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On the Positive Solutions of the Matukuma Equation

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§**1. Introduction.** In 1930, based on his physical intuition, T. Matukama proposed the following equation as a mathematical model to describe the dynamics of globular cluster of stars ([M]),

$$\Delta u + \frac{1}{1+|x|^2} u^p = 0 \text{ in } \mathbf{R}^3 \ , \tag{1.1}$$

where $p > 1$ and $u > 0$ is the gravitational potential with $\int_{\mathbf{R}^3} \frac{u^p}{4\pi(1+|x|^2)} dx$ representing the total mass. His aim was to improve a model given earlier in 1915 by A.S. Eddington. (See [NY 1,2] for a more detailed history of these two models.) Since Matukuma equation (1.1) is rotationally invariant, the structure of positive radial solutions $u(r, \alpha)$ of the corresponding initial value problem

$$\begin{cases} u_{rr} + \frac{2}{r} u_r + \frac{1}{1+r^2} u^p = 0 & \text{in } [0, \infty), \\ u(0) = \alpha > 0, u_r(0) = 0, \end{cases} \tag{1.2}$$

was first studied by Matukuma. He then conjectured that

(i) *if $p < 3$, then $u(r; \alpha)$ has a finite zero for every $\alpha > 0$,*

(ii) *if $p = 3$, then $u(r; \alpha)$ is a positive entire solution with finite total mass for every $\alpha > 0$,*

(iii) *if $p > 3$, then $u(r, \alpha)$ is a positive entire solution with infinite total mass for every $\alpha > 0$.*

In 1938, Matukuma found an interesting exact solution $u(r; \sqrt{3}\,) = \sqrt{3/(1+r^2)}$ for (1.2) with $p = 3$ which confirms part of his conjecture. Since then there seems to be very little mathematical contribution in the literature on this equation until the recent works of W.-M. Ni and S. Yotsutani [NY1,2], Y. Li and W.-M. Ni [LN2], and E.S. Noussair and C.A. Swanson [NS]. First, it was observed in [NY2] and [LN2] that Eddington's model *does not have any positive entire solutions* (which perhaps indicates that Matukuma equation is indeed a better physical model). Concerning Matukuma's conjecture, the following results were established by [NY2] and [LN1,2] which shows that equation (1.2) is perhaps more delicate than Matukuma had expected.

Theorem A. *Let $u = u(r; \alpha)$ be the solution of (1.2).*

(i) *If $1 < p < 5$, then $u(r; \alpha)$ has a finite zero for every sufficiently large $\alpha > 0$.*

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(ii) *If* $1 < p < 5$, *then there exists an* $\alpha^* > 0$ *such that the solution* $u(r;\alpha^*)$ *is positive in* $[0,\infty)$ *with finite total mass.*

(iii) *If* $1 < p < 5$, *then* $u(r;\alpha)$ *is a positive entire solution with infinite total mass for every sufficiently small* $\alpha > 0$.

(iv) *If* $p \geq 5$, *then* $u(r;\alpha)$ *is a positive entire solution with infinite total mass for every* $\alpha > 0$.

Furthermore, for r *near* ∞,

$$\begin{cases} cr^{-1} \leq u(r) \leq c^{-1}r^{-1} & \text{if } u \text{ is in (ii), (fast decay),} \\ c(\log r)^{\frac{-1}{p-1}} \leq u(r) \leq c^{-1}(\log r)^{\frac{-1}{p-1}} & \text{if } u \text{ is in (iii) or (iv), (slow decay),} \end{cases} \tag{1.3}$$

where c *is some positive constant.*

Remark 1.1. The dividing exponent $p = 5$ is the so-called Sobolev critical power $\frac{n+2}{n-2}$ when $n = 3$.

Theorem A gives a nearly complete description on the structure of positive radial solutions of equation (1.2). On the other hand, it is an interesting and natural mathematical question that whether (1.1) possesses *only* positive radial entire solutions. In [LN2,3,4] we settle this problem concerning finite total mass solutions. Attempting to apply the method in [GNN2], one immediately encounters the fact that fundamental tool-(Lemma 2.1 in [GNN2, p.375]), no longer holds when p is close to 1. Our key new idea is to obtain precise asymptotic expansions of solutions at ∞ which turns out to be sufficient to get the "moving plane" process started near ∞. This "moving-plane" method was first devised by A.D. Alexandroff in 1956 and since then has been used by many mathematicians. (See, e.g. [BN], [CGS], [CL1,2], [GNN1,2], [H], [KKL], [Li], [L2] and [S].) The results in [LN2,3,4] yield the following.

Theorem B.

