AN INFORMAL INTRODUCTION TO COMPUTING WITH CHERN CLASSES

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This is a draft version of these notes. When complete, these notes will be my VIGRE Fourth-Year Project; this document will be posted on the UM math department's VIGRE web page.

1. Introduction

Chern classes are objects associated to vector bundles. They describe relationships between the vector bundle and the topology of the base space. They straddle the worlds of topology and algebra or geometry; as such, they can be subtle, and difficult to compute. On the other hand, they satisfy some nice properties, and they are useful tools. One goal of these notes is to give examples of several different applications of Chern classes.

In particular, Chern classes of certain bundles on Grassmanians can be used to solve enumerative problems, and this will be the main goal of these notes.

I hope these notes will be accessible to a student who has some basic knowledge of algebraic geometry, say at the level of Shafarevich [14], and who is familiar with vector bundles.

2. Chow Ring and Rational Equivalence

Let X be a smooth, irreducible projective variety of dimension n over an algebraically closed field. We will define the *Chow ring* of X, which is where Chern classes will take their values. I should warn the reader that the picture is a little different for non-smooth X: here we are taking advantage of smoothness and Poincaré duality (so we will not distinguish between cycles and cocycles).

For $d \geq 0$, let $Z^d(X)$ be the free abelian group generated by irreducible closed subvarieties of X of codimension d. Elements of $Z^d(X)$ are called *d-cocycles* or *$(n-d)$-cycles*.

Definition 1. A **principal** k-cycle is $\operatorname{div}(f)$, where $Y \subseteq X$ has $\dim Y = k+1$, f is a rational function on Y, and $\operatorname{div}(f) = \operatorname{Zeros}(f) - \operatorname{Poles}(f)$, with appropriate multiplicities (considered as a cycle on X rather than on Y).

Two k-cycles are **rationally equivalent** if they differ by a sum of principal k-cycles.

Definition 2. The **dth Chow group** $A^d(X)$ is $Z^d(X)/\sim$, where $\sim$ is rational equivalence. That is, it is $Z^d(X)$ modulo the subgroup generated by principal cycles. The equivalence class of a cycle V will be denoted by $[V]$, or just V by abuse of notation.

Another way to state the definition of rational equivalence is that k-cycles $V, W \in Z^{n-k}(X)$ are rationally equivalent if there is a subvariety Y in $X \times \mathbb{P}^1$ of dimension $k+1$ such that V and W are the fibers of Y over two points of $\mathbb{P}^1$; or, more generally, a sequence of such subvarieties $Y_1, \ldots, Y_r \subset X \times \mathbb{P}^1$ all of dimension $k+1$ and $V = V_0, V_1, \ldots, V_r = W \subseteq X$ all of dimension k such that each Y_i has V_{i-1} and V_i as fibers over two points of $\mathbb{P}^1$.

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Rational equivalence is similar to cobordism in topology. Basically it says we can "move" one subvariety to the other, with the motion parametrized by a $\mathbb{P}^1$, or by a "chain" of $\mathbb{P}^1$s.

A few cases:

(i) $A^1(X)$ is generated by (classes of) Weil divisors; since X is smooth, this is the same as Cartier divisors. Since the only choice for a 0-codimension subvariety ("Y") is X itself, two divisors are rationally equivalent if and only if they are linearly equivalent. So $A^1(X)$ is the Picard group of X.

(ii) $A^0(X)$ is generated by $[X]$, the class of the cycle $1 \cdot X$, where X is considered as a codimension zero subvariety of itself. There is no torsion (no multiple of X is rationally equivalent to the empty set). So $A^0(X) \cong Z^0(X) \cong \mathbb{Z}$.

(iii) For $d > n = \dim(X)$, $A^d(X) = 0$.

(iv) $A^n(X)$ is generated by (classes of) points. Two points P and Q are rationally equivalent if there is a chain of rational curves on X so that you can start at P, go along the first rational curve to a point of intersection with the next rational curve, go along the next rational curve to the next point, and so on, to arrive at Q.

Rational equivalence of points is subtle: the study of rational curves on varieties is already deep, yet the full equivalence relation involves higher genus curves on X. For example, two pairs of points $P + P'$ and $Q + Q'$ are rationally equivalent if (but not only if) there is a hyperelliptic curve in X on which $P + P'$ and $Q + Q'$ both lie in the hyperelliptic pencil; or if there is a chain of such hyperelliptic curves, meeting in an appropriate way.

One thing we can say is that rational equivalence of points preserves the number of points (i.e., the sum of the coefficients of a 0-cycle). Thus there is a well-defined **degree** function $A^n(X) \to \mathbb{Z}$.

Example 3. For $X = \mathbb{P}^n$, $A^1(X) = \operatorname{Pic}(X) = \mathbb{Z}$, generated by the hyperplane class H. Each $A^d(X)$ for $1 \le d \le n$ contains the class of a codimension d linear subspace, which we will denote by H^d: for now, this is just notation, but we will see that it actually means a d-fold product in the Chow ring, which will be defined soon.

Exercise 4. Show that any two codimension d linear subspaces of $\mathbb{P}^n$ are rationally equivalent, so this H^d is a single rational equivalence class.

Now, it turns out that $A^d(X) \cong \mathbb{Z}$ is generated by H^d. Given a cycle Y in $\mathbb{P}^n$ of codimension d (meaning every irreducible component of Y has this same codimension), how can we relate it to H^d? The idea is that we project it down one dimension to a hyperplane (a finite map), then down another dimension (again a finite map), and so on, till we've projected it *onto* a codimension d linear subspace; the degree of this map is the degree of the original subvariety Y. We just have to pick the projections in a reasonable way: don't use a projection so that Y is contracted to a smaller dimension. We can always avoid this. Also, to make the image of Y a cycle, we have to assign the right multiplicities to its irreducible summands.

If $Y = \sum a_i V_i$ is a cycle in $\mathbb{P}^n$ (so the V_i are irreducible) and π is a projection to a hyperplane in $\mathbb{P}^n$ such that for each i, $\dim(\pi(V_i)) = \dim V_i$, (for example, if π is a projection from a point not on any V_i), then we make $\pi(Y)$ a cycle by setting $\pi(Y) = \sum \deg(\pi|V_i) a_i \pi(V_i)$. (More generally, if V_i is a summand in Y with $\dim \pi(V_i) < \dim V_i$, then $\pi(V_i)$ appears in $\pi(Y)$ with coefficient zero.)

Exercise 5. If Y is a cycle in $\mathbb{P}^n$ and π is a linear projection onto a hyperplane in $\mathbb{P}^n$ such that $\pi|V_i$ is finite for each i, then Y and $\pi(Y)$ are rationally equivalent, and

$\deg(Y) := \sum a_i \deg(V_i)$ is equal to $\deg(\pi(Y)) = \sum \deg(\pi|V_i) a_i \deg(\pi(V_i))$. The finiteness condition holds for a general projection π (indeed, for any projection from a point not on Y).

For example, if we start with a twisted cubic in $\mathbb{P}^3$, the first projection to a plane will give (say) a plane nodal cubic curve (it can be cuspidal if you want), and projecting a plane cubic curve to a line gives a three-to-one map to the line, which we interpret as three copies of the line.

We have essentially given the definition of cycle pushforwards: for other maps it is the same as for projections in $\mathbb{P}^n$, as described above. But we will not go into more detail, since we will not use cycle pushforwards again in these notes.

The above argument justifies the claim that $A^d(\mathbb{P}^n)$ is cyclic, generated by H^d, the class of a codimension d linear subspace.

We can add and subtract rational equivalence classes of cycles formally. To give a multiplication rule for (classes of) cycles, on a smooth projective X, it is enough to just multiply (classes of) irreducible subvarieties, and then distribute over sums.

To multiply the classes of two irreducible subvarieties V and W, if they meet **properly**, meaning $\operatorname{codim} V \cap W = \operatorname{codim} V + \operatorname{codim} W$, then the product is just (the class of) the intersection of the subvarieties, with appropriate multiplicities assigned to the components of the intersection (note that the assumption of proper intersection is exactly what is needed to ensure the product takes $A^d(X) \times A^e(X) \to A^{d+e}(X)$). The question of what the multiplicities should be is tricky. But if the intersection is **transversal**, meaning that at each point of intersection, V and W are smooth and their tangent spaces together span the tangent space of X, then the multiplicities are all 1.

If the subvarieties do not meet transversally, or properly, one strategy is to try to move them by rational equivalence so that they do. Since our ambient variety X is smooth and projective, we may apply the *Moving Lemma* of section 11.4 of Fulton's *Intersection Theory* [6], along with Example 11.4.2, which imply the following:

Fact 6.

- If α and β are cycles on X, there is a cycle α' rationally equivalent to α such that α' and β meet transversally (that is, if $\alpha' = \sum a_i' V_i$ and $\beta = \sum b_j W_j$ then each V_i meets each W_j transversally).
- If V and V' are rationally equivalent irreducible subvarieties of X and each is transversal to the irreducible subvariety W, then $V \cap W$ and $V' \cap W$ are rationally equivalent.

