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\textbf{A B S T R A C T}

In this Letter, we propose to use the Cantor-type cylindrical-coordinate method in order to investigate a family of local fractional differential operators on Cantor sets. Some testing examples are given to illustrate the capability of the proposed method for the heat-conduction equation on a Cantor set and the damped wave equation in fractal strings. It is seen to be a powerful tool to convert differential equations on Cantor sets from Cantorian-coordinate systems to Cantor-type cylindrical-coordinate systems.

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\[ K^{2\alpha} \nabla^{2\alpha} T - \rho_c c_0 \frac{\partial^\alpha T}{\partial \tau^\alpha} = 0 \] (1)
or
\[ K^{2\alpha} \left( \frac{\partial^{2\alpha} T}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} T}{\partial y^{2\alpha}} + \frac{\partial^{2\alpha} T}{\partial z^{2\alpha}} \right) - \rho_c c_0 \frac{\partial^\alpha T}{\partial \tau^\alpha} = 0, \] (2)
where
\[ \nabla^{2\alpha} = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial z^{2\alpha}} \]
is the local fractional Laplace operator [4–6], whose local fractional differential operator is denoted as follows [4–11] (for other definitions, see also [12–25]):
\[ f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha} \bigg|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha f(x) - f(x_0)}{(x - x_0)^\alpha}, \] (3)
where \( \Delta^\alpha f(x) - f(x_0) \leq \Gamma(1+\alpha) \Delta^\alpha f(x) - f(x_0) \) and \( f(x) \) is satisfied with the following condition [4,15]:
\[ |f(x) - f(x_0)| \leq \tau^\alpha |x - x_0|^\alpha, \]
so that (see [4–18]).
\[ |f(x) - f(x_0)| < \varepsilon^x \]  

with \(|x - x_0| < \delta\), for \(\varepsilon, \delta > 0\) and \(\varepsilon, \delta \in \mathbb{R}\).

In a similar manner, for a given vector function \(F(t) = F_1(t)e_1^R + F_2(t)e_2^R + F_3(t)e_3^R\), the local fractional vector derivative is defined by \(\text{see} \ [4]\):

\[
E^{(a)}(t_0) = \frac{d^a F(t)}{dt^a} \quad \text{at} \quad t = t_0 = \lim_{t \to t_0} \frac{\Delta^n (F(t) - F(t_0))}{(t - t_0)^a}
\]

where \(e_1^R, e_2^R\) and \(e_3^R\) are the directions of the local fractional vector function.

The aim of this Letter is to investigate the Cantor-type cylindrical-coordinate method within the local fractional vector operator. The layout of the Letter is as follows. In Section 2, we propose and describe the Cantor-type cylindrical-coordinate method. In Section 3, we consider the testing examples. Finally, in Section 4, we present our concluding remarks and observations.

2. Cantor-type cylindrical-coordinate method

For the following Cantor-type cylindrical coordinates \([4]\):

\[
\begin{align*}
x^R &= R^\alpha \cos\theta^\alpha, \\
y^R &= R^\alpha \sin\theta^\alpha, \\
z^R &= z^R,
\end{align*}
\]

with \(R > 0, z \in (-\infty, +\infty), 0 < \theta < 2\pi\) and \(x^2 + y^2 = R^2z^2\), we have the local fractional vector given by

\[
r = R^\alpha \cos\theta^\alpha e_1^R + R^\alpha \sin\theta^\alpha e_2^R + z^R e_3^R,
\]

so that

\[
\begin{align*}
C_1^R &= \frac{1}{\Gamma(1 + \alpha)} \frac{\partial^\alpha r}{\partial R^\alpha} = \cos\theta^\alpha e_1^R + \sin\theta^\alpha e_2^R, \\
C_2^R &= \frac{1}{\Gamma(1 + \alpha)} \frac{\partial^\alpha \mathbf{R}}{\partial R^\alpha} = -\frac{\sin\theta^\alpha e_1^R}{R^\alpha} + \frac{R^\alpha}{\Gamma(1 + \alpha)} \cos\theta^\alpha e_2^R, \\
C_3^R &= \frac{1}{\Gamma(1 + \alpha)} \frac{\partial^\alpha z}{\partial z^R} = e_3^R.
\end{align*}
\]

Therefore, we obtain

\[
\begin{align*}
e_1^R &= \cos\theta^\alpha e_1^R + \sin\theta^\alpha e_2^R, \\
e_2^R &= -\sin\theta^\alpha e_1^R + \cos\theta^\alpha e_2^R, \\
e_3^R &= e_3^R,
\end{align*}
\]

where \(C_1^R = e_1^R, C_2^R = \frac{R^\alpha}{\Gamma(1 + \alpha)} e_2^R, C_3^R = e_3^R\).

