when the investors invest in the option markets, the equilibrium risk premium on the risky underlying asset is affected by the strike price. I plot the equilibrium risk premium as a function of heterogeneous prior precision at a fixed level of the strike price, and as a function of the strike price at a given level of the heterogeneity in beliefs in Fig. 2.

---

**Fig. 2.** The equilibrium risk premium. The equilibrium risk premium on the risky underlying asset is plotted as a function of the heterogeneous prior precision conditional on a given level of the strike price and the strike price conditional on a given level of the heterogeneity in prior precision.

**Fig. 3.** Marginal utility of consumption. The marginal utility of consumption is plotted as a function of the realizations of forthcoming dividend with the strike price $K = -0.005, 0.01, 0.023$ respectively.
From the left panel in Fig. 2, we can see the equilibrium risk premium is not affected by the heterogeneity in beliefs, but decreases with the average prior precision. With homogeneous prior precision, the investor’s confidence on the forthcoming dividend increases with the prior precision, and thus decreases with the requirement of compensation for risk. Whereas the lack of impact of heterogeneity in beliefs is quite likely specific to the particular model with exponential utilities and normally distributed dividends.

The curve in the right panel of Fig. 2 is just like the inverted risk-adjusted expected dividend. With a fixed level of heterogeneity in beliefs, when the strike price is much higher or much lower than the mean, the option resembles nothing or a stock. The equilibrium risk premium on the risky underlying asset converges to that in a benchmark case with a stock and a zero-coupon bond as depicted by the middle line. The intuition of the impact of the intermediate strike price can be gained from the fact that the risk-adjusted expected dividend is the expected dividend plus the covariance of the marginal utility of consumption and the dividend scaled by the expected marginal utility of consumption. See more in Chapter 5 in Christensen and Feltham (2003). Note in the right panel, when the strike price K increases from -0.005 to 0.023, the equilibrium risk premium increases. To understand this fact, I plot the marginal utility of consumption as a function of the realizations of forthcoming dividend in Fig. 3.

We can see from Fig. 3 that when the strike price K increases from -0.005 to 0.023, the slope of the broken line of marginal utility is in general decreasing, this fact signals the covariance of the marginal utility of consumption and the dividend is in general decreasing, thus the value of \( E^d [d] \) decreases, and the equilibrium risk premium increases.

Furthermore, the marginal utility of consumption is determined by the investors’ portfolios. I plot the portfolios in financial markets as functions of the strike price in Fig. 4.

An investor with low prior precision takes a positive position in the option market. Moreover, since with extreme strike price, the option tends to resemble stock or nothing, and thus has a payoff with little convexity. Therefore, from Fig. 4 we can see that, in order to accumulate enough convexity, the investor takes a very high position in the option market. When the option is very deep in the money and resembles a stock, high positive demand in the option leads to a large short position in the stock market and bond market. Relatively, when the option is very out of the money and resembles nothing, the portfolio in the stock market and the bond market converges to that in the benchmark model in which investors only trade in a stock and a bond. The equilibrium portfolios give rise to the state-dependent payoff according to Eq. (3), and thus the marginal utility which is a function of the forthcoming dividend.

3.2.3. Equilibrium price of risky underlying asset

The underlying asset price is endogenized in the equilibrium as a function of the priors and the strike price. I plot the equilibrium underlying asset price as a function of heterogeneous prior precision at a fixed level of the strike price, and as a function of the strike price at a given level of the heterogeneity in beliefs in Fig. 5.

To analyze the properties of the underlying asset price, I first establish the following proposition by deriving the investors’ first-order condition with respect to the portfolio in the underlying asset.

**Proposition 2.** Assume the investors with heterogeneous beliefs can potentially trade in option markets. The ex ante equilibrium price of the risky underlying asset at \( t = 0 \) is equal to the equilibrium riskless discount factor times the risk-adjusted expected dividend, i.e.,

\[
p_0 = p_0 E^d[d].
\]

The risk-adjusted expected dividend is expressed as a function of the prior means and variances, i.e.,

\[
E^d[d] = \left[ \prod_{i=1}^I \left( \frac{\partial \mathbb{E} d_i(b_i, x_0, \theta_b)}{\partial x_0} \right) \right]^{\frac{1}{2}}
\]

\[
= \left[ \prod_{i=1}^I \left( m_{i0} - r X_0 \theta_0 + \frac{\partial f(x_0, \theta_0)}{\partial x_0} \right) \right]^{\frac{1}{2}}.
\]

As the strike price varies, the ex ante equilibrium price of the risky underlying asset can either be higher or lower than that in the benchmark model in which investors trade only in a stock and a zero-coupon bond.

**Proof.** See Appendix A.2.2 for details of the derivation of the equilibrium underlying asset price. □

As indicated by Proposition 2, the left (right) panel in Fig. 5 is a balanced result of the left (right) panel in Figs. 1 and 2. With the parameters in Table 1, the price of the risky underlying asset

\[ d \sim N \left( \left( \begin{array}{c} \frac{1}{2} \delta^2 \frac{\partial \mathbb{E} d_i(b_i, x_0, \theta_b)}{\partial x_0} \end{array} \right), \delta^2 I \right). \]

Note that while the expected dividend under \( Q \) is uniquely determined in equilibrium, the variance of the dividend under \( Q \) is not uniquely determined due to the market incompleteness and, thus, I just take it to be \( \delta^2 \). The lack of the uniqueness of the variance has no consequences in the subsequent analysis.
decreases with the heterogeneity in beliefs. The diagonal line indicates that with homogeneous belief, the risky asset price increases with the prior precision.

Moreover, the right panel in Fig. 5 shows that, with a fixed level of heterogeneous beliefs, when the strike price is much higher or much lower than the mean, the price of the risky underlying asset converges to that in a benchmark case with a risky asset and a zero-coupon bond as depicted by the middle line of around 8.62.

3.2.4. Equilibrium call option price

The European call option price is endogenized in the equilibrium as a function of the priors and the strike price. I plot the equilibrium call option price as a function of heterogeneous prior precision at a fixed level of the strike price, and as a function of the strike price at a given level of the heterogeneity in beliefs in Fig. 6.

Similar to the case of the equilibrium underlying asset price, to analyze the option price, I establish the following proposition by deriving the investors’ first-order condition with respect to the portfolio in the call option.

**Proposition 3.** Assume the investors with heterogeneous beliefs can potentially trade in option markets. The ex ante equilibrium price of the European call option at $t = 0$ is equal to the equilibrium riskless discount factor times the risk-adjusted expected option payment, i.e.,

$$\pi_0 = \beta_0 E^0[\max(d - K, 0)].$$

The risk-adjusted expected dividend is expressed as a function of the prior means and variances, i.e.,

$$E^0[\max(d - K, 0)] = \left[ \prod_{i=1}^{n} \frac{\partial E_0(\theta_0, \theta_0, \gamma_0)}{\partial \theta_0} \right]^{\frac{1}{2}} = \left[ \prod_{i=1}^{n} \frac{\partial f(x_0, \theta_0)}{\partial \theta_0} \right]^{\frac{1}{2}}.$$  

(13)

As we can see from the left panel in Fig. 6, with the parameters in Table 1, the impact of the heterogeneity in beliefs on the equilibrium call option price is almost invisible. The diagonal line indicates that with homogeneous prior precision, the equilibrium call option price decreases with the prior precision. In this case, the investors do not trade in the option markets, and the option is a redundant asset. The model reduces to a benchmark case that matches the result of the Black-Scholes model: The higher stock volatility, the higher value of the option.

Moreover, the right panel in Fig. 6 shows that, of course, the deeper in the money, the more valuable the option is. When the option is very deep out of the money, the equilibrium call option price is nearly zero.