(i) *Let* $1 < p < 5$. *Then every bounded positive entire solution of equation (1.1) with finite total mass is radially symmetric about the origin and* $u_r < 0$ *in* $r > 0$. *Furthermore,*

$$\begin{cases} u(x) \quad = \frac{C}{|x|^{n-2}} + \frac{c}{|x|^{n-2+\gamma}} + \cdots + \frac{c}{|x|^{n-2+(2k+1)\gamma}} + \frac{c}{|x|^{n-1+\gamma}} + \cdots + \\ \qquad\quad + \frac{1}{|x|^{n-1+k\gamma}} + 0\left(\frac{1}{|x|^n}\right) \ \text{near } \infty, \end{cases} \tag{1.4}$$

where $\gamma = (p-1)(n-2)$, k *is the integer that* $k\gamma \leq 1 < (k+1)\gamma$, *and* $C > 0$

(ii) *Let $p \geq 5$. Then every bounded positive entire solution of (1.1) has infinity total mass.*

One of the key ingredients in the proof of Theorem B is a detailed analysis of the asymptotic behavior of finite total mass solutions at ∞ which gets the moving-plane device start near ∞. (See [LN2; Lemma 2.3], [LN3; Theorem 2.8], and [LN4; page 2]), e.g., one of the estimate in [LN2] implies that every bounded positive entire solution $u(x)$ of (1.1) with finite total mass must be bounded above by $c/|x|$ at ∞ for some constant $C > 0$. However, the radial symmetry of positive solutions with infinite total mass of (1.1) is left open in [LN2,3,4] due to the slow decay property of such solutions (see (1.3)). The main purpose of this paper is to settle this case for $1 < p < 5$.

Theorem 1. *Let $1 < p < 5$. Then every positive entire solution u of equation (1.1) is radially symmetric about the origin and $u_r < 0$ in $r > 0$.*

Now to understand the structures of all positive solutions of (1.1) is equivalent to understand the structures of such solutions of (1.2), and for which we have

Theorem 2. *Let $1 < p < 5$ and $u(r;\alpha)$ be the solution of (1.2). Then there exists a unique $\alpha^* > 0$, such that*

(i) if $u(x)$ is a positive entire finite total mass solution of (1.1), then $u(x) = u(|x|;\alpha^*)$ and u can be expanded according to (1.4) at ∞.

(ii) if $u(x)$ is a positive entire solution of (1.1) with infinite total mass, then there exists an $\alpha \in (0,\alpha^*)$ such that

$$u(x) = u(|x|;\alpha) = u(r)$$

and

$$u(x) = \frac{C_1}{(\log|x|)^{\frac{1}{p-1}}} - \frac{p\,C_1}{(p-1)^2(n-2)}\,\frac{\log(\log r)}{(\log r)^{\frac{p}{p-1}}} + 0\left(\frac{1}{(\log r)^{\frac{p}{p-1}}}\right) \tag{1.5}$$

at ∞.

Remark 1.2. The uniqueness of α^* in Theorem 2 is given by [Y]. (See also [KL] and [KYY] for various extensions of results of [Y]), while the expansion (1.5) is derived by [L1].

The crucial ingredients of the proof of Theorem 1 are the followings: first, the

by the study of their asymptotic behavior the maximum principle is observed to be applicable at ∞ to the operator

$$L = \Delta + K(x) \text{ if } K(x) \leq \epsilon |x|^{-2} \text{ at } \infty$$

to start the moving-plane process for solution u as long as the ϵ is sufficiently small . We shall organize this paper as follows. In §2 the asymptotic analysis of solutions of (1.1) is made. In §3 a symmetry result is proved for general nonlinear elliptic equations which includes Theorem 1 as a special case. It is clear from the context that the method used in this paper can be applied to cover a class of more general equations. For example, the symmetry result also holds for the following model proposed by J. Batt , W. Faltenbacher and E. Horst [BFH; page179]

$$\Delta u + \frac{|x|^{\lambda-2}}{(1+|x|^2)^{\lambda/2}} u^p = 0 \quad \text{in } \mathbf{R}^3,\ \lambda > 0\,, 1 < p < 5\ .$$

§2. Asymptotic Analysis.