Now, by making use of Eq. (9), we can write this last result in matrix form as follows:

\[
\begin{pmatrix}
e_1^R \\ e_2^R \\ e_3^R
\end{pmatrix} = \begin{pmatrix}
\cos\theta^\alpha & \sin\theta^\alpha & 0 \\
-\sin\theta^\alpha & \cos\theta^\alpha & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
e_1^R \\ e_2^R \\ e_3^R
\end{pmatrix}.
\]

which leads to

\[
E^{(a)}_i = T^{(a)}_{ij} E^{(a)}_j.
\]

where

\[
E_i^R = \begin{pmatrix}
e_1^R \\ e_2^R \\ e_3^R
\end{pmatrix}, \quad T^{(a)}_{ij} = \begin{pmatrix}
\cos\theta^\alpha & \sin\theta^\alpha & 0 \\
-\sin\theta^\alpha & \cos\theta^\alpha & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
E^{(a)}_i = \begin{pmatrix}
e_1^R \\ e_2^R \\ e_3^R
\end{pmatrix}.
\]

Here \(T^{(a)}_{ij}\) is fractal matrix, which is defined on the generalized Banach space \([5,6]\). The general basis vectors of two fractal spaces are defined, respectively, from the fractal tangent vectors \([4]\), namely,

\[
E^{(a)}_i = \begin{pmatrix}
e_1^R \\ e_2^R \\ e_3^R
\end{pmatrix}, \quad E^{(a)}_j = \begin{pmatrix}
e_1^R \\ e_2^R \\ e_3^R
\end{pmatrix}.
\]

In view of Eqs. (8) and (9), upon differentiating the Cantorian position with respect to the Cantor-type cylindrical coordinates implies that

\[
\begin{align*}
e_1^R &= \frac{1}{\Gamma(1 + \alpha)} \frac{\partial^\alpha \mathbf{r}}{\partial R^\alpha} = \cos\theta^\alpha e_1^R + \sin\theta^\alpha e_2^R, \\
e_2^R &= \frac{1}{R^\alpha \partial \theta^\alpha} = -\sin\theta^\alpha e_1^R + \cos\theta^\alpha e_2^R, \\
e_3^R &= \frac{1}{\Gamma(1 + \alpha)} \frac{\partial^\alpha \mathbf{r}}{\partial z^R} = e_3^R.
\end{align*}
\]

Eq. (14) is orthogonal and normalized everywhere \((\text{see} \ [5,6])\). Hence, we can define a local fractal basis with an orientation, which is derived from one fractal space to another fractal space. Based on this, a local fractional vector field can be defined as follows:

\[
r(R, \theta, z) = r(R, \theta, z) \cdot e_1^R + r_0(R, \theta, z) \cdot e_2^R + r_z(R, \theta, z) \cdot e_3^R
\]

are the projections of \(r\) on the local fractal basis vectors.

The local fractional derivatives with respect to the Cantor-type cylindrical coordinates are given by the local fractional differentiation through the Cantor coordinates as follows:

\[
\frac{\partial^\alpha}{\partial R^\alpha} = \left(\frac{\partial x}{\partial R}\right)^\alpha \frac{\partial}{\partial x^\alpha} + \left(\frac{\partial y}{\partial R}\right)^\alpha \frac{\partial}{\partial y^\alpha} + \left(\frac{\partial z}{\partial R}\right)^\alpha \frac{\partial}{\partial z^\alpha},
\]

\[
\frac{\partial^\alpha}{\partial \theta^\alpha} = R^\alpha \left(\frac{\partial x}{\partial \theta}\right)^\alpha \frac{\partial}{\partial x^\alpha} + \left(\frac{\partial y}{\partial \theta}\right)^\alpha \frac{\partial}{\partial y^\alpha} + \left(\frac{\partial z}{\partial \theta}\right)^\alpha \frac{\partial}{\partial z^\alpha},
\]

and

\[
\frac{\partial^\alpha}{\partial z^R} = R^\alpha \left(\frac{\partial x}{\partial z}\right)^\alpha \frac{\partial}{\partial x^\alpha} + \left(\frac{\partial y}{\partial z}\right)^\alpha \frac{\partial}{\partial y^\alpha} + \left(\frac{\partial z}{\partial z}\right)^\alpha \frac{\partial}{\partial z^\alpha}.
\]

where

\[
\nabla^R = e_1^R \cdot \nabla = \frac{\partial^\alpha}{\partial R^\alpha}, \quad \nabla^\alpha = R^\alpha e_1^R \cdot \nabla = \frac{\partial^\alpha}{\partial R^\alpha},
\]

\[
\frac{\partial^\alpha}{\partial z^R} = e_3^R \cdot \nabla = \frac{\partial^\alpha}{\partial z^R}.
\]

