Local Fractional Integral Equations and Their Applications

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Abstract – This letter outlines the local fractional integral equations carried out by the local fractional calculus (LFC). We first introduce the local fractional calculus and its fractal geometrical explanation. We then investigate the local fractional Volterra/ Fredholm integral equations, local fractional nonlinear integral equations, local fractional singular integral equations and local fractional integro-differential equations. Finally, their applications of some integral equations to handle some differential equations with local fractional derivative and local fractional integral transforms in fractal space are discussed in detail.

Keywords – Local fractional calculus; Volterra/ Fredholm integral equations; Nonlinear integral equations; Singular integral equations; Integro-differential equations

1. Introduction

The theory of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations. However, some initial and boundary value domains are fractal curves, which are everywhere continuous but nowhere differentiable. As a result, we cannot employ the classical calculus, which requires that the defined functions should be differentiable, to process ordinary local fractional differential equation (OLFDE) and local fractional partial differential equation (LFPDE) with fractal conditions. The theory of local fractional integrals and derivatives (fractal calculus)[1-17], as one of useful tools to handle the fractal and continuously non-differentiable functions, was successfully applied in local fractional Volkerenko-Planck equation[1, 2], anomalous diffusion and relaxation equation in fractal space[3, 4], the fractal heat conduction equation[5, 6], fractal-time dynamical systems[7, 8], fractal elasticity[9-10], local fractional diffusion equation[11], local fractional Laplace equation[20], local fractional ordinary differential equations[17, 18], local fractional partial differential equation[17, 18, 20, 26], local fractional integral equations[25, 30], fractal signals[17, 18, 19, 23], fractional Brownian motion in local fractional derivatives sense[16], fractal wave equation[20, 26].

This letter is to suggest some models for integral equations based on the local fractional calculus, and discuss their applications. The structure of this paper is as follows. In section 2, the preliminary results on the local fractional calculus and its fractal geometrical explanation. Local fractional Volterra integral equations and their applications are investigated in section 3. Local fractional Fredholm integral equations and their applications are investigated in section 4. Local fractional nonlinear integral equations and their applications are in section 5. Local fractional singular integral equations and their applications are investigated in section 6. Local fractional integro-differential equations and their applications are in section 7. Conclusions are in section 8.

2. Preliminary results

To begin with we will provide a brief introduction to local fractional calculus.

2.1. Local fractional continuity of functions

Definition 1 If there exists the relation [17-24, 30]
\[ |f(x) - f(x_0)| < \varepsilon|\alpha| \] \hspace{1cm} (2.1)
with \(|x - x_0| < \delta\), for \(\varepsilon, \delta > 0\) and \(\varepsilon, \delta \in \mathbb{R}\). Now \(f(x)\) is called local fractional continuous at \(x = x_0\), denote by \(\lim_{x \to x_0} f(x) = f(x_0)\). Then \(f(x)\) is called local fractional continuous on the interval \((a, b)\), denoted by \(f(x) \in C_{\alpha}(a, b)\).

Definition 2 A function \(f(x)\) is called a non-differentiable function of exponent \(\alpha, 0 < \alpha \leq 1\), which satisfy Hölder function of exponent \(\alpha\), then for \(x, y \in X\) such that [17-24, 30]
\[ |f(x) - f(y)| \leq C|x - y|^\alpha. \] \hspace{1cm} (2.3)

Definition 3 A function \(f(x)\) is called to be continuous of order \(\alpha, 0 < \alpha \leq 1\), or shortly \(\alpha\) continuous, when we have the following relation [17-24, 30]
\[ f(x) - f(x_0) = o \left( (x - x_0)^\alpha \right). \] \hspace{1cm} (2.4)

Remark 1. Compared with (2.4), (2.1) is standard definition of local fractional continuity. Here (2.3) is unified local fractional continuity.

Lemma (See [31])
If \((\Omega, d)\) and \((\Omega', d')\) are metric spaces, \(E \subset \mathbb{R}\) and 
\[ f : E \to \Omega \]
\[ \rho d(x,y) \leq d'(f(x), f(y)) \leq \tau d(x,y) \]  \hspace{1cm} (2.5)
where \(\rho\) and \(\tau\) are positives and finite constants, then 
\[ \rho^\alpha H^\alpha (E) \leq H^\alpha (f(E)) \leq \tau^\alpha H^\alpha (E) \]  \hspace{1cm} (2.6)
where each \(s \geq 0\) and \(H^\alpha\) is the s-dimensional Hausdorff measures.