Let u be a positive entire solution of the following equation

$$\Delta u + \frac{1}{1+|x|^2} u^p = 0 \quad \text{in } \mathbf{R}^n,\ n \geq 3,\ 1 < p < \frac{n+2}{n-2}\ . \tag{2.1}$$

Then we have

Lemma 2.1. *The following always holds*

$$\lim_{x\to\infty} (\log|x|)^{\frac{1}{p-1}} u(x) = \begin{cases} \left(\frac{n-2}{p-1}\right)^{\frac{1}{p-1}}\ , \\ or \\ 0\ . \end{cases} \tag{2.2}$$

Furthermore, if $\lim_{x\to\infty}(\log|x|)^{\frac{1}{p-1}} u(x) = 0$, *then there exists some positive constant* $c_0 > 0$ *such that*

$$\lim_{x\to\infty} |x|^{n-2} u(x) = c_0\ . \tag{2.3}$$

Proof. Let $v(x)$ be the Kelvin transform of u, i.e.

$$v(x) = |x|^{2-n} u(x/|x|^2) \quad \text{for } x \in \mathbf{R}^n \setminus \{0\}\ .$$

Then v satisfies

$$\int \Delta v + \tfrac{|x|^{p(n-2)-n}}{1+|x|^2} v^p = 0 \quad \text{in } \mathbf{R}^n \setminus \{0\},$$

Now Theorem A in [A] (see also [CGS]) implies that

$$\lim_{x\to 0}|x|^{n-2}(-\log|x|)^{\frac{1}{p-1}}v(x)=\begin{cases}\left(\frac{n-2}{p-1}\right)^{\frac{1}{p-1}}\ ,\\ \text{or}\\ 0,\end{cases}$$

$$\text{i.e.}\qquad \lim_{x\to\infty}(\log|x|)^{\frac{1}{p-1}}u(x)=\begin{cases}\left(\frac{n-2}{p-1}\right)^{\frac{1}{p-1}}\ ,\\ \text{or}\\ 0,\end{cases}$$

which gives (2.2).

If $\lim_{x\to\infty}(\log|x|)^{\frac{1}{p-1}}u(x)=0$, then Theorem 2.4 in [LN3] implies (2.3), which completes the proof.

Lemma 2.2. *Let φ be a positive entire solution of (1.2) with infinite total mass, then*

$$\begin{cases}\lim_{r\to\infty}(\log r)^{\frac{1}{p-1}}\varphi(r)=\left(\frac{n-2}{p-1}\right)^{\frac{1}{p-1}}\equiv C_1,\ \textit{and}\\ \lim_{r\to\infty}r(\log r)^{\frac{p}{p-1}}\varphi_r(r)=-C_1/(p-1)\ .\end{cases}\tag{2.4}$$

Proof. (2.4) is a direct consequence of Lemma 5.1 in [L1].

Remark 2.1. If a positive solution u of (1.1) satisfies (2.3), then u has finite total mass and the result in [LN4] shows that u is radially symmetric about the origin and satisfies (1.4). Hence we will only consider the case when $\lim_{x\to\infty}(\log|x|)^{\frac{1}{p-1}}u(x)=C_1$, which gives infinite total mass.

Let (r,θ) be the spherical coordinates of x in $\mathbf{R}^n$. Then for an integrable function $f(x)$, we define $\overline{f}$ to be its spherical mean, i.e.

$$\begin{aligned}\overline{f}(r)&=\frac{1}{n\omega_n r^{n-1}}\int_{|x|=r}f(x)dS_x\\ &=\frac{1}{n\omega_n}\int_{S^{n-1}}f(r,\theta)d\theta\ ,\end{aligned}$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. Also we let Δ_θ be the Laplace-Beltrami operator on S^{n-1}.

Theorem 2.3. *If u is a positive solution of (1.1) such that*

$$\lim_{x\to\infty}(\log|x|)^{\frac{1}{p-1}}u(x)=C_1>0,\ \textit{then}$$

where $\overline{u}(r) = \frac{1}{n\omega_n}\int_{S^{n-1}} u(r,\theta)d\theta$, φ *is as in Lemma 2.2, and* $d(x)$ *has polynomial decay at* ∞*; more precisely,*

$$d(x) = 0(|x|^{-\frac{n-1}{2}+\epsilon}) \text{ at } \infty \text{ for any } \epsilon > 0\,. \tag{2.6}$$

Remark 2.2. Since $\overline{u}$ is a supersolution of (2.1), namely

$$\begin{cases} \overline{u}_{rr} + \frac{n-1}{r}\overline{u}_r + \frac{1}{1+r^2}\overline{u}^p \le 0\,, \\ \overline{u}(0) = u(0) > 0\,, \\ \overline{u}_r(0) = 0\,, \end{cases}$$

it is very easy to conclude that $\overline{u}$ decreases strictly.