3.2.5. Equilibrium expected utilities

The impact of the option on individual utility is similar to that of public signal precision in Christensen and Qin [2013], hence I only clarify the underlying mechanism concisely. The investors’
ex ante expected utilities are affected in two ways by changes of the strike price. First, changes in the strike price affects the gains to trade based on heterogeneously updated posterior beliefs and, thus, the growth in their certainty equivalents. Secondly, the strike price affects the ex ante equilibrium asset prices through the equilibrium interest rate and the equilibrium risk premium and, thus, affects the value of the investors’ individual endowments. The latter may affect the investors in different ways depending on their individual endowments relative to their equilibrium portfolio at \( t = 0 \). Therefore, the investors may not unanimously prefer the option with an intermediate strike price. Since a low equilibrium asset price is of course good if the investor wants to reduce the holding of the asset at \( t = 1 \), but it is bad if the investor wants to reduce the holding of the asset. However, with equilibrium endowments of the market security, the equilibrium prices are independent of the investors’ individual endowments, the investors do not trade at \( t = 0 \) given these endowments, all the investors can benefit from the option with an intermediate strike price.

\[ m_i(y) = -r\sigma^2_i x_i(y) + \frac{\partial f_i(x_i(y),\theta_i(y))}{\partial \theta_i} - p_i(y) = 0, \]  
(14a)  
\[ \frac{\partial f_i(x_i(y),\theta_i(y))}{\partial \theta_i} - \pi_i(y) = 0, \]  
(14b)  
and the market clearing condition for each asset, i.e.,

\[ \sum_{t=1}^{T} \theta_i(y) = 0, \quad \text{and} \quad \sum_{t=1}^{T} x_i(y) = Z. \]  
(14c)

Note the implicit solutions are functions of the public signal \( y \), the public signal precision, the strike price, and the posterior beliefs (hence the prior beliefs). In a two-investor model, there are six equations with six unknowns. The methods to further simplify the equations to the nonlinear equations with only two unknowns and the algorithms to solve for the equilibrium numerically are described in Appendix A.4.

4.2. Ex ante equilibrium at date \( t = 0 \)

I now determine the ex ante equilibrium price and the demands for assets functions at \( t = 0 \), taking the equilibrium at \( t = 1 \) characterized by the system of equations, i.e., Eqs. 14a, 14b, and 14c as given. From the perspective of \( t = 0 \), the date \( t = 2 \) consumption for the investor \( i \) is

\[ c_2 = \theta_i(y) \max(d - K, 0) + x_i(y)d + \gamma_{1i}(y). \]

The period-specific negative exponential utility, the investor \( i \) maximizes his certainty equivalent of \( t = 2 \) consumption conditional on the public information at \( t = 1 \), subject to his budget constraint, i.e.,

\[ \max_{\theta_i(y), y_{1i}(y), y_{2i}(y), \sigma^2_i} \text{CE}_2(\theta_i(y), x_i(y), y_{1i}(y), y_{2i}(y), \sigma^2_i) \]

\[ \text{s.t.} \quad \beta_{1j}(y) + p_j(y)x_j(y) + \pi_j(y)\theta_j(y) \leq \beta_{1j(y)} + p_j(y)x_j(y) + \pi_j(y)\theta_j(y), \]

where \( \text{CE}_2(\theta_i(y), x_i(y), y_{1i}(y), y_{2i}(y), \sigma^2_i) \) can be calculated by Lemma 1.

The ex post certainty equivalent of \( t = 2 \) consumption of the investor \( i \) conditional on the public information at \( t = 1 \) is

\[ \text{Ex post CE}_2(\theta_i(y), x_i(y), y_{1i}(y), y_{2i}(y), \sigma^2_i) \]

\[ = \gamma_{1i}(y) + m_i(y)x_i(y) - \frac{1}{2} \sigma_i^2 x_i(y) + f_i(x_i(y), \theta_i(y)), \]

where

\[ f_i(x_i(y), \theta_i(y)) = \frac{1}{2} \ln \left( \frac{\Phi_{\frac{K - (m_i(y)) - r\sigma^2_i x_i(y)}}{\sigma_i}} \right) \]

\[ + \exp \left( -r_0(i) \left[ K - m_i(y) - \frac{1}{2} \sigma_i^2 x_i(y) - r\sigma^2_i x_i(y) \right] \right) \]

\[ \times \left( 1 - \Phi_{\frac{K - (m_i(y)) - r\sigma^2_i x_i(y) + x_i(y))}{\sigma_i}} \right), \]

and \( \Phi(.) \) is the cumulative distribution function of the standard normal distribution.

I assume without loss of generality that \( \beta_1 = 1 \) since there is no consumption at \( t = 1 \).\(^5\) Solving the investor \( i \)'s optimal portfolio choice problem, the equilibrium portfolios and the prices are the implicit solutions of the system of equations which arise from the first-order conditions for the portfolio of each asset,

\[ m_i(y) = -r\sigma^2_i x_i(y) + \frac{\partial f_i(x_i(y),\theta_i(y))}{\partial \theta_i} - p_i(y) = 0, \]

(14a)  
\[ \frac{\partial f_i(x_i(y),\theta_i(y))}{\partial \theta_i} - \pi_i(y) = 0, \]

(14b)  
and the market clearing condition for each asset, i.e.,

\[ \sum_{t=1}^{T} \theta_i(y) = 0, \quad \text{and} \quad \sum_{t=1}^{T} x_i(y) = Z. \]  
(14c)

\(^5\) In the single-consumption-date ex-post equilibrium, the market prices of the underlying asset and the option are expressed relative to the \( t = 1 \) price of the zero-coupon bond, which acts as the numeraire in the economy. Hence, the riskless interest rate from \( t = 1 \) to \( t = 2 \) is equal to zero and has no substantial meaning in the single-consumption-date economy.
The two above panels are essentially the same plot. The left one shows a bird view of the interest rate surface which exhibits the whole surface, the right one views the surface from the front side. Both panels indicate that, the option and the public signal precision enable the investors to achieve improved side-betting based on their heterogeneously updated posterior beliefs. These gains to trade translate into increased certainty equivalents of $t = 2$ equilibrium consumption and, thus, a higher growth in their certainty equivalents and the ex ante equilibrium interest rate. With an intermediate public signal precision and the option with an intermediate strike price, the highest efficiency of side-betting is achieved, reflected by the unique maximum point of the ex ante equilibrium interest rate.

The left below panel in Fig. 7 shows the ex ante equilibrium interest rate surface from a special angle by which we can see, conditional on a certain level of the public signal precision, the ex ante equilibrium interest rate is bell-shaped with respect to the strike price. Since with very low or very high strike price, the option payment is similar to a stock or nothing, thus the option's ability to facilitate side-betting is limited. However, with an intermediate strike price, the option distinguishes from a fixed income asset and a stock. This kind of option can enhance the market allocation efficiency to the best degree and, thus, leads to the highest growth in certainty equivalents and the highest equilibrium interest rate. Moreover, we can see that when the public signal precision changes from 0 to 10, conditional on each fixed public signal precision, the extreme point of the strike price is always at around 0.009. This result indicates that the extreme point of the ex ante equilibrium interest rate surface is independent of the public signal precision. This finding matches the result in the benchmark model in Christensen and Qin (2013).

The right below panel in Fig. 7 shows the ex ante equilibrium interest rate surface from a special angle by which we can see, conditional on a certain level of the strike price, the ex ante equilibrium interest rate is bell-shaped with respect to the strike price. Furthermore, when the strike price changes from −0.04 to 0.01, conditional on each fixed level of the strike price, the extreme point of the public signal precision moves from around 3.5 to around 4.5. This result indicates that the extreme point of the ex ante equilibrium interest rate surface is a function of the strike price.

The intuition for the fact that the ex ante equilibrium interest rate is bell-shaped with respect to public signal precision conditional on a certain level of the strike price can be gained from Eq. (9). Actually, the equilibrium consumption growth is bell-shaped with respect to the public signal precision. Note when the public signal is highly informative, the investors obtain quite similar posterior beliefs on the dividend, while when the public signal is very uninformative, the investors’ posterior beliefs are similar to the priors. Therefore, in both cases, the investors do not speculate and rebalance their positions actively at $t = 1$. This leads to low side-betting gains at $t = 1$. On the other hand, with intermediate signal precision, the investors update their posterior beliefs differently and speculate actively at $t = 1$. Thus, they can obtain the highest side-betting benefits based on the additional trading at $t = 1$. These gains to trade translate directly into increased certainty equivalents of $t = 2$ consumption and, thus, a higher growth in their certainty equivalents and a higher equilibrium interest rate.