In light of Eq. (20), the local fractional gradient operator is described as follows:

\[
\nabla^R = e_1^R \nabla^R + e_2^R \nabla^\alpha + e_3^R \nabla^z = e_1^R \frac{\partial^\alpha}{\partial R^\alpha} + e_2^R \frac{\partial^\alpha}{\partial \theta^\alpha} + e_3^R \frac{\partial^\alpha}{\partial z^R}.
\]
A local fractional gradient operator in Cantor-type cylindrical-coordinate systems is expressed as follows:

$$
\nabla^\alpha u = e^\alpha_0 \frac{\partial u}{\partial \alpha} + e^\alpha_1 \frac{1}{R^\alpha} \frac{\partial u}{\partial \theta} + e^\alpha_2 \frac{\partial u}{\partial z},
$$

where

$$
\nabla^\alpha u = e^\alpha_0 \Delta^\alpha u(\alpha, \theta, z) + e^\alpha_1 \Delta^\alpha u(\alpha, \theta, z) + e^\alpha_2 \Delta^\alpha u(\alpha, \theta, z).
$$

The local fractional divergence operator of

$$
\mathbf{r} = e^\alpha_0 \mathbf{r}_R + e^\alpha_1 \mathbf{r}_\theta + e^\alpha_2 \mathbf{r}_z
$$
in Cantor-type cylindrical-coordinate systems has the form given by

$$
\nabla^\alpha \cdot \mathbf{r} = \left( e^\alpha_0 \frac{\partial \mathbf{r}_R}{\partial \alpha} + e^\alpha_1 \frac{\partial \mathbf{r}_\theta}{\partial \alpha} + e^\alpha_2 \frac{\partial \mathbf{r}_z}{\partial \alpha} \right) \cdot \left( e^\alpha_0 \mathbf{r}_R + e^\alpha_1 \mathbf{r}_\theta + e^\alpha_2 \mathbf{r}_z \right)
$$

The first term of Eq. (24) becomes

$$
e^\alpha_0 \frac{\partial \mathbf{r}_R}{\partial \alpha} = e^\alpha_0 \left( \frac{\partial^2 \mathbf{r}_R}{\partial \alpha^2} + \frac{\partial \mathbf{r}_\theta}{\partial \alpha} \frac{\partial \mathbf{r}_R}{\partial \theta} + \frac{\partial \mathbf{r}_z}{\partial \alpha} \frac{\partial \mathbf{r}_R}{\partial z} + \frac{\partial \mathbf{r}_z}{\partial \alpha} \right)
$$

The second term of Eq. (24) is given by

$$
e^\alpha_0 \frac{1}{R^\alpha} \frac{\partial \mathbf{r}_\theta}{\partial \alpha} = e^\alpha_0 \left( \frac{\partial^2 \mathbf{r}_\theta}{\partial \alpha^2} + \frac{\partial \mathbf{r}_R}{\partial \alpha} \frac{\partial \mathbf{r}_\theta}{\partial R} + \frac{\partial \mathbf{r}_z}{\partial \alpha} \frac{\partial \mathbf{r}_\theta}{\partial z} - \frac{\partial \mathbf{r}_\theta}{\partial \alpha} \right)
$$

Finally, the third term of Eq. (24) has the following form:

$$
e^\alpha_0 \frac{\partial \mathbf{r}_z}{\partial \alpha} = e^\alpha_0 \left( \frac{\partial^2 \mathbf{r}_z}{\partial \alpha^2} + \frac{\partial \mathbf{r}_R}{\partial \alpha} \frac{\partial \mathbf{r}_z}{\partial R} + \frac{\partial \mathbf{r}_\theta}{\partial \alpha} \frac{\partial \mathbf{r}_z}{\partial \theta} \right)
$$

By combining Eqs. (25) to (27), Eq. (24) can be reformulated as follows:

$$
\nabla^\alpha \cdot \mathbf{r} = e^\alpha_0 \frac{\partial \mathbf{r}_R}{\partial \alpha} + e^\alpha_1 \frac{\partial \mathbf{r}_\theta}{\partial \alpha} + e^\alpha_2 \frac{\partial \mathbf{r}_z}{\partial \alpha}.
$$