So, we define the product $[V][W]$ as follows.

Definition 7. For any cycle $\omega = \sum a_i W_i$ which is rationally equivalent to W and meets V transversally, we set $[V][W] = [V \cap \omega] = [\sum a_i (V \cap W_i)]$.

This gives a well-defined product of rational equivalence classes of cycles, taking $A^d(X) \times A^e(X) \to A^{d+e}(X)$. Note that the class $[X] \in A^0(X)$ is the unit element for this multiplication (X is trivially transversal to any smooth $V \subseteq X$, so $[X][V] = [X \cap V] = [V]$).

Definition 8. For smooth, irreducible, projective X, the **Chow ring** $A(X)$, or $A^*(X)$, of X, is the direct sum of the $A^i(X)$, with a product given by intersection as above. It is commutative, graded by i, and has unit element $1 = [X]$.

Example 9. We see now that H^d, the class of a codimension d linear subspace in $\mathbb{P}^n$, is in fact the d-fold product in $A(\mathbb{P}^n)$ of the class H of a hyperplane. Indeed, going by induction on d, a codimension d linear subspace may be written as the transversal intersection of a hyperplane H with a codimension $d-1$ linear subspace, whose class is H^{d-1}.

Now, $H^{n+1} = 0$ because $A^{n+1}(\mathbb{P}^n) = 0$, or because a general hyperplane (H) does not meet (contain) a point (H^n); if the hyperplane and point do not meet, then they are vacuously transversal, so H^{n+1} is represented by their intersection, which is empty, corresponding to the zero class in $A(\mathbb{P}^n)$. So $H^{n+1} = 0$ is a relation in $A(\mathbb{P}^n)$. On the other hand, $A(\mathbb{P}^n)$ is generated by H and each $A^i(\mathbb{P}^n)$ is cyclic (with no torsion), so there can't be any other relations in degree $n+1$ or lower. Therefore $A(\mathbb{P}^n)$ is $\mathbb{Z}[H]/H^{n+1}$.

(Note, when we are working over $\mathbb{C}$, this is equal to the cohomology ring of $\mathbb{P}^n$. The same thing will happen with Grassmanians, below. It does not, however, happen for other varieties: the Chow and cohomology rings coincide only under certain circumstances.)

3. Chern classes

To a vector bundle E on X of rank e, we associate **Chern classes** $c_1(E) \in A^1(X)$ (the first Chern class of E), $c_2(E) \in A^2(X)$ (the second Chern class of E), and so on up to $c_e(E) \in A^e(X)$. Note that if $e > n$, then some of these Chern classes have to be zero. Sometimes we write $c_{\text{top}}(E)$ for $c_e(E)$ and call it the **top** Chern class of E. The **total Chern class** is $c(E) = 1 + c_1(E) + c_2(E) + \dots c_e(E)$.

The question of which classes they are and how to calculate them will be addressed soon. First, the Chern classes have many formal properties. Here are two to start with.

Fact 10 (Two properties of Chern classes)**.**

(i) If L is a line bundle and D is a Cartier divisor such that $L = \mathscr{O}(D)$, then $c(L) = 1+D$ (in other words, $c_1(L)$ is the class of D). (Notice that if D' is another choice for a divisor that gives L, then it is linearly equivalent to D, hence rationally equivalent, so $c_1(L)$ is well-defined.) Another way of saying this is that any line bundle L is equal to $\mathscr{O}_X(c_1(L))$.

(ii) (Whitney sum) If $0 \to A \to B \to C \to 0$ is a short exact sequence of vector bundles, then $c(B) = c(A)c(C)$. In particular, the total Chern class of a direct sum of bundles is the product of the total Chern classes of the summands (for any number of summands).

Example 11.

Projective space: Every line bundle on $\mathbb{P}^n$ is of the form $\mathscr{O}_{\mathbb{P}^n}(a)$ for some $a \in \mathbb{Z}$, corresponding to the divisor aH; therefore this line bundle has total Chern class $1 + aH$. Hence a direct sum of line bundles $\mathscr{O}(a) \oplus \mathscr{O}(b)$ has total Chern class $(1 + aH)(1 + bH) = 1 + (a + b)H + abH^2$.

From the Euler exact sequence,

$$0 \to \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0,$$

we can find the Chern classes of the tangent bundle to $\mathbb{P}^n$. The total Chern class of the first bundle is $c(\mathscr{O}_{\mathbb{P}^n}) = 1 + 0H = 1$, and for the middle bundle we get $c(\mathscr{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)}) = (1 + H)^{n+1}$. Therefore $c(T_{\mathbb{P}^n}) = (1 + H)^{n+1}$. Since

$A(\mathbb{P}^n) = \mathbb{Z}[H]/H^{n+1}$, we find that

$$c(T_{\mathbb{P}^1}) = 1 + 2H, \quad c(T_{\mathbb{P}^2}) = 1 + 3H + 3H^2,$$
$$c(T_{\mathbb{P}^3}) = 1 + 4H + 6H^2 + 4H^3,$$

and so on.

Note that $1+3H+3H^2$ is not a product of two linear polynomials in $\mathbb{Z}[H]/H^3$. This shows that $T_{\mathbb{P}^2}$ is not an extension of line bundles (meaning just that it cannot fit in the middle of a short exact sequence with a line bundle on each side), much less a direct sum of line bundles.

Similarly, $c(T_{\mathbb{P}^n})$ is irreducible for $n+1$ prime, by Eisenstein's criterion, showing that $T_{\mathbb{P}^n}$ is not an extension of lower-rank bundles. On the other hand, $c(T_{\mathbb{P}^n})$ is reducible for $n+1$ not prime: for example,

$$1 + 4H + 6H^2 + 4H^3 = (1 + 2H + 2H^2)(1 + 2H).$$

It turns out that these tangent bundles can indeed be extensions of smaller-rank bundles for certain n: see [13], section 4.2.

Curves: In fact, every vector bundle on a curve is an extension of line bundles. There's an exercise in Hartshorne about this (Chapter V, Exercise 2.3): Given any vector bundle E on a curve, there is a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_e = E$$

such that E_i/E_{i-1} is a line bundle for each i. Then

$$c(E) = (1 + c_1(E_1))(1 + c_1(E_2/E_1)) \ldots (1 + c_1(E_e/E_{e-1})).$$

On the other hand, every product $(1 + a_1)(1 + a_2)(1 + a_3) \ldots (1 + a_e)$, with $a_1, \ldots, a_e \in A^1(X)$, is the total Chern class of the vector bundle $\mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_e)$.

The following definition of the Chern classes of a vector bundle E was given by Grothendieck [9]. We will not use it directly; instead, we will use a description in terms of degeneracy loci, but we will state the definition for completeness. Let $\dim X = n$ and let E be a vector bundle on X with $\operatorname{rk} E = r$. Let $\mathbb{P}_{\text{quot}}(E) \to X$ be the projective bundle of hyperplanes in E, with projection map π. Then $\pi^*(E)$ contains a canonical rank n subbundle, sometimes written $\mathscr{O}_E(1)$. Its first Chern class ξ is very important. There is a pullback map $\pi^* : A^*(X) \to A^*(\mathbb{P}_{\text{quot}}(E))$, taking each class in X to its preimage in the projective bundle. (More general pullbacks will be discussed in a later section.)

Exercise 12. Check that for this π, the pullback π^* is well-defined on rational equivalence classes, preserves codimension, and commutes with transversal intersections.

This pullback map gives us a map,

$$A^*(X)[\xi] \twoheadrightarrow A^*(\mathbb{P}_{\text{quot}}(E))$$

whose kernel is principal, and has a unique monic generator,

$$\xi^{r+1} + \xi^r \tilde{c}_1 + \xi^{r-1} \tilde{c}_2 + \cdots + \tilde{c}_r$$

where each $\tilde{c}_i$ is the pullback of an element $c_i \in A^i(X)$. Since the pullback map is in fact injective, this c_i is uniquely determined. This is the ith Chern class of E.

Earlier definitions of Chern classes were more analytical: see [2], [8]. Also see [1] for historical information.

4. The splitting principle

Many computations are much easier if the vector bundle at hand happens to be **decomposable**, that is, a direct sum of line bundles, or **split**, that is, filtered so that all the quotients are line bundles, as in the case of bundles on curves. In many cases, the result of such a computation is valid even without the splitting hypothesis: this is called the **splitting principle**—roughly speaking, in many situations, the hypothesis that a vector bundle is split (or decomposable) can be removed freely. Before discussing where this principle comes from and exactly when it is applicable, let's see a few examples of how it can be applied.

Example 13. Let E be a vector bundle and L a line bundle. What are the Chern classes of $E \otimes L$ in terms of those of E and L?