In order to see clearly the impact of the option and the public signal precision on growth in certainty equivalents (increase in allocation efficiency) and the equilibrium interest rate, I compare the equilibrium interest rate in this model with the equilibrium interest rates in some benchmark models in Christensen and Qin (2013) in Fig. 8.

As we can see from Fig. 8, the bottom line is the equilibrium interest rate around 0.036 in the benchmark model in which the

$$-E^{\text{CE}}\left[\exp\left(-\text{rCE}_2(x_0, \theta_0, \gamma_0, \hat{\theta}_0, \hat{p}_0, \pi_0, X_1(y), \tilde{\theta}_1(y), \tilde{p}_1(y), \pi_1(y)|F_1)\right)\right] = -\frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \int_{-\infty}^{\infty} \exp\left(-\text{rCE}_2(y)\right) \exp\left(-\frac{(\mathbf{m}_i - \mathbf{y})^2}{2(\sigma_1^2 + \sigma_2^2)}\right) dy.$$  

Note the expected utility is a function of the ex ante equilibrium portfolios, $\gamma_0, X_0, \theta_0, \pi_0$, the priors $\mathbf{m}_i, \sigma_1^2$, and the public signal precision $h_i$. The expected utility is a deterministic integral over the infinite interval, and thus can be computed at any desired level of accuracy using standard numerical integration methods (see more about the numerical integration methods in the Appendix A.2).

The investor’s $t = 0$ certainty equivalent equals to the date $t = 0$ consumption, i.e.,

$$\text{CE}_0 = c_0 = d_0 x_0 + (\beta_0 - \theta_0) \pi_0 + (\gamma_0 - \gamma_0) \hat{\theta}_0 + (\gamma_0 - \gamma_0) \pi_0.$$  

hence the investor’s decision problem at $t = 0$ can be stated as follows

$$\max_{\theta_0, \gamma_0, x_0} -\exp(-\text{rCE}_0) - \exp(-\delta) \exp(-\text{rCE}_2).$$

Solve the investor’s optimal portfolio choice problem, the ex ante equilibrium portfolios and the prices are implicit solutions of the system of equations which arise from the first-order conditions for the portfolio of each asset, and the market clearing condition for each asset. The equilibrium equations are similar to that in the benchmark model (Section 3.1.1) and, hence, are omitted here.

Note the implicit solutions are functions of the public signal precision, the strike price, and the prior beliefs. In a two-investor model, there are nine equations with nine unknowns. The ex ante equilibrium can be solved for numerically.

4.3. The impact of heterogeneous beliefs, strike price and public signal precision on ex ante asset pricing properties

When the investors update their beliefs with imperfect public signal precision, the public information system shows its influence in facilitating side-betting. Solving for the ex ante equilibrium, I find that with homogeneous prior variance, the investors do not trade in the option markets, and asset pricing is independent of the public information system. The intuition is similar to that in the benchmark model, i.e., trading in the underlying asset and the zero-coupon bond is already able to facilitate efficient side-betting. Furthermore, conditional on a certain level of the public signal precision and the strike price, the impact of the heterogeneous beliefs on the asset pricing properties is quite similar to that in the previous section. The figures and intuition of the panels are almost the same. Thus I do not repeat those results including pictures and explanations again. Therefore in this section, I only plot the asset pricing properties as functions of the public signal precision and the strike price conditional on a certain level of heterogeneity in beliefs.

4.3.1. Ex ante equilibrium interest rate

The ex ante equilibrium interest rate from $t = 0$ to $t = 2$ is defined as $\delta_t = \ln\beta_0$. By the investor’s decision problem in Eq. (17), and the expression of the investor’s certainty equivalents at $t = 0$ and $t = 2$, i.e., Eqs. (15) and (16), the relationship between the equilibrium interest rate and the growth in certainty equivalents in Eq. (9) still establishes.

The ex ante equilibrium interest rate is endogenized in the equilibrium as a function of the priors, the public signal precision and the strike price. To see the impact of the strike price and the public information quality, and plot the ex ante equilibrium interest rate as a function of the strike price and the public signal precision conditional on a certain level of heterogeneity in the prior variance in Fig. 7.
The equilibrium interest rates in different models are plotted as functions of the public signal precision conditional on a certain level of heterogeneity in the prior variance. The scale on the axis of the public signal precision is effective complete market. The scale on the axis of the public signal precision is beliefs with imperfect public signal. The highest horizontal line: interest rate in the maximum point of around 0.063: interest rate when investors can also update 0.059: interest rate when investors trade in an option. The above curve with a maximum point of around 0.052: interest rate when investors trade in only a zero-coupon bond and a stock. Note the increase of the equilibrium interest rate can be viewed as an analog of the gain from side-betting. Thus, this result illustrates the lowest efficiency of side-betting and benefit from the disagreement in the prior variance. When the investors can trade two rounds and update their beliefs with the imperfect public signal, the equilibrium interest rate is even higher showing by the curve with a maximum point of around 0.052. This fact indicates that the imperfect public signal does facilitate side-betting when the investors heterogeneously update their beliefs. When the investors speculate with an option and receiving no public signal or perfect public signal, the equilibrium interest rate is even higher and increases to around 0.059, and the equilibrium interest rate increases again when the investors update their beliefs with imperfect public signal. This result indicates that the option and the public signal are capable of facilitating side-betting, leading to a higher growth in certainty equivalent and thus a higher equilibrium interest rate. Finally, the highest equilibrium interest rate is obtained when the investors trade with the derivative which pays a square of the dividend in the effectively complete market. This fact illustrates the option and the imperfect public signal still cannot make the investors take full advantage of the difference in beliefs, and the efficiency of side-betting is not as high as that in the effectively complete market.

Fig. 8 also indicates that when investors cannot trade in options, the increase in interest rate due to speculations with imperfect public signal (around 1.6%) is much more significant than the increase in interest rate due to imperfect-signal-contingent trading when investors can already trade in options at \( t = 0 \) (around...
The intuition is straightforward: when investors can speculate in options at \( t = 0 \), they already accumulate relatively adequate needed convexity in payoff. As a result, there is less motivation to side-bet based on the imperfect public signal at \( t = 1 \), which leads to less trading signal-contingent gains and a lower increase in certainty equivalent.

Note first, Cuoco and He (1994) demonstrate the equilibrium with a stochastic weight in the representative agent utility is in general not Pareto efficient. Hence markets in Buraschi and Jiltsov (2006) are essentially incomplete, and they do not illustrate how much the option can help to improve the efficiency of side-betting. While, the model in this section shows that, the option and the imperfect public signal can facilitate side-betting, and increase the allocation efficiency of the markets to a better degree than that in some benchmark cases. Second, Buraschi and Jiltsov (2006) plot the asset pricing properties such as stock price and stock volatility as functions of the difference in the updated beliefs scaled by the signal volatility, hence the heterogeneous beliefs in their model carry the effect of the information system. However, I plot the asset pricing properties as functions of the different priors, thus the analysis in this paper probes the impact of the heterogeneity in beliefs and the public information quality separately.

4.3.2. Ex ante equilibrium risk premium

As in the previous section, I define the ex ante equilibrium risk premium on the risky underlying asset as

\[
\sigma^\text{exante} = \ln \overline{m} - \ln E[D],
\]

where the weighted mean \( \overline{m} \) and the risk-adjusted expected dividend, \( E[D] \), have the same definitions as in the previous section. To see the impact of the strike price and the public information quality, I plot the ex ante equilibrium risk premium as a function of the strike price and the public signal precision conditional on a certain level of heterogeneity in prior variance in Fig. 9.