We notice that the local fractional curl operator of

$$
\mathbf{r} = e^\alpha_0 \mathbf{r}_R + e^\alpha_1 \mathbf{r}_\theta + e^\alpha_2 \mathbf{r}_z
$$
in Cantor-type cylindrical-coordinate systems can be computed by

$$
\nabla^\alpha \times \mathbf{r} = \left( e^\alpha_0 \frac{\partial \mathbf{r}_R}{\partial \alpha} + e^\alpha_1 \frac{\partial \mathbf{r}_\theta}{\partial \alpha} + e^\alpha_2 \frac{\partial \mathbf{r}_z}{\partial \alpha} \right) \times \left( e^\alpha_0 \mathbf{r}_R + e^\alpha_1 \mathbf{r}_\theta + e^\alpha_2 \mathbf{r}_z \right)
$$

The first term of Eq. (29) is determined by

$$
e^\alpha_0 \times \frac{\partial \mathbf{r}_R}{\partial \alpha} = e^\alpha_0 \left( \frac{\partial^2 \mathbf{r}_R}{\partial \alpha^2} + \frac{\partial \mathbf{r}_\theta}{\partial \alpha} \frac{\partial \mathbf{r}_R}{\partial \theta} + \frac{\partial \mathbf{r}_z}{\partial \alpha} \frac{\partial \mathbf{r}_R}{\partial z} \right)
$$

The second term of Eq. (29) is represented as follows:

$$
e^\alpha_0 \frac{1}{R^\alpha} \frac{\partial \mathbf{r}_\theta}{\partial \alpha} = e^\alpha_0 \left( \frac{\partial^2 \mathbf{r}_\theta}{\partial \alpha^2} + \frac{\partial \mathbf{r}_R}{\partial \alpha} \frac{\partial \mathbf{r}_\theta}{\partial R} + \frac{\partial \mathbf{r}_z}{\partial \alpha} \frac{\partial \mathbf{r}_\theta}{\partial z} - \frac{\partial \mathbf{r}_\theta}{\partial \alpha} \right)
$$

Finally, the third term of Eq. (24) has the following form:

$$
e^\alpha_0 \frac{\partial \mathbf{r}_z}{\partial \alpha} = e^\alpha_0 \left( \frac{\partial^2 \mathbf{r}_z}{\partial \alpha^2} + \frac{\partial \mathbf{r}_R}{\partial \alpha} \frac{\partial \mathbf{r}_z}{\partial R} + \frac{\partial \mathbf{r}_\theta}{\partial \alpha} \frac{\partial \mathbf{r}_z}{\partial \theta} \right)
$$

Substituting Eqs. (30) to (32) into Eq. (29), it follows that
\[ \nabla^\alpha \times \mathbf{r} = \left( \frac{\partial^\alpha \mathbf{r}_0}{\partial R^\alpha} e_0^\alpha - \frac{\partial^\alpha \mathbf{r}_0}{\partial R^\alpha} e_0^\alpha \right) + \left( - \frac{\partial^\alpha \mathbf{r}_R}{\partial R^\alpha} + \frac{\partial^\alpha \mathbf{r}_R}{\partial R^\alpha} \right) + \left( \frac{\partial^\alpha \mathbf{r}_0}{\partial R^\alpha} \right) \]

Consequently, the form of the local fractional Laplace operator is given by

\[ \nabla^{2\alpha} \phi = \left( e_k^{\alpha} e_0^{\alpha} + e_k^{\alpha} \right) \left( \frac{\partial^\alpha e_0^{\alpha}}{\partial R^\alpha} + e_0^{\alpha} \frac{\partial^\alpha e_0^{\alpha}}{\partial R^\alpha} \right) \]

The first term of Eq. (34) is rewritten as follows:

\[ e_k^{\alpha} \frac{\partial^\alpha e_0^{\alpha}}{\partial R^\alpha} e_0^{\alpha} \left( e_k^{\alpha} + e_0^{\alpha} \right) \frac{\partial^\alpha e_0^{\alpha}}{\partial R^\alpha} \]

\[ = e_k^{\alpha} \left( e_k^{\alpha} + e_0^{\alpha} \right) \frac{\partial^\alpha e_0^{\alpha}}{\partial R^\alpha} \frac{\partial^\alpha e_0^{\alpha}}{\partial R^\alpha} \]

\[ = \frac{\partial^2 \alpha \phi}{\partial R^2} \frac{\partial^2 \alpha \phi}{\partial R^2} \]