Suppose \((\Omega, d)\) and \((\Omega', d')\) are metric spaces. A bijection 
\[ f : (\Omega, d) \to (\Omega', d') \]
is said to be a bi-Lipschitz mapping, if there are constants \(\rho, \tau > 0\) such that for all 
\[ x_1, x_2 \in \Omega, \]
\[ \rho d(x_1, x_2) \leq d' (f(x_1), f(x_2)) \leq \tau d(x_1, x_2). \]  \hspace{1cm} (2.7)

The following lemma is also a standard result in fractal geometry (see for example [32-35]).

**Lemma 2** (See [32])

If \( f : (\Omega, d) \to (\Omega', d') \) is a bi-Lipschitz mapping, then 
\[ \dim_H (A) = \dim_H (f(A)) \]  \hspace{1cm} (2.8)
for all \(A \in \Omega\).

**Lemma 3**

Let \(F\) be a subset of the real line and be a fractal. If 
\[ f : (F, d) \to (\Omega, d') \]
is a bi-Lipschitz mapping, then there is for constants \(\rho, \tau > 0\) and \(F \subset \mathbb{R}\), 
\[ \rho^\alpha H^\alpha (F) \leq H^\alpha (f(F)) \leq \tau^\alpha H^\alpha (F) \]
such that for all \(x_1, x_2 \in F\), 
\[ \rho^\alpha |x_1 - x_2|^\alpha \leq d(f(x_1), f(x_2)) \leq \tau^\alpha |x_1 - x_2|^\alpha. \]  \hspace{1cm} (2.9)
This result is directly deduced from fractal geometry. From **Lemma 1** and **Lemma 2** it is observed that that 
\[ \dim_H (F) = \dim_H (f(F)) = s. \]

**Theorem 4**

Let \(F\) be a subset of the real line and be a fractal. If 
\[ f : (\zeta, \xi, d) \to (\eta, \nu, d') \]
is a bi-Lipschitz mapping, then there is for constants \(\rho, \tau > 0\), 
\[ \rho^\alpha |\zeta - \xi|^\alpha \leq |f(\zeta) - f(\xi)| \leq \tau^\alpha |\zeta - \xi|^\alpha. \]  \hspace{1cm} (2.10)
where \(E = (\eta, \nu)\).

Proof. Let 
\[ H^\alpha (F \cap (\zeta, \xi)) = (\xi - \zeta)' = |\zeta - \xi|^\alpha \]  \hspace{1cm} [36],
by **Theorem 3** we get the result.

**Theorem 5**

Let \(F\) be a subset of the real line and be a fractal. If 
\( f(\Omega)\) is a bi-Lipschitz mapping, then there are any 
\(x_1, x_2 \in \Omega \subset \mathbb{R}\) and positive constant \(\gamma\) such that 
\[ |f(x_1) - f(x_2)| \leq \rho |x_1 - x_2|^\gamma. \]  \hspace{1cm} (2.11)
Proof. By using **Theorem 4**, considering \(\gamma = \max (\rho, \tau)\)
and \(\rho = \gamma^\gamma\) we obtain the result.

**Remark 2.** if \( f(x) \in C_{a} (a, b) \),
then \(\dim_{H} (F \cap (a, b)) = \dim_{H} (C_{a} (a, b)) = \alpha\) and 
\(C_{a} (a, b) = \{ f : f(x) \text{ is local fractional continuous}, x \in F \cap (a, b) \} \).

### 2.2. Local fractional derivatives

**Definition 4** Setting \( f(x) \in C_{a} (a, b) \), local fractional derivative of \( f(x) \) of order \(\alpha\) at \(x = x_{0}\) is defined [17-24, 30, 36, 37]
\[ f^{(\alpha)} (x_{0}) = \frac{d^\alpha f(x)}{dx^\alpha} \bigg|_{x=x_0} = \lim_{x \to x_{0}} \frac{\Delta^{\alpha}_{x_{0}} (f(x) - f(x_{0}))}{(x - x_{0})^\alpha}, \]  \hspace{1cm} (2.12)
where \(\Delta^{\alpha}_{x_{0}} (f(x) - f(x_{0})) \equiv (1 + \alpha) \Delta (f(x) - f(x_{0})).\)