The ex ante risk premium surface in the panel is just like the inverted risk-adjusted expected dividend. With a fixed level of heterogeneous beliefs, when the strike price is much higher or much lower than the mean, the ex ante risk premium on the risky underlying asset converges to that in a benchmark case in which the investors trade only in a stock and a zero-coupon bond, which is independent of the public signal precision. However, with an intermediate strike price, the impact of the public signal precision on the ex ante equilibrium risk premium is nontrivial. This fact suggests that even though the strike price of option affects the ex ante risk premium on the risky underlying asset regardless of the presence of the public signal, the informativeness of the public information system affects the ex ante risk premium only via its relationship with the option.

Compare to the benchmark model in Christensen and Qin (2013), in which the ex ante risk premium is independent of the public signal precision, the ex ante risk premium in this paper is not aligned with the increase in investors’ certainty equivalent and thus welfare (as a function of the signal precision). The equilibrium interest rate and the investor welfare, however, are still perfectly aligned. This fact has an implication that it may be wise to be cautious in making policy statements about, for example, financial reporting regulation, based on empirical measures of equity premia (which are hard to measure reliably anyway).

Similar to the benchmark model in the previous sections, the intuition can be gained from the fact that the risk-adjusted expected dividend is the expected dividend plus the covariance of the marginal utility of consumption and the dividend scaled by

\[
\frac{h}{0.05}\%.
\]

The intuition is straightforward: when investors can speculate in options at \( t = 0 \), they already accumulate relatively adequate needed convexity in payoff. As a result, there is less motivation to side-bet based on the imperfect public signal at \( t = 1 \), which leads to less trading signal-contingent gains and a lower increase in certainty equivalent.

Note first, Cuoco and He (1994) demonstrate the equilibrium with a stochastic weight in the representative agent utility is in general not Pareto efficient. Hence markets in Buraschi and Jiltsov (2006) are essentially incomplete, and they do not illustrate how much the option can help to improve the efficiency of side-betting. While, the model in this section shows that, the option and the imperfect public signal can facilitate side-betting, and increase the allocation efficiency of the markets to a better degree than that in some benchmark cases. Second, Buraschi and Jiltsov (2006) plot the asset pricing properties such as stock price and stock volatility as functions of the difference in the updated beliefs scaled by the signal volatility, hence the heterogeneous beliefs in their model carry the effect of the information system. However, I plot the asset pricing properties as functions of the different priors, thus the analysis in this paper probes the impact of the heterogeneity in beliefs and the public information quality separately.

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As in the previous section, I define the ex ante equilibrium risk premium on the risky underlying asset as

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\]

where the weighted mean \( \overline{m} \) and the risk-adjusted expected dividend, \( E[D] \), have the same definitions as in the previous section. To see the impact of the strike price and the public information quality, I plot the ex ante equilibrium risk premium as a function of the strike price and the public signal precision conditional on a certain level of heterogeneity in prior variance in Fig. 9.

The ex ante risk premium surface in the panel is just like the inverted risk-adjusted expected dividend. With a fixed level of heterogeneous beliefs, when the strike price is much higher or much lower than the mean, the ex ante risk premium on the risky underlying asset converges to that in a benchmark case in which the investors trade only in a stock and a zero-coupon bond, which is independent of the public signal precision. However, with an intermediate strike price, the impact of the public signal precision on the ex ante equilibrium risk premium is nontrivial. This fact suggests that even though the strike price of option affects the ex ante risk premium on the risky underlying asset regardless of the presence of the public signal, the informativeness of the public information system affects the ex ante risk premium only via its relationship with the option.

Compare to the benchmark model in Christensen and Qin (2013), in which the ex ante risk premium is independent of the public signal precision, the ex ante risk premium in this paper is not aligned with the increase in investors’ certainty equivalent and thus welfare (as a function of the signal precision). The equilibrium interest rate and the investor welfare, however, are still perfectly aligned. This fact has an implication that it may be wise to be cautious in making policy statements about, for example, financial reporting regulation, based on empirical measures of equity premia (which are hard to measure reliably anyway).

Similar to the benchmark model in the previous sections, the intuition can be gained from the fact that the risk-adjusted expected dividend is the expected dividend plus the covariance of the marginal utility of consumption and the dividend scaled by
the expected marginal utility of consumption. Actually, the surface of the covariance of the marginal utility of consumption and the dividend is similar to the inverted ex ante risk premium. Furthermore, the marginal utility of consumption is determined by the investors' portfolios. I plot the portfolios in financial markets as functions of the strike price and the public signal precision conditional on a certain level of heterogeneity in prior variance in Fig. 10.

Conditional on a certain level of public signal precision, for the investor with lower prior precision, the portfolio in the option market is U-shaped. The intuition is similar to that in the benchmark models. Conditional on a certain level of strike price, the ex ante equilibrium demand of the option increases with the public signal precision. The intuition is that when the public signal precision increases, the variance of the posterior beliefs of the dividend decreases and, thus, the convexity of the option payoff at \( t = 1 \) decreases. To gain more payoff convexity, the investors need to increase the position in the option market at \( t = 0 \). This result correspondingly increases the short positions in the underlying asset and the zero-coupon bond.

4.3.3. Ex ante equilibrium price of risky underlying asset

The underlying asset price is endogenized in the equilibrium as a function of the priors and the strike price. I plot the ex ante equilibrium underlying asset price as a function of the strike price and public signal precision conditional on a certain level of heterogeneity in prior variance in Fig. 11.

The ex ante equilibrium underlying asset price is jointly determined by the public signal precision and the strike price through their effects on the ex ante equilibrium riskless discount factor and the risk-adjusted expected dividend. Similar to the results in the previous section, Fig. 11 is a balanced result of Figs. 7 and 9. Moreover, in the benchmark model in which the investors update beliefs with imperfect public information and speculate in a stock and a zero-coupon bond, the lowest stock price is attained at an intermediate public signal precision. Differently, after incorporating the option, the lowest price of the underlying asset is attained when there is no or perfect public signal, and at an intermediate strike price.

4.3.4. Ex ante equilibrium call option price

The European call option price is endogenized in the equilibrium as a function of the priors and the strike price. I plot the ex ante equilibrium call option price as a function of strike price and public signal precision conditional on a certain level of heterogeneity in prior variance in Fig. 12.

The ex ante equilibrium call option price is jointly determined by the public signal precision and the strike price through their effects on the ex ante equilibrium riskless discount factor and the risk-adjusted expected option payment. We can see from Fig. 12 that the impact of the public signal precision on the ex ante equilibrium call option price is limited and hardly visible from the plot. The impact of the strike price is quite similar to that in the previous section: The value of the option decreases with the strike price, and converges to zero when the strike price increases to a very high level.

4.3.5. Ex ante equilibrium expected utilities

As mentioned before, the impact of the option and the public information system on individual utility depends on their individual endowments relative to their equilibrium portfolio at

Fig. 11. The ex ante equilibrium underlying asset price. The ex ante equilibrium underlying asset price is plotted as a function of the strike price and the public signal precision conditional on certain level of the heterogeneity in prior variance. The scale on the axis of the public signal precision is \( x = \ln(1 + 0.005h) \).
Appendix B.

the figures, but the derivations of the models are provided in price, and the equilibrium risk premium. Hence, I do not repeat asset pricing properties such as the equilibrium underlying asset and is hardly visible when plotting it. The same happens to other straddles have potential to exert an influence on the incomplete can take a short position to get a concave payoff profile. Thus the can use Gamma strategy by trading straddle to achieve the convex-

The investors, who think the variance of the dividend is high, still dividend in the terminal date and effectively complete the market. 