First, if E is a line bundle, then

$$c(E \otimes L) = 1 + c_1(E \otimes L) = 1 + c_1(E) + c_1(L).$$

If E is decomposable, $E = L_1 \oplus \cdots \oplus L_r$, then we can extend by linearity. We have $c(E) = (1 + c_1(L_1)) \cdots (1 + c_r(L_r))$. Then $E \otimes L = (L_1 \otimes L) \oplus \cdots \oplus (L_r \otimes L)$, so

$$\begin{aligned} c(E) &= (1 + c_1(L_1) + c_1(L)) \cdots (1 + c_1(L_r) + c_1(L)) \\ &= 1 + (c_1(E) + r\, c_1(L)) \\ &\quad + \big(c_2(E) + (r-1)c_1(E)c_1(L) + \binom{r}{2} c_1(L)^2\big) + \cdots \end{aligned}$$

If E is split, so there is a filtration

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_r = E$$

with each quotient E_{i+1}/E_i a line bundle, then

$$c(E) = (1 + c_1(E_1))(1 + c_1(E_2/E_1)) \cdots (1 + c_1(E_r/E_{r-1})).$$

Considering the filtration

$$0 \subsetneq E_1 \otimes L \subsetneq \cdots \subsetneq E_r \otimes L = E \otimes L,$$

we see that each quotient $(E_{i+1} \otimes L)/(E_i \otimes L) = (E_{i+1}/E_i) \otimes L$. This lets us compute that

$$\begin{aligned} c(E \otimes L) = (1 + c_1(E_1) + c_1(L))(1 + c_1(E_2/E_1) + c_1(L)) \cdots \\ \cdots (1 + c_1(E_r/E_{r-1}) + c_1(L)), \end{aligned}$$

and as in the decomposable case, we get

$$\begin{aligned} c(E \otimes L) = 1 + (c_1(E) + r\, c_1(L)) \\ + \big(c_2(E) + (r-1)c_1(E)c_1(L) + \binom{r}{2} c_1(L)^2\big) + \cdots . \end{aligned}$$

The results of these computations are valid even without assuming E splits or is decomposable. By applying the splitting principle, we obtain

Proposition 14. *For any vector bundle E and line bundle L,*

$$\begin{aligned} c(E \otimes L) = 1 + (c_1(E) + r\, c_1(L)) \\ + \big(c_2(E) + (r-1)c_1(E)c_1(L) + \binom{r}{2} c_1(L)^2\big) + \cdots , \end{aligned}$$

as above.

Example 15. What is the relation between $c_1(T_X)$ and the canonical divisor K_X? Suppose $T_X = L_1 \oplus \cdots \oplus L_n$ (where $n = \dim(X)$), and $a_i = c_1(L_i)$. Then

$$c(T_X) = 1 + c_1 + \cdots + c_n = (1 + a_1)\cdots(1 + a_n),$$

so $c_1 = c_1(T_X) = a_1 + \cdots + a_n$. Now, $\Omega^1_X = T_X^\vee = L_1^\vee \oplus \cdots \oplus L_n^\vee$, so

$$\mathscr{O}(K_X) = \Omega^n_X = \bigwedge^n \Omega^1_X = L_1^\vee \otimes \cdots \otimes L_n^\vee = \mathscr{O}(-a_1 - \cdots - a_n).$$

Therefore $K_X = -a_1 - \cdots - a_n$, so $c_1 = c_1(T_X) = -K_X$.

Exercise 16. If T_X is split, then dualizing does not give us a filtration of Ω^1_X by subbundles, but a filtration by quotient bundles: If

$$0 \subset E_1 \subset \cdots \subset E_n = T_X,$$

then, dualizing,

$$\Omega^1_X \twoheadrightarrow E_{n-1}^\vee \twoheadrightarrow \cdots \twoheadrightarrow E_1^\vee.$$

With this we can compute $K_X = c_1(\Omega^1_X) = -c_1(T_X)$.

Even if T_X is not decomposable or split, we have

Proposition 17. *For smooth, irreducible projective X, $c_1(T_X) = -K_X$.*

Note, this matches with our computation for $\mathbb{P}^n$: $c_1(T_{\mathbb{P}^n}) = (n+1)H$, and $K_{\mathbb{P}^n} = -(n+1)H$.

Example 18. We can compute the Chern classes of Sym^3 of a decomposable rank 2 bundle in terms of the Chern classes of the original bundle as follows. Say $\mathrm{rk}(E) = 2$ and $c(E) = 1 + c_1 + c_2$. If $E = \mathscr{O}(a_1) \oplus \mathscr{O}(a_2)$, then $c(E) = (1 + a_1)(1 + a_2)$, so $a_1 + a_2 = c_1$ and $a_1 a_2 = c_2$.

Now,

$$\mathrm{Sym}^3(E) = \mathscr{O}(3a_1) \oplus \mathscr{O}(2a_1 + a_2) \oplus \mathscr{O}(a_1 + 2a_2) \oplus \mathscr{O}(3a_2).$$

Therefore

$$\begin{aligned} c(\mathrm{Sym}^3(E)) &= (1 + 3a_1)(1 + 2a_1 + a_2)(1 + a_1 + 2a_2)(1 + 3a_2) \\ &= 1 + 6c_1 + (11c_1^2 + 10c_2) + (6c_1^3 + 30c_1c_2) + 9c_2(2c_1^2 + c_2). \end{aligned}$$

That is,

$$(*) \qquad \begin{aligned} c_1(\mathrm{Sym}^3(E)) &= 6c_1(E), \\ c_2(\mathrm{Sym}^3(E)) &= 11c_1(E)^2 + 10c_2(E), \\ c_3(\mathrm{Sym}^3(E)) &= 6c_1(E)^3 + 30c_1(E)c_2(E), \\ c_4(\mathrm{Sym}^3(E)) &= 9c_2(E)(2c_1(E)^2 + c_2(E)). \end{aligned}$$

We proved $(*)$ for any bundle E which is a direct sum of two line bundles.

Exercise 19. Suppose E is split, that is, there is a line bundle $L_1 \subset E$ such that $E/L_1 = L_2$ is a line bundle. Show that $\mathrm{Sym}^3(E)$ has the following splitting:

$$0 \subset L_1^3 \subset L_1^2 \otimes E \subset L_1 \otimes \mathrm{Sym}^2(E) \subset \mathrm{Sym}^3(E),$$

with adjacent terms having quotients L_1^3, $L_1^2 \otimes L_2$, $L_1 \otimes L_2^2$, and L_2^3. (Note that $L_1^3 = \mathrm{Sym}^3(L_1)$, and so on.) Check that the Chern classes of $\mathrm{Sym}^3(E)$ in this case are the same as for the direct sum case.

In fact, $(*)$ holds for *any* E of rank 2, regardless of whether or not E is split.

In a similar way, we can work out the Chern classes of $E \otimes F$, $\bigwedge^d E$, $\mathrm{Sym}^d(E)$, $E^\vee$, and $E \oplus F$ in terms of the Chern classes of E and F. The idea is to assume E and F are decomposable (or split), and work out a formula only in terms of the Chern classes of E and F (not their summands or subbundles). Such a formula will be valid even when E and F are not decomposable or split.

The way the splitting principle really works is that you construct a space $\tilde{X}$ and a map $\tilde{X} \to X$ so that the pullback of E splits, and furthermore the pullback of Chow rings $A^*(X) \to A^*(\tilde{X})$ is injective (the pullback of Chow rings will be discussed in the next section). (In giving the definition of Chern classes, we saw that $\mathbb{P}_{\mathrm{quot}}(E) \to X$ satisfies this injectivity hypothesis, although the pullback bundle does not split.) Then computations can be performed in $A^*(\tilde{X})$, where E splits, without any loss of information.

The space $\tilde{X}$ is constructed iteratively: on $\mathbb{P}(E)$, the pullback of E contains a rank 1 subbundle (the tautological line bundle); quotient out by it to get a new bundle of rank one less, and keep going. It has to be checked that the map of Chow rings is injective. This construction preserves any computation that "commutes with pullback", such as the operations listed above (tensor, wedge product, symmetric powers, dual, and direct sums of vector bundles).

Regarding the computation of $\mathrm{Sym}^3(E)$ carried out above, the splitting principle would be applied as follows. For the map $f : \tilde{X} = \mathbb{P}(E) \to X$, the bundle $f^*(E)$ satisfies the following:

- The bundle $f^*(E)$ is split: there is a line bundle $L_1 \subset f^*(E)$ such that $f^*(E)/L_1 = L_2$ is a line bundle.
- The Chern classes of E pull back to the Chern classes of $f^*(E)$: $c(f^*(E)) = f^*(c(E)) = 1 + f^*(c_1(E)) + f^*(c_2(E))$. Similarly, the Chern classes of $\mathrm{Sym}^3(E)$ pull back to the Chern classes of $f^*(\mathrm{Sym}^3(E))$.
- $\mathrm{Sym}^3(f^*(E)) = f^*(\mathrm{Sym}^3(E))$
- The map of Chow rings $f^* : A^*(X) \to A^*(\tilde{X})$ is injective.

The upshot is that the Chern classes of $f^*(E)$ and $f^*(\mathrm{Sym}^3(E))$ are related as in $(*)$; so the pulled-back Chern classes of E and $\mathrm{Sym}^3(E)$ are related as in $(*)$. Since the pullback map of Chow rings is injective, the actual classes on X must have satisfied the relation (that is, before being pulled back to $\tilde{X}$).