For any \(x \in (a, b)\), there exists 
\[ f^{(\alpha)} (x) = D_{a}^{(\alpha)} f(x), \]
denoted by 
\[ f(x) \in D_{a}^{(\alpha)} (a, b). \]
Local fractional derivative of high order is written in the form 
\[ f^{(\alpha)} (x) = D_{a}^{(\alpha)} ... D_{a}^{(\alpha)} f(x), \]  \hspace{1cm} (2.6)
and local fractional partial derivative of high order, 
\[ \frac{\partial^{\alpha}_{x_{0}} f(x)}{\partial x^{\alpha}} = \frac{\partial^{\alpha}_{x_{0}}}{\partial x^{\alpha}} f(x). \]  \hspace{1cm} (2.7)

### 2.3. Local fractional integrals

**Definition 5** Setting \( f(x) \in C_{a} (a, b) \), local fractional integral of \( f(x) \) of order \(\alpha\) in the interval \([a, b]\) is defined [17-30, 36, 37]
\[ \mathcal{I}_{a}^{(\alpha)} f(x) \]
\[ = \frac{1}{\Gamma (1 + \alpha)} \left( I_{a}^{(\alpha)} f(t) (dt)^{\alpha} \right), \]  \hspace{1cm} (2.8)
where \( \Delta_{i} = t_{i+1} - t_{i} \), \( \Delta = \max \{ \Delta_{i}, \Delta_{i+1}, \Delta_{i+2}, ... \} \)
and \([t_{i}, t_{i+1}]\), \( j = 0,...,N-1 \), \( t_{0} = a, t_{N} = b \), is a partition of the interval \([a, b]\).
For any \(x \in (a, b)\), there exists 
\[ \mathcal{I}_{a}^{(\alpha)} f(x), \]
denoted by 
\[ f(x) \in I_{a}^{(\alpha)} (a, b). \]

**Remark 3.** If \( f(x) \in D_{a}^{(\alpha)} (a, b)\), or \( I_{a}^{(\alpha)} (a, b)\), we have 
\[ f(x) \in C_{a} (a, b). \]

Here, it follows that
\[ \mathcal{I}_{a}^{(\alpha)} f(x) = 0 \text{ if } a = b; \]  \hspace{1cm} (2.2)
\[ \mathcal{I}_{a}^{(\alpha)} f(x) = - \mathcal{I}_{a}^{(\alpha)} f(x) \text{ if } a < b; \]  \hspace{1cm} (2.3)
and 
\[ \mathcal{I}_{a}^{(\alpha)} f(x) = f(x). \]  \hspace{1cm} (2.4)
We only need here the following:
For any \( f(x) \in C_a(a,b) \), \( 0 < \alpha \leq 1 \), we have local fractional multiple integrals, which is written as [27]
\[
I_{\alpha}^{(a)} f(x) = \sum_{k=0}^{[a]} I_{\alpha}^{(a)} I_{\alpha}^{(a)} \ldots I_{\alpha}^{(a)} f(x).
\]
(2.5)
For \( 0 < \alpha \leq 1 \), \( f^{(\alpha)}(x) \in C_{\alpha}^{(a)}(a,b) \), then we have [27]
\[
(I_{\alpha} f(x))^{(\alpha)} = f(x),
\]
(2.7)
where \( I_{\alpha}^{(a)} f(x) = \sum_{k=0}^{[a]} I_{\alpha}^{(a)} I_{\alpha}^{(a)} \ldots I_{\alpha}^{(a)} f(x) \) and
\[
f^{(\alpha)}(x) = D_{\alpha}^{(a)} \ldots D_{\alpha}^{(a)} f(x) .
\]
The results are valid [37]:
(1) If \( \psi(x,y) \in C_a(a,b) \times C_a(c,d) \), then
\[
I_{\alpha}^{(a)} I_{\alpha}^{(a)} \psi(x,y) = \sum_{k=0}^{[a]} I_{\alpha}^{(a)} I_{\alpha}^{(a)} \ldots I_{\alpha}^{(a)} \psi(x,y).
\]
(2) If \( \psi(x,y,z) \in C_a(a,b) \times C_a(c,d) \times C_a(e,f) \), then
\[
I_{\alpha}^{(a)} I_{\alpha}^{(a)} I_{\alpha}^{(a)} \psi(x,y,z) = \sum_{k=0}^{[a]} I_{\alpha}^{(a)} I_{\alpha}^{(a)} I_{\alpha}^{(a)} \ldots I_{\alpha}^{(a)} \psi(x,y,z).
\]
2.4. Its fractal geometrical explanation