Proof of Theorem 2.3. Let $\varphi(|x|)$ be as in Lemma 2.2 and $u(x) = \varphi(|x|)(1+\psi(x))$. Then $\overline{u} = \varphi(1+\overline{\psi})$ and $u(x) = \overline{u}(|x|) + \varphi d$ where $d = \psi - \overline{\psi}$ such that $\psi, \overline{\psi}$ and d satisfy

$$\begin{cases} \Delta\psi + \frac{2\nabla\varphi\nabla\psi}{\varphi} + \frac{\varphi^{p-1}}{1+|x|^2}[(p-1)\psi + h(\psi)] = 0, \\ \lim_{x\to\infty}\psi(x) = 0\,, \end{cases} \tag{2.7}$$

with $h(t) = (1+t)^p - 1 - pt$, and so $h(t) = \frac{p(p-1)}{2}t^2 + 0(t^3)$ near $t = 0$, or,

$$\psi_{rr} + \left(\frac{n-1}{r} + 2\frac{\varphi_r}{\varphi}\right)\psi_r + \frac{\Delta_\theta\psi}{r^2} + \frac{\varphi^{p-1}}{1+r^2}[(p-1)\psi + h(\psi)] = 0, \tag{2.8}$$

$$\overline{\psi}_{rr} + \left(\frac{n-1}{r} + 2\frac{\varphi_r}{\varphi}\right)\bar{\psi}_r + \frac{\varphi^{p-1}}{1+r^2}[(p-1)\overline{\psi} + \overline{h(\psi)}] = 0, \tag{2.9}$$

and

$$d_{rr} + \left(\frac{n-1}{r} + 2\frac{\varphi_r}{\varphi}\right)d_r + \frac{\Delta_\theta d}{r^2} + \frac{\varphi^{p-1}}{1+r^2}[(p-1)d + h(\psi) - \overline{h(\psi)}] = 0, \tag{2.10}$$

(Note that $\overline{d} \equiv 0$).

Now, if we define

$$D(r) = \sqrt{\overline{\psi}^2} + \left(\frac{1}{\ldots}\int_{\ldots}(\psi - \overline{\psi})^2 d\theta\right)^{\frac{1}{2}} \tag{2.11}$$

then we can derive a differential inequality for $D(r)$ near $r=\infty$ similar to the ones in [K] by first multiplying (2.10) by d and integrating over S^{n-1}.

$$\int dd_{rr}+(\frac{n-1}{r}+2\frac{\varphi_r}{\varphi})\int dd_r+\frac{1}{r^2}\int d\Delta_\theta d+\frac{\varphi^{p-1}}{1+r^2}\int\{(p-1)d^2+d[h(\psi)-\overline{h(\psi)}]\}=0$$

Since $D^2=\frac{1}{n\omega_n}\int_{S^{n-1}}d^2d\theta$, we obtain $DD_r=\frac{1}{n\omega_n}\int_{S^{n-1}}dd_rd\theta$, and hence $DD_{rr}+D_r^2=\frac{1}{n\omega_n}\int_{S^{n-1}}(dd_{rr}+d_r^2)\geq\frac{1}{n\omega_n}\int_{S^{n-1}}dd_{rr}+D_r^2$ so that

$$DD_{rr}\geq\frac{1}{n\omega_n}\int_{S^{n-1}}dd_{rr}\ , \tag{2.12}$$

and

$$\begin{aligned}|\frac{1}{n\omega_n}\int_{S^{n-1}}d[h(\psi)-\overline{h(\psi)}]|&\leq(\frac{1}{n\omega_n}\int_{S^{n-1}}|h(\psi)-\overline{h(\psi)}|^2)^{1/2}(\frac{1}{n\omega_n}\int_{S^{n-1}}d^2)^{1/2}\\&\leq D(\frac{1}{(n-1)n\omega_n}\int_{S^{n-1}}|\nabla_\theta h(\psi)|^2)^{1/2}\end{aligned}$$

by the Poincare inequality on S^{n-1}. Therefore, we have

$$\begin{aligned}&DD_{rr}+(\frac{n-1}{r}+2\frac{\varphi_r}{\varphi})DD_r-\frac{1}{r^2}\frac{1}{n\omega_n}\int_{S^{n-1}}|\nabla_\theta d|^2+\\&+\frac{\varphi^{p-1}}{1+r^2}[(p-1)D^2+D(\frac{1}{(n-1)n\omega_n}\int_{S^{n-1}}|\nabla_\theta h(\psi)|^2)^{1/2}]\geq 0\ .\end{aligned}$$

