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A New Successive Approximation to Non-homogeneous Local Fractional Volterra Equation

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Abstract – A new successive approximation approach to the non-homogeneous local fractional Volterra equation derived from local fractional calculus is proposed in this paper. The Volterra equation is described in local fractional integral operator. The theory of local fractional derivative and integration is one of useful tools to handle the fractal and continuously non-differentiable functions, was successfully applied in engineering problem. We investigate an efficient example of handling a non-homogeneous local fractional Volterra equation.

Keywords: Approximation; Non-homogeneous local fractional Volterra equation; Local fractional operator; local fractional calculus

1. Introduction

Engineering problems can be mathematically described by differential equations, and thus differential equations play very important roles in the solution of practical problems. Many initial and boundary value problems associated with differential equations can be transformed into problems of solving some approximate integral equations. However, some initial and boundary value domains are fractal curves, which are everywhere continuous but nowhere differentiable [1-10]. As a result, we cannot employ the classical approximate theory, which requires that the defined functions should be differentiable, to process fractal integral equations [11-13].

The theory of local fractional calculus [1-10], as one of useful tools to handle the fractal and continuously non-differentiable functions, was successfully applied in local fractional Fokker-Planck equation[1], the fractal heat conduction equation [3], fractal elasticity [4], local fractional Laplace equation [10], local fractional ordinary differential equations[8, 10], local fractional partial differential equation[8, 10], local fractional integral equations[12, 13], fractal signals [10, 14], local fractional Mellin transform in fractal space [5], local fractional Z transform in fractal space [6], local fractional short time transform in fractal space [9, 10, 14], local fractional wavelet transform in fractal space [9, 10, 14], Yang-Fourier Transforms in fractal space[7, 14], Yang-Laplace Transforms in fractal space [8, 14, 15], local fractional Fourier analysis [16], local fractional Stieltjes transform in fractal space [17], and the generalized Newton iteration method via generalized local fractional Taylor series[18].

In this paper we investigate a new successive approximation solution to the non-homogeneous local fractional Volterra integral equation of the second kind [12]. The structure of this paper is as follows. In section 2, we introduce the preliminary results on the local fractional calculus. The method of successive approximations is proposed in section 3. An illusive example is shown in section 4.

2. Preliminary Results

2.1. Local fractional continuity of functions

Definition 1
If there is [6-10]

\[ |f(x) - f(x_0)| < \varepsilon^\alpha \] (1)

with \(|x - x_0| < \delta\) for \(\varepsilon, \delta > 0\). Here, \(f(x)\) is so called local fractional continuous at \(x = x_0\), denote by \(\lim_{x \to x_0} f(x) = f(x_0)\). Then \(f(x)\) is called local fractional continuous on the interval \((a, b)\), denoted by

\[ f(x) \in C_\alpha(a,b) \] (2)

2.2. Local fractional derivatives

Definition 2
Setting \(f(x) \in C_\alpha(a,b)\), local fractional derivative of \(f(x)\) of order \(\alpha\) at \(x = x_0\) is defined [6-16]

\[ f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha} \bigg|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha(f(x) - f(x_0))}{(x-x_0)^\alpha} \] (3)

where \(\Delta^\alpha(f(x) - f(x_0)) \geq \Gamma(1+\alpha)\Delta(f(x) - f(x_0))\).

For any \(x \in (a,b)\), there exists

\[ f^{(\alpha)}(x) = D^\alpha_x f(x) \] (4)

denoted by
Local fractional derivative of high order is written in the form [11]
\[ f^{(k\alpha)}(x) = D_{x}^{\alpha} f^{(k \times \alpha)}(x), \] (6)
and local fractional partial derivative of high order is [11]
\[ \frac{\partial^{k\alpha} f(x)}{\partial x^{k\alpha}} = \frac{\partial^{\alpha} f^{(k \times \alpha)}}{\partial x^{\alpha}}. \] (7)

2.3. Local fractional integrals

Definition 3
Let \( f \in C_{\alpha}(a, b) \). Local fractional integral of \( f \) of order \( \alpha \) in the interval \([a, b]\) is [6-18]
\[ a I_{b}^{\alpha} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_{a}^{b} f(t)(dt)^{\alpha}, \] (8)
\[ = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta \to 0} \sum_{j=0}^{N-1} f(t_{j})(\Delta t_{j})^{\alpha}, \]
where \( \Delta t_{j} = t_{j+1} - t_{j}, \Delta = \max \{ \Delta t_{1}, \Delta t_{2}, \ldots \} \) and \([t_{j}, t_{j+1}]\), \( j = 0, 1, \ldots, N-1 \), \( t_{0} = a, t_{N} = b \), is a partition of the interval \([a, b]\).