Definition 6 Let \( \alpha \) be an arbitrary but fixed real number. The integral staircase function \( S_{x,\alpha}(x) \) of order \( \alpha \) for a set \( F \) is given by [7, 8, 36]
\[
S_{x,\alpha}(x) = \begin{cases} \gamma^\alpha[F,a,x], & \text{if } x \geq a; \\ -\gamma^\alpha[F,x,a], & \text{if } x < a. \end{cases}
\]
Then we have the following results:
(a) The fractal mass function \( \gamma^\alpha[F,a,b] \) can written as
\[
\gamma^\alpha[F,a,b] = \frac{1}{\Gamma(1+\alpha)} \int_x^b (dt)^\alpha
\]
\[
= \frac{1}{\Gamma(1+\alpha)} H^\alpha(F \cap (a,b)).
\]
(3.1)
(b) we have [36]
\[
S_{x,\alpha}^a(x) - S_{x,\alpha}^b(x) = \gamma^\alpha[F,x,y],
\]
(3.2)
(c) if \( a < b < c \), we have
\[
\gamma^\alpha[F,a,b] + \gamma^\alpha[F,b,c] = \gamma^\alpha[F,a,c].
\]
Remark 4. From formula (a) we obtain that
\[
\gamma^\alpha[F,a,b]
\]
\[
= \frac{1}{\Gamma(1+\alpha)} \int_x^b (dt)^\alpha
\]
\[
= \frac{1}{\Gamma(1+\alpha)} \sum_{j=0}^{[\alpha]} (\Delta t_j)^\alpha
\]
\[
= (b-a)^\alpha
\]
(3.3)
Remark 5. From formula (c) we deduce to
\[
(b-a)^\alpha + (c-b)^\alpha = (c-a)^\alpha,
\]
which is called the theory of fractional set [18, 38]. Hence, we can understand it by fractal geometry:
\[
H^\alpha(F \cap (c-b)) + H^\alpha(F \cap (c-b)) = H^\alpha(F \cap (c-a))
\]
\[
\text{ie. } 1^\alpha + 2^\alpha = 3^\alpha.
\]
That is, the fractal geometric representation is that cantor set [0, 3] is equivalent to the sum of cantor set [0, 1] and cantor set [1, 3]. The dimension of cantor set is \( \alpha \), for \( 0 < \alpha \leq 1 \) and, \( 1^\alpha \), \( 2^\alpha \) and \( 3^\alpha \) are real line numbers on a fractional set [18, 38, 39].

3. Local fractional Volterra integral equations

3.1. Local fractional Volterra integral equations

The most standard form of Volterra linear local fractional integral equations is of the form [30]
\[
u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_x^\infty K(x,t) u(t)(dt)^\alpha
\]
(3.1)
\[
or f(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_x^\infty K(x,t) u(t)(dt)^\alpha
\]
(3.2)
where \( K(x,t) \) is the kernel of the local fractional integral equation, \( f(x) \) a local fractional continuous function of \( x \), and \( \lambda^\alpha \) a parameter. The limits of integration are function of \( x \) and the unknown function \( u(x) \) appears linearly under the integral sign. The equation (3.1) is called Volterra local fractional integral equation of second kind; the equation (3.2) is called Volterra local fractional integral equation of first kind.

3.2. Applications of local fractional Volterra integral equations

We directly observe that the local fractional differential equation of \( \alpha \) order
\[
\frac{d^\alpha \phi}{dx^\alpha} = f(x,\phi), (a \leq x \leq b)
\]
can be written immediately as the local fractional Volterra integral equation of second kind
\[
\phi(x) = \phi(a) + \frac{1}{\Gamma(1+\alpha)} \int_a^x f(t,\phi(t))(dt)^\alpha.
\]
(3.4)
We observe that the local fractional differential equation of $2\alpha$ order
\[ \frac{d^{2\alpha} \phi}{dx^{2\alpha}} = f(x,\phi), (a \leq x \leq b) \] (3.5)
carrying out an integration by parts, can be expressed immediately as
\[ \phi(x) = \phi(a) + \frac{(x-a)^{\alpha}}{\Gamma(1+\alpha)} \phi^{(\alpha)}(a) + \frac{1}{\Gamma(1+\alpha)} \int_{a}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(1+\alpha)} \int_{a}^{s} f(t,\phi(t))(dt)^{\alpha} \] (3.6)

4. Local fractional Fredholm integral equations

4.1. Local fractional Fredholm integral equations

The most standard form of Fredholm linear local fractional integral equations is given by the form
\[ u(x) = f(x) + \frac{\lambda^{a}}{\Gamma(1+\alpha)} \int_{a}^{b} K(x,t)u(t)(dt)^{\alpha} \] (4.1)
or
\[ f(x) = \frac{\lambda^{a}}{\Gamma(1+\alpha)} \int_{a}^{b} K(x,t)u(t)(dt)^{\alpha} \] (4.2)
where $K(x,t)$ is the kernel of the local fractional integral equation, $f(x)$ a local fractional continuous function of $x$, and $\lambda^{a}$ a parameter. The limits of integration $a$ and $b$ are constants and the unknown function $u(x)$ appears linearly under the integral sign.