4.3.6. Case when investors update beliefs and speculate with straddles

As mentioned in Christensen and Qin (2013), in an incomplete market setting with heterogeneous beliefs about the risks on the underlying asset, straddles, i.e., long positions in both a call and a put option with the same strike price can play an important role to facilitate side-betting. Since straddle can resemble the payoff profile of the dividend derivative which pays off the square of dividend in the terminal date and effectively complete the market. The investors, who think the variance of the dividend is high, still can use Gamma strategy by trading straddle to achieve the convex-

ity of its payoff while the investors, who think the variance is low, can take a short position to get a concave payoff profile. Thus the straddles have potential to exert an influence on the incomplete market settings with heterogeneous beliefs about the risks on the underlying assets. I derive the models when the investors speculate with straddle, receiving no public signal, imperfect public signal or perfect public signal, and solve for the equilibrium numerically. I find that the put option is able to facilitate side-betting, increase the growth in certainty equivalents and equilibrium interest rate. However, compare to the change of the level of equi-

librium interest rate, the increase of the equilibrium interest rate which due to the incorporation of the put option is very small, and is hardly visible when plotting it. The same happens to other asset pricing properties such as the equilibrium underlying asset price, and the equilibrium risk premium. Hence, I do not repeat the figures, but the derivations of the models are provided in Appendix B.

5. Concluding remarks

Under the assumptions of time-additive negative exponential investors and normally distributed dividend, this paper studies the relationship between the heterogeneous beliefs, the strike price and the public signal precision in the option markets. I demonstrat that the option market is important to facilitate financial market allocation efficiency when the investors hold heterogeneous beliefs. Moreover, the option market also makes the role of public signal precision more complicated and sophisticated. The public signal precision affects the ex ante risk premium on the risky underlying asset via this relationship with the option. When the investors do not speculate with the option, the public signal precision is independent of the ex ante risk premium. Com-

bine with the right intermediate public signal precision and the right intermediate strike price, i.e., the right type of option, the highest allocation efficiency of the market can be attained, reflected by the unique maximum point of the increase in certainty equivalent and the ex ante equilibrium interest rate surface.

In the effectively complete market, an additional asset of the right type eliminates the need for dynamic trading based on public signals and enables the investors to take full advantage of the heterogeneity in beliefs. Compare to that extreme case, I use option to facilitate side-betting and show that in this intermediate case, the public signal still has room to show its potential to facilitate im-

proved dynamic trading opportunities and yields a more efficient market structure.

More trading rounds based on a sequence of the public signals may lead to more efficient side-betting based on the heterogeneous beliefs. To solve for such a multi-period ex ante equilibrium, the algorithms in Dumas and Lyasoff (2012) which solve for equi-

librium recursively on an event tree can be helpful. Furthermore, as another extreme case, continuous trading may dynamically effect-

tively complete the financial market with heterogeneous beliefs under the assumptions of two consumption dates (Qin (2013)). In the dynamically complete market, the payoff of option can be replicated by trading continuously in only a riskless bond and a stock and, thus, the option becomes redundant. Therefore, the mar-

ket has to be effectively incomplete (statically or dynamically) to make the option valuable for enhancing allocation efficiency.

With heterogeneous beliefs, introducing an intermediate consum-

ption date at \( t = 1 \) will bring the need of a short term option (mature at \( t = 1 \)), and both the short term option and the long term option (mature at \( t = 2 \)) should play a role to improve allocation efficiency of the markets. Correspondingly, assume that the under-

lying risky asset also pays an aggregate dividend at \( t = 1 \) for inves-

tors to consume. Heterogeneity in beliefs on the dividend paid at both \( t = 1 \) and \( t = 2 \) will create the demand for speculative trading in both the short term option and the long term option, and the trading gains are expected to increase the allocation efficiency reflected by the changes in the riskless interest rates: Both of the riskless interest rates from \( t = 0 \) to \( t = 1 \) and from \( t = 1 \) to \( t = 2 \) should still be increasing functions of growth in certainty equivalents. After introducing options at \( t = 1 \) and \( t = 2 \), both of the risk-

less interest rates are expected to increase comparing to that without options. Moreover, with the intermediate consumption date, the spot interest rate from \( t = 1 \) to \( t = 2 \) reflects the intertem-

poral substitution of consumption from \( t = 1 \) to \( t = 2 \) and depends on the stochastic signal (in contrast to riskless rate in the single-consumption-date ex-post equilibrium which is equal to zero and has no substantial meaning). This also gives rise to the signal-con-

tent growth in certainty equivalent form \( t = 1 \) to \( t = 2 \). One the other hand, in an incomplete market with the intermediate consum-

ption date, information released a \( t = 1 \) should enable investors to better allocate intertemporal consumption. See chapter 7 in Christensen and Feltham (2003) for a further discussion on the role of information for better implementing intertemporal consump-

tion streams.

Another direction to extend this paper is to consider a CRRA-

lognormal specification. The CARA-normal setting in this paper does buy some analytical tractability. However, more realistic pre-

ference and dividend distribution may come up with more empiri-

cally testable implications. Besides, the model can be generalized to a heterogeneous risk-aversion case. With identical prior
precision, the least risk-averse investors are expected to tolerate risk and short options, while opposite speculative positions are taken by the most risk-averse investors.

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Appendix A. Proofs and Algorithms

A.1. Proof of Lemma 1

Since the investor i’s consumption at t = 2 is
\[ c_{2i} = \theta_{i0} \max(d - K, 0) + x_{0d} d + \gamma_{i0}, \]
and let the probability density function of a normal distribution 
\[ N(\mu, \sigma^2) \]
be
\[ f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \]

Hence, the t = 2 expected utility for the investor i is
\[ -\mathbb{E}^i[\exp[-r_0 \theta_{i0} \max(d - K, 0) + x_{0d} d + \gamma_{i0}]] \]
\[ = -\int_{-\infty}^{\infty} \exp[-r_0 \theta_{i0} \max(x - K, 0) + x_{0d} x + \gamma_{i0}] f(x; m_0, \sigma_0^2) dx \]
\[ = -\int_{-\infty}^{K} \exp[-r_0 \theta_{i0} x + \gamma_{i0}] f(x; m_0, \sigma_0^2) dx \]
\[ -\int_{K}^{\infty} \exp[-r_0 \theta_{i0} x + \theta_{i0} K + \gamma_{i0}] f(x; m_0, \sigma_0^2) dx \]
\[ = -\exp\left[-r_0 \gamma_{i0} + m_0 x_{0d} - \frac{1}{2} r_0 \sigma_0^2 x_{0d}^2\right] \times \int_{-\infty}^{K} f(x; m_0 - r_0 \sigma_0^2 x, \sigma_0^2) dx \]
\[ -\exp\left[-r_0 \gamma_{i0} - \theta_{i0} K + m_0 (\theta_{i0} + x_{0d}) - \frac{1}{2} r_0^2 \sigma_0^2 (\theta_{i0} + x_{0d})^2\right] \times \int_{K}^{\infty} f(x; m_0 - r_0 \sigma_0^2 x, \sigma_0^2) dx \]
\[ = -\exp\left[-r_0 \gamma_{i0} + m_0 x_{0d} - \frac{1}{2} r_0 \sigma_0^2 x_{0d}^2\right] \times \int_{-\infty}^{K} \phi(x) dx \]
\[ -\exp\left[-r_0 \gamma_{i0} - \theta_{i0} K + m_0 (\theta_{i0} + x_{0d}) - \frac{1}{2} r_0^2 \sigma_0^2 (\theta_{i0} + x_{0d})^2\right] \times \int_{K}^{\infty} \phi(x) dx \]
\[ = -\exp\left[-r_0 \gamma_{i0} + m_0 x_{0d} - \frac{1}{2} r_0 \sigma_0^2 x_{0d}^2\right] \times \phi(K - (m_0 - r_0 \sigma_0^2 x_{0d}))/\sigma_0 \]
\[ -\exp\left[-r_0 \gamma_{i0} - \theta_{i0} K + m_0 (\theta_{i0} + x_{0d}) - \frac{1}{2} r_0^2 \sigma_0^2 (\theta_{i0} + x_{0d})^2\right] \times \left(1 - \phi(K - (m_0 - r_0 \sigma_0^2 x_{0d}))/\sigma_0 \right) \]
\[ \times \left(1 - \phi(K - (m_0 - r_0 \sigma_0^2 x_{0d}))/\sigma_0 \right). \]