The following results are valid:
(1) If \( a = b \), then we have [8, 9]
\[ a I_{b}^{\alpha} f(x) = 0. \] (9)
(2) If \( a < b \), then we have [8, 9]
\[ a I_{b}^{\alpha} f(x) = -b I_{a}^{\alpha} f(x). \] (10)
(3) The Mittag-Leffler function in fractal space can be written as [6, 7]
\[ E_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}. \] (11)

2.4. Local fractional Volterra integral equations

The most standard form of Volterra linear local fractional integral equations is of the form [12]
\[ u(x) = f(x) + \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)} \int_{0}^{x} K(x, t) u(t)(dt)^{\alpha}, \] (12)
where \( K(x, t) \) is the kernel of the local fractional integral equation, \( f(x) \) a local fractional continuous function of \( x \), and \( \lambda^{\alpha} \) a parameter. The limits of integration are function of \( x \) and the unknown function \( u(x) \) appears linearly under the integral sign. The equation (11) is called Volterra local fractional integral equation of second kind; the equation (12) is called local fractional Volterra integral equation of first kind.

3. The Method of Successive Approximations

In this method, we replace the unknown function \( u(x) \) under the integral sign of the local fractional Volterra equation (11) by any selective real-valued local fractional continuous function \( u_{0}(x) \), which is so called the zeroth approximation. This substitution will give the first approximation \( u_{1}(x) \) by
\[ u_{1}(x) = f(x) + \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)} \int_{0}^{x} K(x, t) u_{0}(x)(dt)^{\alpha}. \] (13)
It is obvious that \( u_{1}(x) \) is local fractional continuous if \( f(x) \), \( K(x, t) \), and \( u_{0}(x) \) are local fractional continuous. The second approximation \( u_{2}(x) \) can be written as
\[ u_{2}(x) = f(x) + \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)} \int_{0}^{x} K(x, t) u_{1}(x)(dt)^{\alpha}. \] (14)
In like manner, we obtain an infinite sequence of functions, which can be written as
\[ u_{n}(x), u_{1}(x), u_{2}(x), \ldots, u_{n}(x), \ldots \] (15)
that satisfies the recurrence formula
\[ u_{n}(x) = \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)} \int_{0}^{x} K(x, t) u_{n-1}(t)(dt)^{\alpha}. \] (16)
Consider the relation
\[ u_{2}(x) - u_{1}(x) = \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)} \int_{0}^{x} K(x, t) f(t)(dt)^{\alpha} \]
\[ = \lambda^{2\alpha} \psi_{2}(x), \] (17)
where \( \psi_{2}(x) \) is
\[ \psi_{2}(x) = \frac{1}{\Gamma(1+\alpha)} \int_{0}^{x} K(x, t)(dt)^{\alpha} \] (18)
Thus, it can be easily observed that
\[ u_{n}(x) = \sum_{m=0}^{\infty} \lambda^{mn} \psi_{m}. \] (19)
If \( u_0(x) = f(x) \), and further that
\[
\psi_m(x) = \frac{1}{(1+\alpha)^m} \int_0^x K(x,t)\psi_{m-1}(t)(dt)^\alpha \tag{20}
\]
where \( m = 1, 2, 3, \cdots \) and hence
\[
\psi_1(x) = \frac{1}{1+\alpha} \int_0^x K(x,t)f(t)(dt)^\alpha \tag{21}
\]
Thus interchanging the order of integration, we have that
\[
\psi_2(x) = \frac{1}{(1+\alpha)^2} \int_0^x K(x,t)(dt)^\alpha \left[ \frac{1}{(1+\alpha)} \int_0^t K(t,\tau)f(\tau)(d\tau)^\alpha \right] \tag{22}
\]
where
\[
K_2(x,\tau) = \frac{1}{1+\alpha} \int_0^\tau K_1(x,\tau)f(\tau)(d\tau)^\alpha.
\]
Similarly, we find in general
\[
\psi_m(x) = \frac{1}{(1+\alpha)^m} \int_0^x K_m(x,\tau)f(\tau)(d\tau)^\alpha, \quad m = 1, 2, 3, \cdots \tag{23}
\]
where the iterative kernels \( K_1(x,\tau) = K(x,\tau) \), \( K_2(x,\tau), K_3(x,\tau), \cdots \) are defined by the recurrence formula
\[
K_{m+1}(x,\tau) = \frac{1}{1+\alpha} \int_0^\tau K(x,\tau)K_m(\tau,\tau)(d\tau)^\alpha, \quad m = 1, 2, 3, \cdots \tag{24}
\]
Thus, we give the solution for \( u_n(x) \), which can be written as
\[
u_n(x) = f(x) + \sum_{m=1}^n \lambda^{(m)} \psi_m \tag{25}
\]
We can obtain the following equation:
\[
u_n(x) = f(x) + \sum_{m=1}^n \lambda^{(m)} \int_0^x K_m(x,\tau)f(\tau)(d\tau)^\alpha \tag{26}
\]
Hence it is also plausible that the solution, which is given by as \( n \to \infty \)
\[
\lim_{n \to \infty} \nu_n(x) = u(x)
\]
\[
u(x) = \frac{1}{1+\alpha} \left[ \lim_{n \to \infty} \sum_{m=1}^n \lambda^{(m)} K_m(x,\tau) \right] f(\tau)(d\tau)^\alpha \tag{27}
\]
\[
u(x) = \frac{\lambda^u}{1+\alpha} \int_0^x H(x,\tau;\lambda,\alpha)(d\tau)^\alpha
\]
where
\[
H(x,\tau;\lambda,\alpha) = \sum_{m=1}^{\infty} \lambda^{(m)} K_m(x,\tau)
\]
is known as the resolvent kernel.