This argument is pretty standard, and tends to not be spelled out in this much detail! For purposes of computation, the result of the splitting principle is more important than the detailed application, or even the construction of the space $\tilde{X}$.

Here is one way to avoid the drudgery of the above argument: Even if E does not split, we define the **Chern roots** formally to be a_i such that $c(E) = (1 + a_1) \cdots (1 + a_e)$. The Chern classes of E are the elementary symmetric functions of its Chern roots. If E is decomposable, its direct summands are its Chern roots; if E is split, the line bundle quotients of the splitting filtration (the E_{i+1}/E_i) are its Chern roots. Otherwise, the Chern roots are just formal objects.

They behave as in the decomposable or split cases. For example, $E \otimes L$ has Chern roots $a_i + c_1(L)$, $E^\vee$ has Chern roots $-a_i$, and so on—taking the elementary symmetric functions, we can find the Chern classes of these bundles in terms of the Chern classes of E. We regard the Chern roots as virtual elements of $A^1(X)$. (It would be more accurate to say they are elements of $A^1(\tilde{X})$, so they are elements adjoined to the ring $A^*(X)$.) The advantage of this formalism is that we can avoid dealing with the space $\tilde{X}$.

5. Pulling back and restricting

Say you have a map $f : X \to Y$ and a bundle E on Y. Is there some relation between the Chern classes of E and of $f^*(E)$?

The easiest statement would be something like: $c_i(f^*(E)) = f^{-1}(c_i(E))$, where we would like $f^{-1}(c_i(E))$ to mean the scheme-theoretic fiber of f over $c_i(E)$. Unfortunately, this is not well-defined up to rational equivalence.

For example, say $Y = \mathbb{P}^2$, $X = \mathrm{Bl}_p(\mathbb{P}^2)$ is the blowup of $\mathbb{P}^2$ at p, and $f : X \to Y$ is the blowdown, with exceptional divisor E. Then for any point $x \neq p$ in Y, $f^{-1}(x)$ is a point, the point in $X - E$ lying over x, but $f^{-1}(p) = E$. Now, in $\mathbb{P}^2$, p and x are rationally equivalent, but $f^{-1}(p)$ has codimension 1 and $f^{-1}(x)$ has codimension 2, so they cannot be rationally equivalent.

The upshot is that there is not necessarily a well-defined pullback map from $A^*(Y)$ to $A^*(X)$. But there is still a pullback map of divisors, or their rational equivalence classes (i.e., line bundles), $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$. What can we do with this?

If E is a decomposable bundle, then we can pull back the direct sum decomposition, which means f^*E splits into line bundles which are the pullbacks of the summands of E. Those line bundles (or their first Chern classes) are the Chern roots of E; so this is saying that the Chern roots of $f^*(E)$ are the pullbacks of the Chern roots of E. Therefore the Chern classes of $f^*(E)$ are the same symmetric functions of the pullbacks of the Chern roots of E. For example, if $\mathrm{rank}(E) = 2$ and $c_2(E) = a_1a_2$, then $c_2(f^*(E)) = f^*(a_1)f^*(a_2)$—but this is different from intersecting first and then pulling back!

Similarly, if E is split, then pulling back the splitting (filtration) gives a splitting of f^*E. So the Chern roots of f^*E are the pullbacks of the Chern roots of E. As before, the Chern classes of f^*E are not the pullbacks of the Chern classes of E, but at least there is a description.

So, in the split or decomposable case we can say something about the Chern classes of f^*E. Unfortunately, if E does not split on Y, the Chern roots are just formal things (they are really divisors on some other space $\tilde{Y}$ over Y), so it is harder to say anything. But according to the philosophy of the splitting principle, it is still true that many computations carried out for pullbacks of split bundles will be true for pullbacks of non-split bundles.

One "safe" case is pulling back via the inclusion of a closed subvariety. If X is a smooth closed subvariety of Y, the way to restrict (pull back) Chern classes of bundles on Y is just to move them so they are transversal to X and then intersect with X. Why? Because to restrict a line bundle or divisor, move the divisor so it is transversal to X, and then intersect with X; so, according to the previous discussion, that's what we do with the Chern roots of a bundle. So we can "split" a bundle on Y, restrict its Chern roots, then take the same old symmetric functions of them to get the Chern classes of the restricted bundle. But once we've moved the divisors, we can intersect them together first and then intersect with X—in this context, intersecting does commute with restricting (pulling back)—as long as everything is transversal.

This means we can just use the Chern classes on Y directly and intersect them down to X. This is what we need to get information out of short exact sequences like the following. Let X be a smooth subvariety of a smooth variety Y. Then $N_{X/Y}$ denotes the normal bundle of X in Y, and $I_{X/Y}$ denotes the ideal sheaf of X in Y; the restriction $I_{X/Y}|X$ is locally free, and called the conormal bundle of X in Y. The following short exact sequences are standard:

$$0 \to T_X \to T_Y|X \to N_{X/Y} \to 0$$
$$0 \to I_{X/Y}|X \to \Omega^1_Y|X \to \Omega^1_X \to 0$$
$$0 \to N_{B/A} \to N_{C/A} \to N_{C/B}|A \to 0$$

(the last is for $A \subset B \subset C$ closed subvarieties, all smooth).

Example 20. Let X be any variety, $\dim(X) = g$, such that the tangent bundle of X is trivial. (For example, any abelian variety has trivial tangent bundle.) Suppose X is embedded in projective space $\mathbb{P}^n$. Then in the short exact sequence

$$0 \to T_X \to T_{\mathbb{P}^n}|X \to N_{X/\mathbb{P}^n} \to 0,$$

we have $c(T_X) = 1$, so $c(N_{X/\mathbb{P}^n}) = c(T_{\mathbb{P}^n}|X)$.

I claim $c_g(T_{\mathbb{P}^n}|X) \neq 0$. Indeed, $c_g(T_{\mathbb{P}^n}) = \binom{n+1}{g} H^g$, so $c_g(T_{\mathbb{P}^n}|X) = \binom{n+1}{g} H^g \cap X$ (once we choose an H^g meeting X transversally). But $H^g \cap X$ is a finite set of points on X whose number is the degree of X in the embedding $X \subset \mathbb{P}^n$. Therefore $c_g(T_{\mathbb{P}^n}|X)$ consists of $\binom{n+1}{g} \deg(X)$ points in X, so it is not rationally equivalent to zero. (Rational equivalence of sets of points preserves the number of points.)

Therefore $c_g(N_{X/\mathbb{P}^n}) \neq 0$ as well. So we must have $\operatorname{rank}(N_{X/\mathbb{P}^n}) \geq g$.

This shows that if X is embedded in projective space, the codimension of X must be at least as great as the dimension of X, so $n \geq 2g$.

By looking at this more closely, Van de Ven [15] was able to say more about embeddings of abelian varieties: he showed that an embedding of an abelian variety with $n = 2g$ must be an elliptic curve in $\mathbb{P}^2$ or an abelian surface in $\mathbb{P}^4$. See also [11].

6. Degeneracy Loci

We have seen how to work with Chern classes formally. The last ingredient we need for our computations is a way of actually finding Chern classes "from scratch", at least sometimes.

Let E be a vector bundle such that $\operatorname{rank}(E) \leq \dim X$, and with "lots" of global sections; say, globally generated.[1] Let $s_1, \ldots, s_i \in \Gamma(X, E)$ be general sections, where $i \leq e = \operatorname{rank}(E)$. Equivalently, let $u : \mathscr{O}_X^{\oplus i} \to E$ be a general map; and this is equivalent to saying u is a general global section of $E \otimes (\mathscr{O}^i)^\vee = E \otimes \mathscr{O}^i = E^{\oplus i}$.

At each point $x \in X$, we get vectors $s_1(x), \ldots, s_i(x)$ in the fiber $E(x)$. For general $s_1, \ldots, s_i$, we expect these vectors to be linearly independent, at least for "most" $x \in X$. The **degeneracy locus** $D(u)$ or $D(s_1, \ldots, s_n)$ is the locus of points x at which the vectors are *not* independent.

Equivalently, $D(u)$ is defined by the equation $s_1(x) \wedge \ldots \wedge s_i(x) = 0$. In local coordinates, and choosing a trivialization of E, we can represent u by an $e \times i$ matrix (in which the jth column is s_j); then $D(u)$ is the locus where this matrix has rank less than i. Therefore $D(u)$ is defined locally by algebraic equations (the vanishing of all the $i \times i$ minors of this matrix).

So $D(u)$ is not just a set of points but really an actual cycle: that is, it's a closed subscheme of X and its irreducible components have multiplicities (independent of the choices of trivializations). What is the codimension of this cycle?

[1] If E is not globally generated, we may be able to twist E up with an ample line bundle until it is; or, it might be possible to do something with meromorphic sections of E.

- If $i = 1$, the matrix representing u is $e \times 1$. For this to be linearly dependent, all the entries need to vanish; that is e independent equations, so $\operatorname{codim} D(u) = e$.
- If $i = e$, the matrix representing u is square. Then $D(u)$ is cut out by a single equation the vanishing of the determinant of this matrix, so $\operatorname{codim} D(u) = 1$.
- The codimension of $D(u)$ ought to be a linear function of i, the number of sections of E we are taking. From the previous two cases we can see that $\operatorname{codim} D(u)$ is $e + 1 - i$.