Therefore, the certainty equivalent of t = 2 consumption for the investor i is
\[ CE_2 = -\frac{1}{r} \ln\left(-\mathbb{E}^i[\exp[-r_0 \theta_{i0} \max(d - K, 0) + x_{0d} d + \gamma_{i0}]])\right) \]
\[ = \gamma_{i0} + m_0 x_{0d} - \frac{1}{2} \sigma_0^2 x_{0d}^2 - \frac{1}{r} \ln\left(\frac{\phi(K - (m_0 - r_0 \sigma_0^2 x_{0d}))/\sigma_0}{\phi(K - (m_0 - r_0 \sigma_0^2 x_{0d}))/\sigma_0} \right) \]
\[ \times \left(1 - \phi(K - (m_0 - r_0 \sigma_0^2 x_{0d}))/\sigma_0 \right) \]
\[ \times \left(1 - \phi(K - (m_0 - r_0 \sigma_0^2 x_{0d}))/\sigma_0 \right). \]

This completes the proof. □

A.2. Derivations of the first derivatives in equilibrium equations and equilibrium price of underlying asset

A.2.1. First derivatives in equilibrium equations

This section provides the calculation of the first derivatives in first-order condition. Let
\[ a_i = \frac{K - (m_0 - r_0 \sigma_0^2 x_{0d})}{\sigma_0}, \]
\[ b_i = \frac{K - (m_0 - r_0 \sigma_0^2 (\theta_{i0} + x_{0d}))}{\sigma_0}, \]
\[ c_i = \exp\left[-r_0 \theta_{i0} \left(-K + m_0 - \frac{1}{2} r_0 \sigma_0^2 (\theta_{i0} + x_{0d})\right)\right], \]
then,
\[ f(x_{0d}, \theta_{0i}) = -\frac{1}{r} \ln(\psi_i). \] (A1)

where \( \psi_i = \Phi(a_i) + c_i \times (1 - \Phi(b_i)) \), and \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal distribution. Hence,
\[ \frac{\partial f(x_{0d}, \theta_{0i})}{\partial x_{0d}} = \frac{\phi(a_i)\sigma_0 - \phi(b_i)\sigma_0 c_i + (1 - \Phi(b_i)) c_i r_0 \sigma_0^2}{\Phi(a_i) + (1 - \Phi(b_i)) c_i}, \]
\[ \frac{\partial f(x_{0d}, \theta_{0i})}{\partial \theta_{0i}} = \frac{\phi(b_i)\sigma_0 c_i + (1 - \Phi(b_i)) c_i (m_0 - K - r_0^2 \sigma_0^2 (\theta_{i0} + x_{0d}))}{\Phi(a_i) + (1 - \Phi(b_i)) c_i}, \]
where \( \phi(x) \) denotes the standard normal probability density function, hence
\[ \phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right). \]

A.2.2. Equilibrium price of underlying asset

Now I derive the price of the underlying asset as the product of equilibrium riskless discount factor and the risk-adjusted expected dividend. The investor i’s first-order condition gives
\[ \frac{\partial}{\partial x_{0d}} (\exp(-rCE_0) - \exp(-\lambda) \exp(-rCE_2)) = 0 \iff \]
\[ -r \exp(-rCE_0) p_0 \frac{\partial}{\partial x_{0d}} (\exp(-\lambda) \exp(-rCE_2)) = 0 \iff \]
\[ r p_0 \exp(-rCE_0) = -\frac{\partial}{\partial x_{0d}} (\exp(-\lambda) \exp(-rCE_2)) \iff \]
\[ \ln(r p_0) - rCE_0 = \ln\left(-\frac{\partial}{\partial x_{0d}} (\exp(-\lambda) \exp(-rCE_2))\right)\]
and sum over i, yields
\[ I \ln(p_i) - r(CE_0 + CE_0) \]
\[ = \sum_{i=1}^{n} \left( -\frac{\partial(\exp(-\rho_0 CE_0))}{\partial \rho_0} \right) \]
\[ \ln(p_0) = \frac{1}{r} \int \sum_{i=1}^{n} \left( -\frac{\partial(\exp(-\rho_0 CE_0))}{\partial \rho_0} \right) \exp\left( \frac{1}{r} \sum_{i=1}^{n} CE_0 \right) \]
\[ = \frac{1}{r} \left( \prod_{i=1}^{n} \left( m_0 - \sigma_0^2 \frac{\partial f_i(\theta_0, \theta_0)}{\partial \theta_0} \right) \right) \]
\[ \times \exp \left( -\frac{1}{r} \sum_{i=1}^{n} CE_0 \right) \exp \left( \frac{1}{r} \sum_{i=1}^{n} CE_0 \right) \]
\[ = \frac{1}{r} \left( \prod_{i=1}^{n} \left( m_0 - \sigma_0^2 \frac{\partial f_i(\theta_0, \theta_0)}{\partial \theta_0} \right) \right) \]
\[ \times \exp \left( -\frac{1}{r} \sum_{i=1}^{n} CE_0 \right) \exp \left( \frac{1}{r} \sum_{i=1}^{n} CE_0 \right) \]
\[ = \beta_0 \exp^G[d], \text{ where } E^G[d] = \left( \prod_{i=1}^{n} \left( m_0 - \sigma_0^2 \frac{\partial f_i(\theta_0, \theta_0)}{\partial \theta_0} \right) \right) \]
\[ \text{A.3. Algorithm to compute the ex ante expected utility at } t = 2 \]

The expectation of the utility for the investor i at t = 2 can be expressed as
\[ -E_i[\exp(-rCE_0(\theta_0, \theta_0, \gamma_0, \beta_0, \rho_0, \sigma_0, \alpha_i(y), \theta_1(y), p_1(y), \pi_1(y))|\mathcal{F}_1)] \]
\[ = \frac{1}{2\pi(\sigma_i^2 + \sigma_i^2)} \int_{-\infty}^{\infty} \exp[-rCE_0(y)] \exp\left( -\frac{(m_i - y)^2}{2(\sigma_i^2 + \sigma_i^2)} \right) dy. \]

The integral over infinite interval can then be evaluated by ordinary integration methods:
\[ \int_{-\infty}^{\infty} f(t) \, dt \approx \frac{2H}{n} \left( f(-H) + f(H) + \sum_{k=1}^{n-1} \left( a + kH/2 \right) \right), \]

when \( H \to \infty \) and \( n \to \infty \) \( \int_{-\infty}^{\infty} f(t) \, dt \approx \int_{-\infty}^{\infty} f(t) \, dt \).

Note that for each public signal y, a corresponding ex post equilibrium can be solved for and yield the ex post equilibrium portfolios \( \theta_1, \theta_1, \gamma_1, \beta_1, \) and the ex post prices, \( p_1(y) \) and \( \pi_1(y) \). Sum up \( \frac{1}{\sqrt{2\pi(\sigma_i^2 + \sigma_i^2)}} \exp[-rCE_0(y)] \exp\left( -\frac{(m_i - y)^2}{2(\sigma_i^2 + \sigma_i^2)} \right) \) over all the public signal y yields the expectation.

Also note that the public signal y ~ \( N(m_i, \sigma_i^2 + \sigma_i^2) \), hence, let \( u = \frac{y - m_i}{\sqrt{\sigma_i^2 + \sigma_i^2}} \) then \( u \sim N(0,1) \). This transformation can make the probability density of the normal distribution be independent of the public signal precision, and bring some conveniences when programming.