4. A Simple Example

Solve the following Volterra integral equation
\[
K_{m+1}(x,\tau) = \frac{1}{\Gamma(1+\alpha)(1+\alpha)} \int_0^\tau K(x,\tau)K_m(\tau,\tau)(d\tau)^\alpha, \quad m = 1, 2, 3, \cdots \tag{28}
\]
Here, the kernel is
\[
K(x,\tau) = E_\alpha(x-\tau)^\alpha.
\]
The solution by the successive approximation is
\[
u(x) = f(x) + \frac{\lambda^u}{\Gamma(1+\alpha)} \int_0^x H(x,\tau;\lambda,\alpha)(d\tau)^\alpha
\]
where the resolvent kernel is given by
\[
u(x) = f(x) + \frac{\lambda^u}{\Gamma(1+\alpha)} \int_0^x H(x,\tau;\lambda,\alpha)(d\tau)^\alpha
\]
in which
\[
K_{m+1}(x,\tau) = \frac{1}{\Gamma(1+\alpha)} \int_0^\tau K(x,\tau)K_m(\tau,\tau)(d\tau)^\alpha, \quad m = 1, 2, 3, \cdots \tag{32}
\]
It is to be noted that \( K_1(x,\tau) = K(x,\tau) \). Therefore, we obtain that
\[
u_2(x,\tau) = \frac{1}{\Gamma(1+\alpha)} \int_0^\tau H(x,\tau;\lambda,\alpha)(d\tau)^\alpha
\]
\[
u_2(x,\tau) = \frac{\lambda^{(\alpha)}}{\Gamma(1+\alpha)} \int_0^\tau (d\tau)^\alpha
\]
\[
u_2(x,\tau) = \frac{(x-\tau)^\alpha}{\Gamma(1+\alpha)} E_\alpha (x-\tau)^\alpha
\]
Similarly, proceeding in this manner, we have that
\[
u(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^x E_\alpha (x-\tau)^\alpha E_\alpha (x-\tau)^\alpha (d\tau)^\alpha
\]
\[
u_2(x,\tau) = \frac{(x-\tau)^\alpha}{\Gamma(1+\alpha)} E_\alpha (x-\tau)^\alpha
\]
\[
u_2(x,\tau) = \frac{(x-\tau)^\alpha}{\Gamma(1+\alpha)} E_\alpha (x-\tau)^\alpha
\]
\[
u_2(x,\tau) = \frac{(x-\tau)^\alpha}{\Gamma(1+\alpha)} E_\alpha (x-\tau)^\alpha
\]
Therefore the resolvent kernel is
\[ H(x, \tau; \lambda, \alpha) = \sum_{m=0}^{\infty} \lambda^{m \alpha} K_{m+1}(x, \tau) \]

\[ = E_\alpha \left( (x-\tau)^\alpha \sum_{m=0}^{\infty} \frac{\lambda^m (x-\tau)^{\max}}{\Gamma(1+m\alpha)} \right) \]

\[ = E_\alpha \left( (x-\tau)^\alpha (1+\lambda)^\alpha \right). \]

5. Conclusions

We investigated the local fractional integrals and its results. Based on the fractional integral operator, we derive a new successive approximation method for solving a class of local fractional integral equation. We take in the local fractional integral operator. Special attention is devoted to the methodology for processing local fractional integral equations in a way for accessible to applied scientists and engineers. An illustrative example is given to elaborate the accuracy, simplification and reliable results. Our method can be applied to various industrial methods [19-21].

References


Vitae

Mr. Yang Xiao-Jun was born in 1981. He worked as a scientist and engineer in CUMT. His research interest includes Fractal mathematics (Geometry, applied mathematics and functional analysis), fractal Mechanics (fractal elasticity and fractal fracture mechanics, fractal rock mechanics and fractional continuous mechanics in fractal media), fractional calculus and its applications, fractional differential equation, local fractional integral equation, local fractional differential equation, local fractional integral transforms, local fractional short-time analysis and wavelet analysis, local fractional calculus and its applications and local fractional functional analysis and its applications.

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