Since $\psi\to 0$ as $x\to\infty$, we have

$$\begin{aligned}\int|\nabla_\theta h(\psi)|^2&=\int|h'(\psi)|^2|\nabla_\theta\psi|^2\\&=\int|h'(\psi)|^2|\nabla_\theta d|^2=o(1)\int|\nabla_\theta d|^2,\end{aligned}\tag{2.13}$$

for $h'(\psi)=p(p-1)\psi+o(\psi)$. Therefore it again follows from the Poincare inequality $(n-1)\int|d|^2\leq\int|\nabla_\theta d|^2$ that

$$\begin{cases}D_{rr}+(\frac{n-1}{r}+2\frac{\varphi_r}{\varphi})D_r-[\frac{n-1}{r^2}-\frac{\varphi^{p-1}}{1+r^2}(p-1+o(1))]D\geq 0,\\ \lim\limits_{r\to\infty}D(r)=0.\end{cases}\tag{2.14}$$

Let L be the linear operator defined in (2.14). Then it can be checked by (2.4) that for any $\varepsilon>0$, there exists $R_\varepsilon>0$ such that

$$\begin{cases}L(r^{1-n}\varphi^{-(1+p(1-\frac{2}{n})+\varepsilon)})\leq 0 \qquad \text{in } [R_\varepsilon,\infty),\\ n-1 \qquad \varphi^{p-1}\end{cases}\tag{2.15}$$

Then the maximum principle implies that

$$D(r) \le K r^{1-n} \varphi^{-(1+p(1-\frac{2}{n})+\varepsilon)}(r) \quad \text{in } [R_\varepsilon, \infty) \tag{2.16}$$

where $K = D(R_\varepsilon) R_\varepsilon^{n-1} \phi^{(1+p(1-\frac{2}{n})+\varepsilon)}(R_\varepsilon)$.

Now for any pair of large numbers $R_1 < R_2 < R_1 + 2$ we have

$$\begin{aligned} &\int_{R_1<|x|<R_2} d^2(x)dx \\ &= \int_{R_1}^{R_2} r^{n-1} dr \int d^2(r,\theta) d\theta = n\omega_n \int_{R_1}^{R_2} D^2(r) r^{n-1} dr \\ &\le n\omega_n K^2 \int_{R_1}^{R_2} r^{1-n} \phi^{-2(1+p(1-\frac{2}{n})+\varepsilon)}(r) dr \\ &\le C \frac{\phi^{-2(1+p(1-\frac{2}{n})+\varepsilon)}(R_1)}{R_1^{n-1}} \quad \text{for some constant } C > 0 \end{aligned} \tag{2.17}$$

Next we want to estimate the difference of the last two terms in (2.10)

$$\begin{aligned} h(\psi) - \overline{h(\psi)} &= h(d+\overline{\psi}) - \overline{h(d+\overline{\psi})} \\ &= h(d+\overline{\psi}) - h(\overline{\psi}) + h(\overline{\psi}) - \overline{h(d+\overline{\psi})} \\ &= h'(\overline{\psi})d + \frac{h''(\overline{\psi})+o(1)}{2} d^2 + \overline{h(\overline{\psi}) - h(d+\overline{\psi})} \\ &= h'(\overline{\psi})d + \frac{p(p-1)+o(1)}{2} d^2 + \overline{[-h'(\overline{\psi})d - \frac{h''(\overline{\psi})+o(1)}{2} d^2]} \\ &= h'(\overline{\psi})d + \frac{p(p-1)+o(1)}{2} d^2 - \frac{p(p-1)+o(1)}{2} n\omega_n D^2 \end{aligned}$$

since $\overline{d} = 0$.

Therefore, d satisfies from (2.10) and the above that

$$\Delta d + 2\frac{\nabla\varphi\nabla d}{\varphi} + \frac{\varphi^{p-1}}{1+|x|^2}[(p-1+o(1))d + \delta(r)] = 0 \tag{2.9$'$}$$

where $\delta(r) = 0((r^{1-n}\varphi^{-(1+p(1-\frac{2}{n})+\varepsilon)}(r))^2)$ at $r = \infty$.

Then for any x large, we have from the standard bootstrap method (see [GT]) that

$$\sup_{B_1(x)} |d(y)| \le C\{(\int_{B_2(x)} d^2)^{1/2} + [\int_{B_2(x)} (\frac{\varphi^{p-1}}{1+|x|^2} g(r))^{\frac{n+1}{2}}]^{\frac{2}{n+1}}\}$$

for some constant $C > 0$. Therefore (2.17) and the bound for g give

$$|d(x)| \le C \frac{\varphi^{-(1+p(1-\frac{2}{n})+\varepsilon)}(|x|)}{|x|^{\frac{n-1}{2}}} \quad \text{at } \infty \tag{2.18}$$

§3. Radial Symmetry.