Exercise 21. Prove this.

The degeneracy loci allow us to find the Chern classes of bundles, provided there are plenty of global sections to choose from.

Theorem 22. *For a globally generated bundle E of rank e, and i general global sections $s_1, \ldots, s_i$, the cycle $D(s_1, \ldots, s_i)$ is equal (rationally equivalent) to $c_{e+1-i}(E)$.*

In particular, if u' is another general i-tuple of sections of E, then $D(u)$ and $D(u')$ are rationally equivalent. We will use this theorem constantly, but its proof is beyond the scope of these notes.

Example 23 (Bézout's Theorem)**.** Let $X = \mathbb{P}^2$ and $E = \mathscr{O}_{\mathbb{P}^2}(a) \oplus \mathscr{O}_{\mathbb{P}^2}(b)$, with $a, b > 0$. Then $c(E) = (1 + aH)(1 + bH)$ (where $H = c_1(\mathscr{O}_{\mathbb{P}^2}(1))$ is the class of a line).

Let $s \in \Gamma(\mathbb{P}^2, E) = \Gamma(\mathbb{P}^2, \mathscr{O}(a)) \oplus \Gamma(\mathbb{P}^2, \mathscr{O}(b))$, so $s = (F_a, F_b)$. Then $\operatorname{Zeros}(s) = \operatorname{Zeros}(F_a) \cap \operatorname{Zeros}(F_b)$. If F_a and F_b are general, then $\operatorname{Zeros}(s) = D(s)$ is a closed cycle in $\mathbb{P}^2$ representing $c_2(E)$. In this case, "general" means the curves F_a and F_b have no common components. Since we know the total Chern class $c(E)$, we know $c_2(E) = abH^2$. So we expect $\#\operatorname{Zeros}(s) = \deg c_2(E) = ab$, in agreement with Bézout's theorem.

Example 24 (Gauss-Bonet theorem)**.** (Now working over the field $k = \mathbb{C}$.) Suppose there is an $s \in \Gamma(X, T_X)$ with isolated zeros. Since s is a vector field on X, the Gauss-Bonet theorem states that if we assign a certain index to each zero of s, the sum of the indices will be the topological Euler characteristic of X, $\chi_{\text{top}}(X)$. The zero set of s is $D(s) = c_n(T_X)$, and in fact the indices are the same multiplicities assigned by the cycle structure on $D(s)$. Summing these multiplicities means finding $\deg c_n(T_X)$, which is often written $\int_X c_n(T_X)$. So $\chi_{\text{top}}(X) = \int_X c_n(T_X)$. (Indeed, this turns out to be true even if there is no vector field on X with isolated zeros.)

Example 25 (Topological Euler characteristic of projective hypersurfaces)**.** (Working over the field $k = \mathbb{C}$.) Let $X = X_d \subset \mathbb{P}^n$ be a hypersurface of degree d. Let H be (the class of) a hyperplane in $\mathbb{P}^n$, and let h be the restriction of H to X (a hyperplane section), so $h = c_1(\mathscr{O}_X(1))$. From the short exact sequence

$$0 \to T_X \to T_{\mathbb{P}^n}|X \to N_{X/\mathbb{P}^n} = \mathscr{O}_X(d) \to 0$$

we see $c(T_X)c(\mathscr{O}_X(d)) = c(T_{\mathbb{P}^n}|X)$. We know $c(T_{\mathbb{P}^n}) = (1 + H)^{n+1}$, so $c(T_{\mathbb{P}^n}|X) = (1 + h)^{n+1}$. Similarly, $c(N_{X/\mathbb{P}^n}) = 1 + dh$. Therefore

$$\begin{aligned} c(T_X) &= (1 + h)^{n+1}(1 + dh)^{-1} \\ &= (1 + h)^{n+1}(1 - dh + d^2h^2 - d^3h^3 + \cdots) \end{aligned}$$

In particular,

$$\begin{aligned} c_{n-1}(T_X) &= \binom{n+1}{n-1} h^{n-1} - \binom{n+1}{n-2} h^{n-2} \cdot dh + \cdots \\ &= h^{n-1} \left(\binom{n+1}{n-1} - \binom{n+1}{n-2} d + \binom{n+1}{n-3} d^2 - \cdots \right) \end{aligned}$$

Note that $\dim X = n-1$, so this is the top Chern class of T_X. From the Gauss-Bonet theorem, we know that the topological Euler characteristic of X is the degree of $c_{\mathrm{top}}(T_X)$. Since X is a degree d subvariety in $\mathbb{P}^n$, $\deg h^{n-1} = d$. Therefore the topological Euler characteristic of X is

$$d \left(\binom{n+1}{n-1} - \binom{n+1}{n-2} d + \binom{n+1}{n-3} d^2 - \cdots \right).$$

For example, a degree d curve in $\mathbb{P}^2$ has topological Euler characteristic $d\left(\binom{3}{1} - \binom{3}{0} d\right) = 3d - d^2$.

7. Lines in $\mathbb{P}^3$

Let $\mathbb{P}^3 = \mathbb{P}(V)$ be the set of *one-dimensional quotients* of V, that is, surjections $V \twoheadrightarrow k$, where $\dim V = 4$ (and where we identify $f, f' : V \twoheadrightarrow k$ if they are scalar multiples of one another). Each line $\mathbb{P}^1 \subset \mathbb{P}^3$ corresponds to a surjection $V \twoheadrightarrow W$ where $\dim W = 2$, and $\mathbb{P}^1 = \mathbb{P}(W) \subset \mathbb{P}(V) = \mathbb{P}^3$. Note that a point $p \in \mathbb{P}^1 = \mathbb{P}^1_W$ iff the surjection $V \twoheadrightarrow k$ corresponding to p factors through $V \twoheadrightarrow W$, that is, the following diagram can be filled in so that it commutes:

$$\begin{array}{ccc} V & \longrightarrow & W \\ & \searrow & \vdots \\ & & k \end{array}$$

Let $\mathbb{G} = \mathbb{G}(\mathbb{P}^1, \mathbb{P}^3)$ be the Grassmanian of projective lines in 3-space, so $\mathbb{G}$ is 4-dimensional. Let $V_{\mathbb{G}} = V \times \mathbb{G}$ be the trivial bundle of rank four on $\mathbb{G}$. There is a tautological rank 2 bundle on $\mathbb{G}$ whose fiber over the point in $\mathbb{G}$ corresponding to $V \twoheadrightarrow W$ is W; call this bundle Q. There is a surjective map $V_{\mathbb{G}} \twoheadrightarrow Q$ which over the point $V \twoheadrightarrow W$ in $\mathbb{G}$ maps the fibers of the bundles by $V \twoheadrightarrow W$.

Exercise 26. Check the following.

(a) Fix a basis for $V \cong k^4$. Then $\mathbb{G}$ is equal to the set of 2×4 matrices of full rank, modulo the action of GL_2 on the left, corresponding to changing the choice of basis for $W \cong k^2$.
(b) Find a covering of $\mathbb{G}$ by open affine patches, and coordinates; find the transition functions.
(c) Find the transition functions of Q.
(d) Find the map $V_{\mathbb{G}} \twoheadrightarrow Q$ in terms of the coordinates on the affine patches and check that these matrices are compatible with the transition functions of Q.

Since $V_{\mathbb{G}}$ is globally generated, so is Q, by the following exercise.

Exercise 27. On any space X, the surjective image of a globally generated sheaf is globally generated. (This is true both for $\mathscr{O}_X$-modules and for arbitrary sheaves of abelian groups.)

Let's find the Chern classes of Q. To find $c_2(Q)$, we'll take a single section of Q and look at its zero locus. Start by picking a 2-plane $\Pi = \mathbb{P}^2 \subset \mathbb{P}^3$. This corresponds to a surjection $V \twoheadrightarrow k^3$, so we for each line $V \twoheadrightarrow W$, we get a diagram

$$\begin{array}{ccc} k & & \\ \downarrow & \searrow & \\ V & \longrightarrow & W \\ \downarrow & & \\ k^3 & & \end{array}$$

Globalizing, we have a similar diagram for bundles,

$$\begin{array}{ccc} \mathscr{O}_{\mathbb{G}} & & \\ \downarrow & \overset{s}{\searrow} & \\ V_{\mathbb{G}} & \longrightarrow & Q \\ \downarrow & & \\ \mathscr{O}_{\mathbb{G}}^3 & & \end{array}$$

This map s is the section of Q whose zero locus we will take to represent $c_2(Q)$. Over a given $V \twoheadrightarrow W$ in $\mathbb{G}$, the section is zero iff there is a factorization

$$\begin{array}{ccc} k & & \\ \downarrow & \overset{0}{\searrow} & \\ V & \twoheadrightarrow & W \\ \downarrow & \nearrow & \\ k^3 & & \end{array}$$

which means the line $\mathbb{P}^1 = \mathbb{P}(W)$ is contained in the 2-plane $\Pi = \mathbb{P}(k^3)$ we picked at the start. Therefore $\mathrm{Zeros}(s) = \{\, \ell \subset \mathbb{P}^3 \mid \ell \subset \Pi \,\}$. Therefore $c_2(Q)$ is represented by this set (assuming s is a general section of Q). The set is called a *Schubert variety*; write σ_Π for it,

$$\sigma_\Pi = \{\, \ell \subset \mathbb{P}^3 \mid \ell \subset \Pi \,\}.$$

Exercise 28. Show that σ_Π is irreducible and closed in $\mathbb{G}$. Show that for two $\mathbb{P}^2$'s Π and Π' in $\mathbb{P}^3$, the Schubert varieties σ_Π and $\sigma_{\Pi'}$ are rationally equivalent.