A.4. Algorithm to solve nonlinear equations in ex post and ex ante equilibriums

A.4.1. Nonlinear least-squares algorithms

Denote the system of the nonlinear equations as \( G(x) \), where x is a vector and \( G(x) \) is a function that returns a vector value. I use nonlinear least-squares algorithms to find x that is a local minimizer to a function that is a sum of squares, i.e.,
\[ \min_x ||G(x)||^2 = \min_x \sum_i^N \alpha_i^2(x). \]

Specifically, in this paper, I have some manipulations corresponding to the properties of the equations. First, I set a larger weight \( z_i \) to the first-order condition equations than the market clearing equations, and the weights depend on the value of the parameters. This is because the first-order condition equations are more nonlinear and difficult to converge properly when iterating. Second, since the coefficients of the portfolio unknowns are very small when the public signal precision is comparatively low, the value of the equations are not sensitive to the change of the value of the portfolio and thus the portfolio unknowns are more difficult to converge. To solve this problem, first time the portfolio unknowns by a large constant, \( N \), to make the value of the portfolios more influential. After solving the equations, I time the same constant, \( N \), again to the solutions of the portfolios and obtain better converged value of portfolios. The value of the constant, \( N \), depends on the value of parameters. With the parameters used in this paper, \( N \in [1, 10,000] \), can be chosen regarding difference equilibriums.

Regarding the nonlinear least-squares algorithms, I first use the Gauss–Newton method. The Gauss–Newton method is more efficient when searching for a good starting point, but not robust to converge to a solution. If the solution is not well converged, then I turn to the Levenberg–Marquardt method. The robustness of the Levenberg–Marquardt method compensates for its occasional poor efficiency. Above algorithms can be implemented by employing the function fmin in MATLAB. Set parameters ‘LargeScale’, ‘off’, ‘NonEqnAlgorithm’, ‘on’ when using the Levenberg–Marquardt method. Robustness measures are included in the method.

For the iteration formulas refer to e.g.,

The residuals can be very small when solving the equations with the parameters used in this paper. It is possible to obtain the value of \( ||G(x)||^2 < 10^{-50}, ||G(x)|| < 10^{-10} \), for almost all the ex post and ex ante equilibriums. Hence the solutions converge properly.

A.4.2. Reduce unknowns in nonlinear equations

To reduce the computing time, in a two-investor model, I rearrange the equilibrium equations at \( t = 1 \) and \( t = 0 \). Note the equations are partly nonlinear, for instance, the market clearing condition is linear, and the asset price in the first order condition is also linear. Hence in the ex post equilibriums (see Section 3.2.1), I can rewrite the equilibrium equations at \( t = 1 \) into nonlinear equations with only two unknowns, i.e., the portfolios in the risky asset markets of one investor, \( x_1, \theta_1 \),
\[ m_1(y) - r\sigma_1^2 x_1 + \frac{\partial f_i(x_1, \theta_1)}{\partial \theta_1} = m_1 - r\sigma_1^2 (Z - x_1) + \frac{\partial f_i(z - x_1, \theta_1)}{\partial \theta_1}, \]
\[ \frac{\partial f_i(x_1, \theta_1)}{\partial x_1} = \frac{\partial f_i(z - x_1, \theta_1)}{\partial \theta_1}. \]

After solving the two unknown equations and obtaining the value of the portfolios, one can use the following linear relationships to calculate the values of the other unknowns:
\[ x_1 = Z - x_1, \]
\[ \theta_1 = -\theta_1, \]
\[ p_1 = m_1(y) - r\sigma_1^2 x_1 + \frac{\partial f_i(x_1, \theta_1)}{\partial \theta_1}, \]
\[ \pi_1 = \frac{\partial f_i(x_1, \theta_1)}{\partial x_1}. \]
Similarly, the nonlinear equations at \( t = 0 \) can be written into nonlinear equations with only three unknowns, i.e., the portfolios in the risky and riskless asset markets of one investor. With fewer unknowns in the nonlinear equations, the computing time reduces greatly, however, the accuracy of the solutions may also be affected slightly.

A.4.3. Set threshold

There are some other practical issues worth mentioning. When I solve for the equilibrium at \( t = 1 \), I set the value of the public signal \( y \) vary widely, since \( y \) is normally distributed. For some value of the public signal \( y \) and the public signal precision \( h_y \), the value of the cumulative distribution function of the standard normal distribution, \( \Phi(\cdot) \) can be very close to 1 or 0 during the iteration. The computer may have difficulty to tell if it is 1/0 or a number very close to 1/0 due to the limited computing precision. If this situation happens, the solutions may not converge properly. To solve this problem, I set a threshold \( q = 10^{-10} \), when \( \Phi(\cdot) < q \), let \( \Phi(\cdot) = 0 \), then according to the expression of \( \partial f_i(x_{1i}(y), 0_{1i}(y)) \) in Eq. (A1), yields

\[
\frac{1}{2} \ln c_i = \frac{1}{2} \ln c_i \left[-K + m_{0i} \right] - \frac{1}{2} \sigma^2_{0i} \theta_{1i}(y) - \frac{1}{2} \sigma^2_{0i} x_{1i}(y) \right].
\]

When \( 1 - \Phi(\cdot) < q \), let \( \Phi(\cdot) = 1 \), and thus \( f_i(x_{1i}(y), 0_{1i}(y)) = 0 \). Hence, at \( t = 1 \), the function \( f_i(x_{1i}(y), 0_{1i}(y)) \) follows different kinds of form conditional on the value of \( \Phi(\cdot) \) in the iteration. Regarding each form, new equations need to be solved. This classification can enhance the accuracy of the solution and reduce computing time considerably.

Appendix B. Derivation of the model when investors update beliefs and speculate with straddles

As indicated in the introduction, in incomplete market settings with heterogeneous beliefs about the risks on the underlying asset, the straddles, i.e., long positions in both a call and a put option with the same strike price can play an important role to facilitate side-betting.

Based on the model in Sections 3.1 and 4.1, I now add an additional European put option in zero net supply with payoff, max \((K - d, 0)\), at \( t = 2 \), and the prices \( v_{10} \) and \( v_{01}(y) \) at \( t = 0 \) and \( t = 1 \), respectively. The investors have endowments \( z_i \) of this asset at \( t = 0 \), and let \( z_{it} \) be its the units of the put option held after trading at date \( t \) satisfying the market clearing conditions

\[
\sum_{i=1}^{t} z_{it} = \sum_{i=1}^{t} z_{it} = 0, t = 0, 1.
\]

B.1. Benchmark case when investors speculate with straddles receiving no or perfect public signal

I first derive a benchmark model in which the investors receive no public signal or perfect public signal at \( t = 1 \), hence the model is equivalent to a single period model.

B.1.1. Equilibrium in a single period economy

From the perspective of \( t = 0 \), the date \( t = 2 \) consumption for the investor \( i \) is

\[
c_2 = \theta_0 \max(d - K, 0) + z_{10} \max(K - d, 0) + x_{0d} + \gamma_{0}.
\]

Given the period-specific negative exponential utility, the investor \( i \)'s \( t = 0 \) certainty equivalent of \( t = 2 \) consumption, receiving no or perfect public information at \( t = 1 \), \( CE_2(z_{0i}, \theta_{0i}, x_{0d}, \gamma_{0}) \) can be calculated according to Lemma 2.

**Lemma 2.** Assume the investors receive no or perfect public information at \( t = 1 \), given the portfolios in the underlying asset markets, the call option markets, the put option markets, and the zero-coupon bond markets at \( t = 0 \), the investor \( i \)'s certainty equivalent of \( t = 2 \) consumption is

\[
CE_2(z_{0i}, \theta_{0i}, x_{0d}, \gamma_{0}) = \gamma_{0i} + m_{0i}x_{0d} - \frac{1}{2} \sigma^2_{0i} + g_i(x_{0i}, \theta_{0i}, \gamma_{0i}).
\]

where

\[
g_i(x_{0i}, \theta_{0i}, \gamma_{0i}) = -\frac{1}{2} \ln \left[ \exp \left[ -r_{0i} \left( K - m_{0i} - \frac{1}{2} \sigma^2_{0i} + \gamma_{0i} \right) \right] \right]
\]

\[
\times \Phi \left( \frac{K - (m_{0i} - \sigma^2_{0i}(x_{0i} - \gamma_{0i}))}{\sigma_{0i}} \right)
\]

\[
+ \exp \left[ -r_{0i} \left( K - m_{0i} - \frac{1}{2} \sigma^2_{0i}(x_{0i} + \gamma_{0i}) \right) \right]
\]

\[
\times \left( 1 - \Phi \left( \frac{K - (m_{0i} - \sigma^2_{0i}(x_{0i} + \gamma_{0i}))}{\sigma_{0i}} \right) \right),
\]

and \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal distribution.