We will now establish a symmetry result in this section which include Theorem 1 in the introduction as a special case. But first we need to introduce a few notations. For $x \in \mathbf{R}^n$ and $\lambda > 0$, let x^λ be the reflection of x with respect to the hyperplane $x_1 = \lambda$. i.e. $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$. And let $x' = (x_2, x_3, \ldots, x_n) \in \mathbf{R}^{n-1}, T_\lambda = \{(x_1, x') \in \mathbf{R}^n | x_1 = \lambda\}$ and $\Sigma_\lambda = \{(x_1, x') \in \mathbf{R}^n | x_1 < \lambda\}$. Then it is easy to see that

$$|x| - |x^\lambda| = \frac{4\lambda(\lambda - x_1)}{|x| + |x^\lambda|} > 0 \quad \text{in } \Sigma_\lambda \ . \tag{3.1}$$

Let u be a positive solution of the following fully nonlinear equation

$$\begin{cases} F[u] = F(x, u(x), Du(x), D^2u(x)) = 0 \text{ in } \mathbf{R}^n, n \geq 3 \ , \\ \lim\limits_{|x| \to \infty} u(x) = 0. \end{cases} \tag{3.2}$$

where F satisfies the following assumptions.

F 1. F is continuous in all its variables, C^1 in p_{ij}, and Lipschitz in p_i and s where p_{ij}'s are position variable for $\frac{\partial^2 u}{\partial x_i \partial x_j}, p_i$ for $\frac{\partial u}{\partial x_i}$ and s for u.

F 2. There exist e_1 and e_2 — two positive constants such that

$$e_1|\xi|^2 \leq F_{p_{ij}}(x, s, p_i, p_{ij})\xi_i\xi_j \leq e_2|\xi|^2, \text{ for } \xi \in \mathbf{R}^n \text{ and } (x, s, p_i, p_{ij}) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n^2} \ .$$

F 3. $F(x, s, p_i, p_{ij}) = F(|x|, s, p_i, p_{ij})$, and F is nonincreasing in $|x|$.

F 4. $F(x, s, p_1, p_2, \ldots, p_{i_0-1}, -p_{i_0}, p_{i_0+1}, \ldots, p_n, p_{11}, \ldots, -p_{i_0 j_0}, \ldots, -p_{j_0 i_0} \ldots p_{nn}) =$

$F(x, s, p_i, p_{ij})$ for $1 \leq i_0 \leq n, 1 \leq j_0 \leq n$ and $i_0 \neq j_0$.

F 5. There exist two functions f_1 and $f_2$defined for large x such that

$$\begin{cases} -\dfrac{x_i}{|x|^2} \cdot F_{pi}(x, s, p_i, p_{ij}) \leq f_1(|x|, s, |p_i|, |p_{ij}|) & \text{for } x \text{ near } \infty \\ F_s(x, s, p_i, p_{ij}) \leq f_2(|x|, s, |p_i|, |p_{ij}|) & \text{for } x \text{ near } \infty \end{cases}$$

where both f_1 and f_2 are nondecreasing in their last three arguments, namely in $s, |p_i|$ and $|p_{ij}|$.

Theorem 3.1. Let u be a positive C^2 solution of (3.2) with

where g_i's are nonincreasing function in their variables at ∞. Suppose that F satisfies $(F1-5)$ such that

$$\begin{cases} f_1(|x|, g_1(|x|), g_2(|x|), g_3(|x|)) \leq \alpha_1 |x|^{-2} \text{ at } \infty, \\ f_2(|x|, g_1(|x|), g_2(|x|), g_3(|x|)) \leq \alpha_2 |x|^{-2} \text{ at } \infty \end{cases} \tag{3.3}$$

for some real number α_1 and α_2.

Then u must be radially symmetric about some point x_0 in $\mathbf{R}^n$ and $u_r < 0$ for $r = |x - x_0| > 0$ provided that there exists a positive number α_0 such that for any $\lambda > 0$ and any rotational matrix T, we have

$$\varliminf_{\substack{x \to \infty \\ x \in \Sigma_\lambda}} |x|^{\alpha_0} [u(Tx) - u((Tx)^\lambda)] \geq 0 \tag{3.4}$$

with

$$e_2 \alpha_0^2 + (2e_2 + \alpha_1 - ne_1)\alpha_0 + \alpha_2 \leq 0 \,. \tag{3.5}$$

Remark 3.1. The proof of Theorem 3.1, in spirit, is close to the ones given in [Li] and [LN5]. In particular, it is an improvement of [Li; Theorem 4, p.589]. The crucial new elements are that first in Theorem 3.1, the decay rate of the coefficients of the linearized equation of (3.2) are not required to be strictly greater than 2 (see (3.3)) which are very often crucial in dealing with slow-decay solutions, second, just as in the case for the slow-decay solutions of Matukuma equation, the solution itself does not necessarily decay polynomially, but some estimate may be obtained to improve the decay rate of the lower bound of their differences, (see Theorem 2.3, (3.4) and (3.10)), so it is very important to utilize such fact.