We will meet several more types of Schubert varieties.

To compute $c_1(Q)$, we'll find two sections of Q and find the degeneracy locus. Start by picking a line $\ell_0 \subset \mathbb{P}^3$. This corresponds to a short exact sequence $0 \to k^2 \to V \to k^2 \to 0$. Globalizing, we get a map of bundles on $\mathbb{G}$,

$$\begin{array}{ccc} \mathscr{O}_{\mathbb{G}}^2 & & \\ \downarrow & \searrow & \\ V_{\mathbb{G}} & \longrightarrow & Q \\ \downarrow & & \\ \mathscr{O}_{\mathbb{G}}^2 & & \end{array}$$

The diagonal arrow (composition of $\mathscr{O}^2 \to V_{\mathbb{G}} \to Q$) gives our two sections of Q. Where does this map drop rank? At $\ell = \mathbb{P}(W) \in \mathbb{G}$, the map of fibers is

$$\begin{array}{ccc} k^2 & & \\ {\scriptstyle i}\big\downarrow & \searrow^{c} & \\ V & \xrightarrow{p} & W \\ {\scriptstyle j}\big\downarrow & & \\ k^2 & & \end{array}$$

where p corresponds to ℓ and j corresponds to ℓ_0. If $\operatorname{rank}(c) < 2$, then $\ker(V \to W) + \ker(V \to k^2)$ is at most 3-dimensional in V. If it is 2-dimensional, then in fact $\ell = \ell_0$; otherwise it is 3-dimensional and we can fill in the diagram

$$\begin{array}{ccc} V & \twoheadrightarrow & W \\ \big\downarrow & & \big\downarrow \\ k^2 & \twoheadrightarrow & k \end{array}$$

showing ℓ meets ℓ_0. Therefore $c_1(Q) = \{\, \ell \subset \mathbb{P}^3 \mid \ell \text{ meets a fixed } \ell_0 \subset \mathbb{P}^3 \,\}$. This is also a Schubert variety; write σ_{ℓ_0} for it.

Notice that $\sigma_{\mathbb{P}^2} \cong \mathbb{G}(\mathbb{P}^1, \mathbb{P}^2)$ is 2-dimensional, so it is codimension 2 in $\mathbb{G} = \mathbb{G}(\mathbb{P}^1, \mathbb{P}^3)$. Also, σ_{ℓ_0} is (almost) a $\mathbb{P}^2$ bundle over $\ell_0 = \mathbb{P}^1$ because there are a $\mathbb{P}^2$ of lines ℓ through any given point in $\ell_0 \subset \mathbb{P}^3$. The only thing keeping this from being a $\mathbb{P}^2$ bundle is the fact that we only include ℓ_0 once in σ_{ℓ_0}, instead of once for each point in ℓ_0. Apart from this single point, we can see that σ_{ℓ_0} is 3-dimensional, hence codimension 1 in $\mathbb{G}$.

We will need one more type of Schubert variety: for $p \in \mathbb{P}^3$, let σ_p be the locus in $\mathbb{G}$ of lines through p. Note $\dim \sigma_p = 2$, so $\operatorname{codim}_{\mathbb{G}}(\sigma_p) = 2$.

Exercise 29. As before, check that σ_ℓ and σ_p are closed and irreducible, and for any other line m and point q, σ_ℓ is rationally equivalent to σ_m (in fact, linearly equivalent) and σ_p is rationally equivalent to σ_q.

Now we know the Chern classes of Q:

$$c(Q) = 1 + \sigma_\ell + \sigma_\Pi$$

for a line ℓ and a 2-plane Π.

We can work out all the products of Schubert varieties σ_p, σ_ℓ, σ_Π.

- First, $\sigma_\ell^2 = \sigma_\Pi + \sigma_p$. To see this, replace one ℓ with ℓ' which is distinct from ℓ but meets ℓ at a point p. Then $\sigma_\ell \cap \sigma_{\ell'}$ is the locus in $\mathbb{G}$ of lines m in $\mathbb{P}^3$ that meet both ℓ and ℓ'. But m meets ℓ and ℓ' iff either m is in the 2-plane $\Pi = \Pi(\ell, \ell')$ spanned by ℓ and ℓ', or m passes through p. Therefore $\sigma_\ell \cap \sigma_{\ell'} = \sigma_\Pi \cup \sigma_p$. The intersection is proper, so it does compute the product in the Chow ring; the intersection is not transversal, but in this case the multiplicities of the components σ_Π and σ_p will be 1 because the intersection is generically transversal. So
$$\sigma_\ell^2 = \sigma_\ell \sigma_{\ell'} = \sigma_\ell \cap \sigma_{\ell'} = \sigma_\Pi + \sigma_p.$$
(If we choose an ℓ' skew to ℓ, then the locus of m's we get is still rationally equivalent to $\sigma_\Pi + \sigma_p$, though it is less obvious geometrically.)
- $\sigma_\Pi \sigma_p = 0$: Pick $p \notin \Pi$. Then a line $\ell \subset \Pi$ cannot pass through p. (If $p \in \Pi$, then $\sigma_\Pi \cap \sigma_p$ is 1-dimensional, but it is not a proper intersection.)

- $\deg \sigma_\Pi^2 = 1$: Given two distinct $\mathbb{P}^2$'s in $\mathbb{P}^3$, their intersection is 1-dimensional, so there is exactly one line contained in both planes.
- $\deg \sigma_p^2 = 1$: Similarly, for two distinct points, there is exactly one line through both.
- The products $\sigma_\Pi \sigma_\ell$ and $\sigma_p \sigma_\ell$ are each equal to yet another type of Schubert variety: given a plane Π and a point $p \in \Pi$, let $\sigma_{p,\Pi} = \{\, \ell | p \in \ell \subset \Pi \,\}$. Then $\sigma_\Pi \sigma_\ell = \sigma_p \sigma_\ell = \sigma_{p,\Pi}$. The verification is left as an exercise. As with the other Schubert varieties, $\sigma_{p,\Pi}$ is irreducible and independent (up to rational equivalence) of the choice of p and Π.

This will allow us to compute with the Chern classes of Q. For example, $\deg c_1(Q)^4 = \deg \sigma_\ell^4 = \deg(\sigma_\Pi + \sigma_p)^2 = 2$.

Remark 30. The Schubert varieties $\sigma_\ell \in A^1(\mathbb{G})$, $\sigma_p, \sigma_\Pi \in A^2(\mathbb{G})$, and $\sigma_{p,\Pi} \in A^3(\mathbb{G})$ generate $A^*(\mathbb{G})$ as a k-algebra, and over $\mathbb{C}$, the corresponding cohomology classes generate the cohomology ring $H^*(\mathbb{G}) \cong A^*(\mathbb{G})$.

The study of Schubert varieties leads one to the vast area known as Schubert calculus. For more information, see [7].

Going in a different direction, the geometry of $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^3)$ contains much more than just the Schubert varieties; its study is one of the classical foundations of algebraic geometry. For an introduction, see [4].

For the rest of these notes, we will apply a very little bit of Schubert calculus to Chern classes of appropriate bundles on Grassmanians in order to answer enumerative questions.

8. Lines on a cubic surface in $\mathbb{P}^3$

Let F be a polynomial defining a cubic surface X in $\mathbb{P}^3$. Then $F \in \mathrm{Sym}^3(V)$. (Remember, since $\mathbb{P}^3$ is the set of quotients of V, the homogeneous coordinates on $\mathbb{P}^3$ come from V, not V^*.) A line $\ell = \mathbb{P}(W) \subset \mathbb{P}^3$ lies in X iff $F|\ell = 0$; from $V \twoheadrightarrow W$, we get

$$\begin{array}{ccc} \mathrm{Sym}^3 V & \ni & F \\ \downarrow & & \downarrow \\ \mathrm{Sym}^3 W & \ni & F|\ell \end{array}$$

Which W have $F|\ell = 0$? Globalizing over $\mathbb{G}$, we have the same map $V_\mathbb{G} \twoheadrightarrow Q$, hence $\mathrm{Sym}^3 V_\mathbb{G} \twoheadrightarrow \mathrm{Sym}^3 Q$, and F defines a map $k \to \mathrm{Sym}^3 V$, hence a global section $\mathscr{O}_\mathbb{G} \to V_\mathbb{G}$. We have a diagram

$$\begin{array}{ccc} \mathrm{Sym}^3 V_\mathbb{G} & \twoheadrightarrow & \mathrm{Sym}^3 Q \\ \uparrow & \nearrow & \\ \mathscr{O}_\mathbb{G} & & \end{array}$$

and the locus of lines on X is the zero locus of the dashed line!