**Proof.** See Appendix B.3. The investors' \( t = 0 \) certainty equivalent equals to the date \( t = 0 \) consumption, i.e.,

\[
CE_0 = c_0
\]

\[
= d_0 z_i + (\bar{z}_{10} - \bar{z}_{0i}) + (\bar{t}_0 - \bar{t}_{0i})\pi_0 + (\bar{y}_{10} - \bar{y}_{0i})d_0 + (\bar{z}_i - x_{0i})p_0
\]

the investor \( i \)'s decision problem at \( t = 0 \) can be stated as follows

\[
\max_{x_{it}, z_{it}, y_{it}, X_{0i}} \bar{U}_i(z_{0i}, \theta_{0i}, y_{1i}, X_{0i}) \text{ where } \bar{U}_i
\]

\[
\equiv - \exp(-cE_{E_0}) - \exp(-\delta) \exp(-cE_{E_2}).
\]

To solve for the equilibrium, I first solve the investor \( i \)'s optimal portfolio choice problem,

\[
\frac{\partial U_i}{\partial x_{it}} = 0, \quad \frac{\partial U_i}{\partial z_{it}} = 0, \quad \frac{\partial U_i}{\partial y_{it}} = 0, \quad \text{ and } \frac{\partial U_i}{\partial \pi_{0i}} = 0.
\]

Note more details of the calculation of the first derivatives in the first-order condition are provided in Appendix B.4. Then the equilibrium portfolios and the prices are the implicit solutions of the system of equations which arise from the first-order conditions for the portfolio of each asset, and the market clearing condition for each asset, i.e.,

\[
\sum_{i=1}^{t} \pi_{it} = 0, \quad \sum_{i=1}^{t} y_{it} = 0, \quad \sum_{i=1}^{t} z_{it} = 0, \quad \text{and } \sum_{i=1}^{t} x_{it} = Z.
\]

B.2. Case when investors update beliefs with imperfect public signal and speculate with straddles

I now derive a dynamic trading model in which the investors trade in both call and put option markets and update their beliefs at \( t = 1 \) with imperfect public information.

B.2.1. Equilibrium prices at \( t = 1 \)

I first derive the ex post equilibrium at \( t = 2 \) conditional on the posterior beliefs. From the perspective of \( t = 1 \), date \( t = 2 \) consumption for the investor \( i \) is

\[
c_2 = \theta_1(y) \max(d - K, 0) + z_{1i}(y) \max(K - d, 0) + x_{11}(y) d + \gamma_{11}(y).
\]
Given the period-specific negative exponential utility, the investor \( i \) maximizes his certainty equivalent of \( t = 2 \) consumption (conditional on his public information at \( t = 1 \), subject to his budget constraint, i.e.,

\[
\max_{x_i(t)} \text{CE}_2(\tilde{c}_i(y), \theta_i(t), x_i(t), \gamma_i(y)(\text{conditional on the public information at } t = 1), \text{subject to his budget constraint, i.e.}, \text{where } \text{CE}_2(\tilde{c}_i(y), \theta_i(t), x_i(t), \gamma_i(y)(\text{conditional on the public information at } t = 1), \text{subject to his budget constraint, i.e.}) \]
\]

\[
\text{subject to } \frac{\partial \gamma_i(y)}{\partial x_i(t)} = 0, \quad \frac{\partial \theta_i(t)}{\partial x_i(t)} = 0, \quad \text{and the market clearing condition for each asset, i.e.,} \]

\[
\sum_{i=1}^{n} \tilde{c}_i(y) = 0, \quad \sum_{i=1}^{n} \theta_i(t) = 0, \quad \text{and } \sum_{i=1}^{n} x_i(t) = Z. \quad \text{(B4)}
\]

**B.3. Proof of Lemma 2**

Since the investor \( i \)'s consumption at \( t = 2 \)

\[
c_d = \theta_0 \text{max}(K - d, 0) + \theta_0 \text{max}(K - d, 0) + \lambda x_d + \gamma_d:
\]

hence the \( t = 2 \) expected utility for the investor \( i \) is

\[
E^\theta[\text{CE}_2(\tilde{c}_i(y), \theta_i(t), x_i(t), \gamma_i(y)(\text{conditional on the public information at } t = 1), \text{subject to his budget constraint, i.e.})]:
\]

\[
E^\theta[\text{CE}_2(\tilde{c}_i(y), \theta_i(t), x_i(t), \gamma_i(y)(\text{conditional on the public information at } t = 1), \text{subject to his budget constraint, i.e.})]
\]

\[
\frac{\partial \gamma_i(y)}{\partial x_i(t)} = 0, \quad \frac{\partial \theta_i(t)}{\partial x_i(t)} = 0, \quad \text{and the market clearing condition for each asset, i.e.,}
\]

\[
\sum_{i=1}^{n} \tilde{c}_i(y) = 0, \quad \sum_{i=1}^{n} \theta_i(t) = 0, \quad \text{and } \sum_{i=1}^{n} x_i(t) = Z. \quad \text{(B4)}
\]
This completes the proof. □

B.4. Calculate the first derivatives in Model B.1 and B.2

This section provides the calculation of the first derivatives in the first-order condition in model B.1 and B.2. Let

\[ a_i = \frac{K - (m_{0} - r\sigma_{a}^{2}(x_{0} - \zeta_{a}))}{\sigma_{a}} \],

\[ b_i = \frac{K - (m_{0} - r\sigma_{b}^{2}(\theta_{0} + x_{0}))}{\sigma_{b}} \],

\[ c_i = \exp \left[-r\theta_{0} - K + m_{0} - \frac{1}{2}r\sigma_{b}^{2}(\theta_{0} + x_{0})\right] \]

\[ d_i = \exp \left[-r\theta_{0} - K - m_{0} - \frac{1}{2}r\sigma_{a}^{2}(x_{0} - \zeta_{a})\right] \]

therefore,

\[ g_{i}(x_{0}, \theta_{0}, \zeta_{a}) = -\frac{1}{r} \ln(a_i \times \phi(a_i) + c_i \times (1 - \Phi(\beta_i))) \]

and \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal distribution. Hence

\[ \frac{\partial g_{i}(x_{0}, \theta_{0}, \zeta_{a})}{\partial \theta_{0}} = -\frac{d_{i}\phi(a_i)\sigma_{a} \Phi(a_i) - d_{i}\Phi(a_i)\sigma_{a} r\sigma_{b}^{2}(x_{0} - \zeta_{a})}{d_{i}\Phi(a_i) + (1 - \Phi(\beta_i))\zeta_{a}} \]

\[ \frac{\partial g_{i}(x_{0}, \theta_{0}, \zeta_{a})}{\partial \theta_{0}} = \frac{\phi(\beta_{i})\sigma_{b}(-1 + (1 - \Phi(\beta_{i}))\zeta_{a} + r\sigma_{b}^{2}(x_{0} - \zeta_{a})))}{\phi(\beta_{i})\sigma_{b} d_{i} + \phi(\beta_{i})(K - m_{0} - r\sigma_{b}^{2}(\zeta_{a} + r\sigma_{b}^{2}(x_{0} - \zeta_{a})\zeta_{a})}{d_{i}\Phi(a_i) + (1 - \Phi(\beta_i))\zeta_{a}} \]

where \( \phi(x) \) denotes the standard normal probability density function, hence

\[ \phi(x) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{x^{2}}{2}\right) \].

References