Remark 3.2. If $\alpha_2 \geq 0$ (which is true for many cases), then a necessary condition for (3.5) to hold is $2e_2 < ne_1$ and $0 < \alpha_0 < n-2$. Hence Theorem 3.1 does not apply in such situations when $n = 2$.

Remark 3.3. If F is strictly decreasing in $|x|$, then it follows from the proof of Theorem 3.1 that u must be radially symmetric about the origin. On the other hand if F does depend on $|x|$ and is only nonincreasing in $|x|$ as in F3, then an example in [LN5] shows the symmetric point needs not to be the origin.

Proof of Theorem 3.1. First we define

By (3.3) and $\lim_{x\to\infty} u = 0$, there exists a $r_1 > r_0 > 0$ such that

$$\begin{cases} f_1 \le \alpha_1 |x|^{-2} \text{ and } f_2 \le \alpha_2 |x|^{-2} \text{ for } |x| \ge r_0, \\ \max_{\mathbf{R}^n - B_{r_1}(0)} u(x) < \min_{\bar{B}_{r_0}(0)} u. \end{cases} \tag{3.6}$$

Step 1. $[r_1, \infty] \subset \Lambda$.

For each $\lambda \ge r_1$, let $v(x) = u(x) - u(x^\lambda)$. Then in Σ_λ

$$\begin{aligned} & F(x, u(x), Du(x), D^2u(x)) - F(x, u(x^\lambda), Du(x^\lambda), D^2u(x^\lambda)) \\ \le\ & F(x, u(x), Du(x), D^2u(x)) - F(x^\lambda, u(x^\lambda), Du(x^\lambda), D^2u(x^\lambda)) \\ =\ & F(x, u(x), Du(x), D^2u(x)) - F(x^\lambda, u(x^\lambda), (Du)(x^\lambda), (D^2u)(x^\lambda)) \end{aligned}$$

by $(F3-4)$ and (3.1). Therefore, it follows from the assumptions of F that

$$\begin{cases} Lv \le 0 & \text{in } \Sigma_\lambda, \\ v = 0 & \text{on } T_\lambda, \end{cases} \tag{3.7}$$

where $L = a_{ij}D_{ij} + b_iD_i + c$ with

$$\begin{aligned} a_{ij}(x) &= \int_0^1 F_{p_{ij}}(x, u(x), Du(x), D^2u(x^\lambda) + t(D^2u(x) - D^2u(x^\lambda)))dt, \\ b_i(x) &= \int_0^1 F_{p_i}(x, u(x), Du(x^\lambda) + t(Du(x) - Du(x^\lambda)), D^2u(x^\lambda))dt, \\ \text{and } c(x) &= \int_0^1 F_s(x, u(x^\lambda) + t(u(x) - u(x^\lambda)), Du(x^\lambda), D^2u(x^\lambda))dt. \end{aligned}$$

Now let $w(x) = |x|^{\alpha_0} v(x)$ in Σ_λ. Since $\lambda \ge r_1$, (3.4) and (3.6), we have

$$\begin{cases} L_{\alpha_0} w \le 0 & \text{in } \Sigma_\lambda, \\ w > 0 & \text{on } \partial(\Sigma_\lambda \backslash B_{r_0}(0)), \\ \varliminf_{\substack{x\to\infty \\ x\in\Sigma_\lambda}} w \ge 0, \end{cases}$$

where $L_{\alpha_0} = a_{ij}D_{ij} + (b_i - 2\alpha_0|x|^{-2}a_{ij}x_j)D_i + c(\alpha_0, x)|x|^{-2}$ and $c(\alpha_0, x) = |x|^2 c(x) + \alpha_0(\alpha_0+2)|x|^{-2}a_{ij}x_ix_j - \alpha_0 b_i x_i - \alpha_0 \Sigma_{0=1}^n a_{ii}$. From the assumptions on F, u and (3.1), (3.3), (3.5), we conclude in $\Sigma_\lambda \backslash B_{r_0}(0)$ that

$$\begin{aligned} c(\alpha_0, x) &\le |x|^2 f_2 + \alpha_0(\alpha_0+2)e_2 + \alpha_0|x|^2 f_1 - ne_1\alpha_0 \\ &\le \alpha_2 + \alpha_0(\alpha_0+2)e_2 + \alpha_0\alpha_1 - ne_1\alpha_0 \le 0. \end{aligned}$$