Since $\mathrm{rank}(\mathrm{Sym}^3 Q) = 4$, the zero locus of the section has expected codimension 4, equal to the dimension of $\mathbb{G}$, so we expect the zero locus to be finite.

Well, we expect the zeros of this section to represent (be rationally equivalent to) $c_{\mathrm{top}}(\mathrm{Sym}^3 Q)$. We have worked out what this is in terms of the Chern classes of Q: it is

$c_4(\mathrm{Sym}^3 Q) = 9c_2(Q)(2c_1(Q)^2 + c_2(Q))$. Therefore

$$\begin{aligned} \deg c_4(\mathrm{Sym}^3 Q) &= \deg 9\sigma_{\mathbb{P}^2}(2(\sigma_{\mathbb{P}^2} + \sigma_p) + \sigma_{\mathbb{P}^2}) \\ &= \deg(27\sigma_{\mathbb{P}^2}^2 + 18\sigma_{\mathbb{P}^2}\sigma_p) \\ &= 27. \end{aligned}$$

That is, we expect the section to have 27 zeros—so we expect the cubic surface X to contain 27 lines.

The cubic surface is very special:

Exercise 31. Use Chern classes to explain why a quadratic surface is expected to have a 1-dimensional family of lines and a surface of degree ≥ 4 is expected to have no lines.

9. Lines on hypersurfaces in $\mathbb{P}^4$

Now, let $\dim V = 5$, and let $\mathbb{P}^4$ be the set of one-dimensional quotients $V \twoheadrightarrow k$. Let $\mathbb{G} = \mathbb{G}(\mathbb{P}^1, \mathbb{P}^4)$ be the Grassmanian of lines in $\mathbb{P}^4$, corresponding to surjections $V \twoheadrightarrow W$ with $\dim W = 2$. Note $\dim \mathbb{G} = 6$. As before, we have the trivial rank 5 bundle $V_{\mathbb{G}} = V \times \mathbb{G}$ and a rank 2 bundle Q on $\mathbb{G}$ with a surjection $V_{\mathbb{G}} \twoheadrightarrow Q$, where over $V \twoheadrightarrow W$ in $\mathbb{G}$, the fiber of Q is W and the bundle map is given over this point by $V \twoheadrightarrow W$.

Let F be a degree d polynomial defining a hypersurface X in $\mathbb{P}^4$, so $F \in \mathrm{Sym}^d V$. Then F defines a map $\mathscr{O}_{\mathbb{G}} \to \mathrm{Sym}^d V_{\mathbb{G}} \to \mathrm{Sym}^d Q$, where the zero locus of this map is the collection of lines in $\mathbb{P}^4$ lying in X. We expect the zero locus of this map to represent the top Chern class of $\mathrm{Sym}^d Q$, which is a bundle of rank $d+1$.

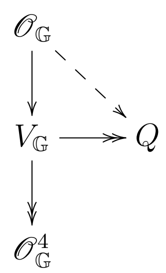
Claim 32. *We expect the set*

$$\{\text{ lines in } X\ \} = D(F : \mathscr{O}_{\mathbb{G}} \to \mathrm{Sym}^d Q)$$

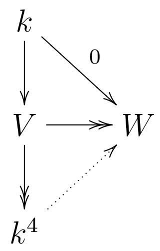
to be rationally equivalent to the top Chern class of $\mathrm{Sym}^d Q$.

Therefore the collection of lines lying on X is a subvariety of $\mathbb{G}$ which we expect to have codimension $d+1$. In particular, if $d = 4$, we expect X to have a one-dimensional family of lines on it; if $d = 5$, we expect X to have finitely many lines. In this section we will first find the expected number of lines on a quintic 3-fold, then examine the family of lines on a quartic 3-fold.

First, what are the Chern classes of Q? To find $c_2(Q)$, choose a hyperplane $\Sigma = \mathbb{P}^3 \subset \mathbb{P}^4$ corresponding to $V \twoheadrightarrow k^4$. Globalizing over $\mathbb{G}$ as before, we get a diagram of bundles



Then $c_2(Q)$ is represented by the zero locus of the map represented by the dashed arrow. This is zero at a particular $V \twoheadrightarrow W$ in $\mathbb{G}$ iff there is a factorization



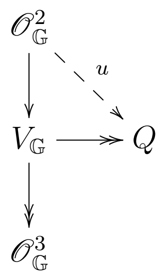
and any such factorization (any way to fill in the dotted arrow) must be a surjection. Therefore $c_2(Q)$ is represented by the locus of lines contained in Σ. For this Schubert variety we will write σ_Σ:

$$\sigma_\Sigma = \{\, \ell | \ell \subset \Sigma \,\}$$

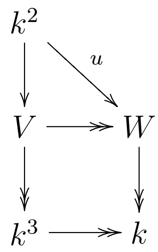
Exercise 33. As before, σ_Σ is closed and irreducible, and for any other hyperplane Σ', σ_Σ and $\sigma_{\Sigma'}$ are rationally equivalent.

Since $\sigma_\Sigma \cong \mathbb{G}(\mathbb{P}^1, \mathbb{P}^3)$ is 4-dimensional, it has codimension 2 in $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^4)$, which is just what we expect for $c_2(Q)$.

To find $c_1(Q)$, choose a 2-plane $\Pi = \mathbb{P}^2 \subset \mathbb{P}^4$ corresponding to $V \twoheadrightarrow k^3$. The diagram of bundles this time is



Now we expect $c_1(Q)$ to be represented by the degeneracy locus $D(u)$. The section u is zero at a line $\ell \in \mathbb{G}$ iff $\ell \subset \Pi$. And u is degenerate but not zero at ℓ corresponding to $V \twoheadrightarrow W$ iff the map has rank equal to 1, which happens iff there is a commutative diagram



This means there is a point $(V \twoheadrightarrow k)$ which is in both the line ℓ (since $W \twoheadrightarrow k$) and Π (since $k^3 \twoheadrightarrow k$). Therefore $c_1(Q)$ is represented by the locus of lines which meet the chosen Π. For this Schubert variety we write σ_Π; it is closed and irreducible, and again a different choice of Π leads to a rationally equivalent cycle on $\mathbb{G}$. Geometrically σ_Π is "almost" a $\mathbb{P}^3$ bundle over $\mathbb{P}^2$, so it has dimension 5 and codimension 1 in $\mathbb{G}$, as it should.

As before, it is useful to have on hand a few more types of Schubert varieties. For a line ℓ, let σ_ℓ be the locus of lines which meet ℓ, and for a point $p \in \mathbb{P}^4$, let σ_p be the locus of lines passing through p. Then σ_ℓ has codimension 2 in $\mathbb{G}$ and σ_p has codimension 3 in $\mathbb{G}$. (There are other types of Schubert varieties in $A^*(\mathbb{G})$, but we will not use them in these notes.)

Now we have worked out the Chern classes of Q. We can work out the top Chern class of $\mathrm{Sym}^5 Q$, which as we saw before, is expected to represent the lines on a quintic 3-fold in $\mathbb{P}^4$. Let a_1, a_2 be Chern roots of Q, so

$$c(Q) = 1 + \sigma_\Pi + \sigma_\Sigma = (1 + a_1)(1 + a_2).$$

Then

$$\mathrm{Sym}^5 Q = \mathscr{O}_{\mathbb{G}}(5a_1) \oplus \mathscr{O}_{\mathbb{G}}(4a_1 + a_2) \oplus \cdots \oplus \mathscr{O}_{\mathbb{G}}(5a_2),$$

so $c(\mathrm{Sym}^5 Q) = (1 + 5a_1)(1 + 4a_1 + a_2) \cdots (1 + 5a_2)$. Therefore the top Chern class of $\mathrm{Sym}^5 Q$ is

$$(5a_1)(4a_1 + a_2)(3a_1 + 2a_2)(2a_1 + 3a_2)(a_1 + 4a_2)(5a_2),$$

and using the relations $a_1 + a_2 = \sigma_\Pi$, $a_1 a_2 = \sigma_\Sigma$, we see that this is equal to

$$25\sigma_\Sigma(4\sigma_\Pi^2 + 9\sigma_\Sigma)(6\sigma_\Pi^2 + \sigma_\Sigma).$$

Let's start figuring out this product. First, up to rational equivalence, $\sigma_\Pi^2 = \sigma_\Pi \sigma_{\Pi'}$ is the locus of lines meeting two distinct 2-planes Π and Π'. By counting dimensions, we see that Π and Π' must meet in at least a point, and this is the generic case. But if we choose 2-planes that meet in a line ℓ, then we see that Π and Π' span a 3-dimensional subspace Σ, and a line m meets both Π and Π' iff either m is contained in Σ, or m meets the line ℓ. Therefore $\sigma_\Pi^2 = \sigma_\Pi \sigma_{\Pi'} = \sigma_\Sigma + \sigma_\ell$.