The second term of Eq. (34) is represented by

\[ e_0^{\alpha} \frac{\partial^\alpha e_0^{\alpha}}{\partial R^\alpha} \]

\[ = \frac{\partial^\alpha e_0^{\alpha}}{\partial R^\alpha} \frac{\partial^\alpha e_0^{\alpha}}{\partial R^\alpha} \]

\[ = \frac{\partial^2 \alpha \phi}{\partial R^2} \frac{\partial^2 \alpha \phi}{\partial R^2} \]

The third term of Eq. (34) is computed by means of the following formula:

\[ e_0^{\alpha} \frac{\partial^\alpha e_0^{\alpha}}{\partial R^\alpha} \]

\[ = \frac{\partial^2 \alpha \phi}{\partial R^2} \frac{\partial^2 \alpha \phi}{\partial R^2} \]

As a result of our computations, the expression for the local fractional Laplace operator assumes the following form:

\[ \nabla^{2\alpha} \phi = \frac{\partial^2 \alpha \phi}{\partial R^2} + \frac{\partial^2 \alpha \phi}{\partial y^2} + \frac{\partial^2 \alpha \phi}{\partial z^2} \]

(38)

3. Testing examples

In order to demonstrate the effectiveness of the proposed method, we have chosen several differential equations on Cantor sets.

Example 1. Let us consider the heat-conduction equation on Cantor sets without heat generation in fractal media, namely,

\[ K^{2\alpha} \left( \left( \frac{\partial^2 \alpha \phi}{\partial x^2} + \frac{\partial^2 \alpha \phi}{\partial y^2} + \frac{\partial^2 \alpha \phi}{\partial z^2} \right) \frac{\partial^2 \alpha \phi}{\partial x^2} \frac{\partial^2 \alpha \phi}{\partial y^2} \frac{\partial^2 \alpha \phi}{\partial z^2} \right) \]

\[ - \rho c \frac{\partial \alpha \phi}{\partial x^2} = 0. \]

By using Eq. (38), we find from Eq. (39) that

\[ K^{2\alpha} \left( \left( \frac{\partial^2 \alpha \phi}{\partial x^2} + \frac{\partial^2 \alpha \phi}{\partial y^2} + \frac{\partial^2 \alpha \phi}{\partial z^2} \right) \frac{\partial^2 \alpha \phi}{\partial x^2} \frac{\partial^2 \alpha \phi}{\partial y^2} \frac{\partial^2 \alpha \phi}{\partial z^2} \right) \]

\[ - \rho c \frac{\partial \alpha \phi}{\partial x^2} = 0, \]

which is the form of the heat-conduction equation on Cantor sets in Cantor-type cylindrical-coordinate systems.

Example 2. Consider the damped wave equation in fractal strings as given below:

\[ \frac{\partial^2 \alpha u(x, y, z, t)}{\partial t^2} - \left. \left( \frac{\partial \alpha u(x, y, z, t)}{\partial x^2} + \frac{\partial \alpha u(x, y, z, t)}{\partial y^2} + \frac{\partial \alpha u(x, y, z, t)}{\partial z^2} \right) \right) = 0. \]

Applying Eq. (38) to Eq. (41), we get

\[ \frac{\partial^2 \alpha u(R, \theta, Z, t)}{\partial t^2} - \left. \left( \frac{\partial \alpha u(R, \theta, Z, t)}{\partial R^2} + \frac{\partial \alpha u(R, \theta, Z, t)}{\partial \theta^2} \right) \right) = 0, \]

which is the form of the damped wave equation in fractal strings in Cantor-type cylindrical-coordinate systems.

4. Concluding remarks and observations

In our present investigation, we have proposed and developed a new Cantor-type cylindrical-coordinate method. The equivalent forms of differential equations on Cantor sets are then investigated within the proposed method. We notice that this method is different from the fractional complex-transform method on the fractional differential operator (see, for details, [8,26–28]). The former is an equivalent form of differential equations on Cantor sets converting from Cantorian-coordinate systems to Cantor-type cylindrical-coordinate systems, while the latter is one form
from Cantorian-coordinate systems to Cartesian-coordinate systems, because it is always most convenient to view curvilinear coordinate systems through the “eyes” of particular global Cartesian-coordinate systems in a flat Euclidean space. However, the Cantor-type cylindrical-coordinate method is used to view fractal curvilinear coordinate systems through the “eyes” of particular global Cantorian-coordinate systems in a flat fractal space. Several examples for the heat-conduction equation on a Cantor set and the damped wave equation in fractal strings were tested by applying the Cantor-type cylindrical-coordinate method. It is similar to the method based upon conversion from the Cartesian-coordinate systems to the cylindrical-coordinate systems.

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