Hence the maximum principle (see, e.g. [PW]) implies that $w > 0$ in $\Sigma_\lambda \backslash B_{r_0}(0)$. Then we conclude that $v > 0$ in Σ_λ together with the fact that $v > 0$ already in $B_{r_0}(0)$. Since v satisfies (3.7) with $v > 0$ in Σ_λ and $v = 0$ on T_λ, the Hopf boundary lemma applies here and we have $\frac{\partial u}{\partial x_1} = \frac{1}{2}\frac{\partial v}{\partial x_1} < 0$ on T_λ, which proves

Step 2. Λ is open in $(0,\infty)$.

Step 3. $\Lambda \cap (0,\infty) = (0,\infty)$ or $u(x) \equiv u(x^{\lambda_1})$ for some $\lambda_1 \geq 0$. That is, either

$$\begin{cases} u(x) & \equiv u(x^{\lambda_1}) \quad \text{for } x_1 < \lambda_1 \text{ and some } \lambda_1 \geq 0, \\ \frac{\partial u}{\partial x_1} < & 0 \text{ for } x_1 > \lambda_1, \end{cases} \tag{3.8}$$

or

$$u(x_1, x_2, \ldots, x_n) > u(-x_1, x_2, \ldots, x_n) \text{ for } x_1 < 0. \tag{3.9}$$

Since these two steps are almost identical to the corresponding ones in the section 2 of [LN5], we will omit the proof here.

Now if (3.8) occur, then u is symmetric in the x_1 direction about the hyperplane $T_{\lambda 1}$ and $\frac{\partial u}{\partial x_1} < 0$ for $x_1 > \lambda_1$. On the other hand, if (3.9) occur, then we can repeat the previous Steps 1–3 for u for the negative x_1-direction to conclude that either

$$u(x) \equiv u(x^{\lambda_2}) \text{ and } \frac{\partial u}{\partial x_1} > 0 \text{ for } x_1 < \lambda_2 \text{ with } \lambda_2 \leq 0, \tag{3.8$'$}$$

or

$$u(x_1, x_2, \ldots, x_n) < u(-x_1, x_2, \ldots, x_n) \text{ for } x_1 < 0. \tag{3.9$'$}$$

But (3.9) and (3.9′) can not happen the same time. Therefore u must be radially symmetric about some hyperplane T_λ and be strictly decreasing away from T_λ. Since the equation (3.2) and the assumptions in Theorem 3.1 are rotationally invariant, we can apply the above argument to every direction to conclude that u must be symmetric about some point x_0 in $\mathbf{R}^n$ and $u_r < 0$ for $r = |x - x_0| > 0$.

This completes the proof.

As a corollary of Theorem 3.1, we have

Corollary 3.2. Let $1 < p < 5$. Suppose that u is a positive solution of equation (1.1). Then u must be radially symmetric about the origin, $u_r < 0$ for $r > 0$ where $r = |x|$. Furthermore u satisfies either (1.4) or (1.5).

Proof. We need only consider the slow decay case. (See Remark 2.1) Let $v(x) = u(x) - u(x^\lambda)$. Then for $x \in \Sigma_\lambda$ and $\lambda > 0$, we have

$$v = \bar{u}(|x|) - \bar{u}(|x^\lambda|) + \phi(|x|)d(x) - \varphi(|x^\lambda|)d(x^\lambda)$$

by (2.5) and the fact that $\bar{u}$ decreases strictly. (See Remark 2.2.). Then (2.6) implies that for any $\lambda > 0$

$$\lim_{\substack{x \to \infty \\ x \in \Sigma_\lambda}} |x|^{\frac{n-1}{4}} v(x) \geq 0. \tag{3.10}$$

On the other hand, v satisfies

$$\Delta v + c(x) v \leq 0 \quad \text{ in } \Sigma_\lambda \ , \tag{3.11}$$

where

$$c(x) = \frac{p(tu(x) + (1-t)u(x^\lambda))^{p-1}}{1 + |x|^2} = 0(\frac{1}{|x|^2 \log |x|}) \text{ at } \infty.$$

for some $t \in (0,1)$. Therefore Theorem 3.1 can be applied with $e_1 = e_2 = 1, \alpha_0 = \frac{n-1}{4}$ and $\alpha_1 = 0, \alpha_2$ being very small in (3.11) to conclude that u is radially symmetric. It then follows from the Remark 3.3 that u must be radial about the origin, which completes the proof.

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