This shows that the top Chern class of $\mathrm{Sym}^5 Q$ is equal to

$$25\sigma_\Sigma(13\sigma_\Sigma + 4\sigma_\ell)(7\sigma_\Sigma + 6\sigma_\ell) = 25\sigma_\Sigma(91\sigma_\Sigma^2 + 106\sigma_\Sigma\sigma_\ell + 24\sigma_\ell^2).$$

There are three products here: σ_Σ^3, $\sigma_\Sigma^2\sigma_\ell$ and $\sigma_\Sigma\sigma_\ell^2$.

- $\deg \sigma_\Sigma^3 = 1$: If we pick three independent hyperplanes Σ_1, Σ_2 and Σ_3, then their intersection has codimension 3 in $\mathbb{P}^4$, so their intersection is a line. Therefore there is precisely one line contained in each Σ_i, so $\sigma_{\Sigma_1}\sigma_{\Sigma_2}\sigma_{\Sigma_3}$ is a single point in $\mathbb{G}$.
- $\sigma_\Sigma^2\sigma_\ell = 0$ (so the degree is also zero): For this one, first, $\sigma_\Sigma^2 = \sigma_\Sigma\sigma_{\Sigma'}$ is the locus of lines contained in the 2-plane $\Sigma \cap \Sigma'$. If we pick the line ℓ to be disjoint from this 2-plane (which is possible in $\mathbb{P}^4$) then it is impossible for any line m to be contained in $\Sigma \cap \Sigma'$ and meet ℓ.
- $\deg \sigma_\Sigma\sigma_\ell^2 = 1$: Take two lines ℓ, ℓ' which meet the 3-plane Σ in two distinct points (since we are in $\mathbb{P}^4$, this is the generic case). Then there is exactly one line contained in Σ meeting each line ℓ and ℓ', namely the line through the two points $\Sigma \cap \ell$ and $\Sigma \cap \ell'$.

Therefore

$$\deg c_6(\mathrm{Sym}^5 Q) = 25 \cdot 91 \cdot 1 + 25 \cdot 106 \cdot 0 + 25 \cdot 24 \cdot 1 = 2{,}875.$$

We see that the expected number of lines in a quintic 3-fold is 2,875.

Let's take another look at the quartic 3-fold, which we saw is expected to have a one-dimensional family of lines on it. So, the lines on a quartic 3-fold X sweep out a surface S in $\mathbb{P}^4$. What is the degree of this surface (assuming it really is a surface)?

The locus of lines on X is expected to be in the rational equivalence class of $c_5(\mathrm{Sym}^4 Q)$. By the splitting principle again, if $c(Q) = (1 + a_1)(1 + a_2)$, then

$$c(\mathrm{Sym}^4 Q) = (1 + 4a_1)(1 + 3a_1 + a_2)(1 + 2a_1 + 2a_2)(1 + a_1 + 3a_2)(1 + 4a_2).$$

Again using $a_1 + a_2 = c_1(Q) = \sigma_\Pi$ and $a_1a_2 = c_2(Q) = \sigma_\Sigma$, we see that

$$\begin{aligned} c_5(\mathrm{Sym}^4 Q) &= (4a_1)(3a_1 + a_2)(2a_1 + 2a_2)(a_1 + 3a_2)(4a_2) \\ &= 32(a_1a_2)(a_1 + a_2)(3(a_1 + a_2)^2 + 4(a_1a_2)) \\ &= 32\sigma_\Sigma\sigma_\Pi(3\sigma_\Pi^2 + 4\sigma_\Sigma) \\ &= 32\sigma_\Sigma\sigma_\Pi(7\sigma_\Pi + 3\sigma_\ell). \end{aligned}$$

Now, the degree of S is equal to the number of points in $S \cap H_1 \cap H_2$ for general hyperplanes H_1 and H_2. Say $H_1 \cap H_2 = \Pi \cong \mathbb{P}^2$. Then each line ℓ contained in X either meets Π at a point, or is contained in it, or does not meet it at all. If any line is contained in Π, then Π meets S at infinitely many points, so we have to pick different hyperplanes H_1 and H_2. So we can assume no line in S is contained in Π; then the degree of S is the number of lines which meet Π. But since the lines on X are parametrized by $c_5(\mathrm{Sym}^4 Q)$, this is just the degree of the cycle $\sigma_\Pi \cdot c_5(\mathrm{Sym}^4 Q)$, which is

$$32\sigma_\Sigma\sigma_\Pi^2(7\sigma_\Sigma + 3\sigma_\ell) = 32\sigma_\Sigma(\sigma_\Sigma + \sigma_\ell)(7\sigma_\Sigma + 3\sigma_\ell).$$

As before, we use $\deg \sigma_\Sigma^3 = \deg \sigma_\Sigma\sigma_\ell^2 = 1$ and $\sigma_\Sigma^2\sigma_\ell = 0$ to see that the degree of this zero cycle is 320. Therefore, for a general quartic 3-fold X, the surface S which is the union of the lines on X is expected to have degree 320 in $\mathbb{P}^4$.

10. Interpretations and Limitations

One must be careful to label all Chern class computations of the types shown above as "expected" values. Why?

Let's look a little more closely at the computation of the number of lines on a cubic surface. We started with a cubic polynomial on $\mathbb{P}^3$, that is, $F \in \mathrm{Sym}^3(V)$, where $\dim V = 4$. From this we obtained a global section of the bundle $\mathrm{Sym}^3(Q)$ on the Grassmanian $\mathbb{G}$. Now, we *expect* the zero locus of this section to represent the top Chern class of $\mathrm{Sym}^3(Q)$. By applying the splitting principle and a little elbow grease, we know what this top Chern class is in terms of Schubert varieties, which in particular tells us its degree.

The only step that's not fully rigorous is the one where we say that the zero locus of the section represents the top Chern class of the bundle. Provided the bundle satisfies some hypothesis—for example, it is sufficient that the bundle be globally generated—general global sections of the bundle will have the property that their degeneracy locus represents the right thing.

So the first question is whether this bundle $\mathrm{Sym}^3(Q)$ is globally generated. The answer is "yes": $\mathrm{Sym}^3(Q)$ is the surjective image of the trivial bundle $\mathrm{Sym}^3(V_\mathbb{G})$, which is globally generated.

The second question is whether this section coming from the cubic polynomial F really constitutes a *general* section of the bundle. In effect, what we have is a function

$$\mathrm{Sym}^3(V) \longrightarrow H^0(\mathbb{G}, \mathrm{Sym}^3(Q)),$$

and in fact it's a linear function between these two vector spaces. Therefore the image is a linear subspace, which is a Zariski closed subset. We know that the "good" set of global sections whose zeros represent the top Chern class of the bundle contains a Zariski open set in H^0. It could happen that the image of our map completely misses the "good" set, in which case *no* cubic surface has the right number of lines on it; or, the "good" set could contain the image, or the part of the image that corresponds to nonsingular cubic surfaces, or just some open set in the image. In these cases, all cubic surfaces, all nonsingular cubic surfaces, or just general cubic surfaces, would have the expected number of lines.

Which is the case? By displaying a single example of a cubic surface with the expected number of lines, we show that the image subspace and the "good" open set do meet, therefore meet along an open subset of the image subspace. For such an example, see [10], section 12.6; or [4]. Therefore, general cubic surfaces have 27 lines.

There are certainly cubic surfaces with the wrong number of lines: for example, a union of three hyperplanes has infinitely many lines. On the other hand, it turns out that every smooth cubic surface has exactly 27 lines.

Similarly, for quintic 3-folds, an example is needed to show that the expected number actually occurs. In this case, it is not even true that all smooth quintic 3-folds have 2,875 lines; for a "bad" nonsingular example:

Exercise 34. (From [3].) For $n \geq 4$, $d \geq 1$, the Fermat hypersurfaces $X_{n,d} \subset \mathbb{P}^n$ defined by $X_0^d + \cdots + X_n^d = 0$ are nonsingular and contain infinitely many lines.

("Hint": Pick a partition of $0, \ldots, n$ into pieces of size at least two each. For each piece $i_1, \ldots, i_r$, pick values of $X_{i_1}, \ldots, X_{i_r}$, not all zero, such that $\sum_{s=1}^r X_{i_s}^d = 0$. Each piece of the partition now gives a point in $\mathbb{P}^n$ with these values in the corresponding coordinates and zeros elsewhere; the linear span of these points lies in $X_{n,d}$ and contains infinitely many lines as long as at least one piece of the partition has size at least 3.)

For a much more detailed investigation of curves on the quintic 3-fold, see [12].

Similarly, we can produce finitely many lines on the $X_{3,d}$'s; in particular, we can produce 27 lines on $X_{3,3}$. A separate argument is needed, however, to show these are the only lines.

The moral is that in any computation of an "expected" number, an example is needed to show that the expected number actually occurs: otherwise it is possible that it never occurs. This issue is discussed from a different perspective in [10], section 12.6.

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