The equation (4.1) is called Fredholm local fractional integral equation of second kind; this equation (4.2) is called Fredholm local fractional integral equation of first kind.

4.2. Applications of local fractional Fredholm integral equations

We consider the following boundary value problem:
\[ \frac{d^{\alpha} \phi}{dx^{\alpha}} = f(x,\phi), a \leq x \leq b \] (4.3)
carrying out an integration by parts, can be expressed immediately as
\[ \phi(x) = \phi(a) + \frac{(x-a)^{\alpha}}{\Gamma(1+\alpha)} \left[ \phi(b) - \phi(a) \right] + \frac{1}{\Gamma(1+\alpha)} \int_{a}^{b} K(x,t)f(t,\phi(t))(dt)^{\alpha} \] (4.4)
where
\[ K(x,t) = \begin{cases} \frac{(b-x)^{\alpha}}{\Gamma(1+\alpha)}, & 0 \leq x \leq t \leq b \\ \frac{(b-t)^{\alpha}}{\Gamma(1+\alpha)}, & 0 \leq t \leq x \leq b \end{cases} \]

5. Local fractional nonlinear integral equations

5.1. Local fractional nonlinear integral equations

If the unknown function $u(t)$ appearing under the integral sign is given in the functional form $F(u(t))$ such as the power of $u(t)$ is no longer unity. Then the Volterra and Fredholm local fractional integral equations are classified as nonlinear local fractional integral equations. In general, a nonlinear local fractional integral equation is defined as given in the following equations:
\[ u(x) = f(x) + \frac{\lambda^{a}}{\Gamma(1+\alpha)} \int_{a}^{b} K(x,t)F(u(t))(dt)^{\alpha} \] (5.1)
or
\[ u(x) = f(x) + \frac{\lambda^{a}}{\Gamma(1+\alpha)} \int_{a}^{b} K(x,t)F(u(t))(dt)^{\alpha} \] (5.2)

Equations (5.1) and (5.2) are called nonlinear Volterra local fractional integral equations and nonlinear Fredholm local fractional integral equations, respectively.

If we set $f(x) = 0$, in Volterra or Fredholm local fractional integral equations, then the resulting
\[ u(x) = \frac{\lambda^{a}}{\Gamma(1+\alpha)} \int_{a}^{b} K(x,t)F(u(t))(dt)^{\alpha} \] (5.3)
and
\[ u(x) = \frac{\lambda^{a}}{\Gamma(1+\alpha)} \int_{a}^{b} K(x,t)F(u(t))(dt)^{\alpha} \] (5.4)
The equations (5.3) and (5.4) are called homogeneous local fractional integral equation; otherwise it is called nonhomogeneous local fractional integral equation.

5.2. Applications of local fractional nonlinear integral equations

The Volterra or Fredholm nonlinear local fractional integral equation can be written in the form
\[ u(x) = f(x) + \frac{\lambda^{a}}{\Gamma(1+\alpha)} \int_{0}^{t} K(x,t)u^{n}(t)(dt)^{\alpha} \] (5.5)
or
\[ u(x) = f(x) + \frac{\lambda^{a}}{\Gamma(1+\alpha)} \int_{a}^{b} K(x,t)u^{n}(t)(dt)^{\alpha}. \] (5.6)

Equations (4.1) and (4.2) are called nonlinear Volterra local fractional integral equations and nonlinear Fredholm local fractional integral equations, respectively.

6. Local fractional singular integral equations

6.1. Local fractional singular integral equations

A singular local fractional integral equation is defined as an integral with the infinite limits or when the kernel.
7.2. Applications of local fractional integro-differential equations

The Volterra local fractional integro-differential equation can be written in the form

\[ u^{(2a)}(x) + u^{(a)}(x) = E_a\left(x^a\right) + \frac{x^a}{\Gamma(1+a)} \int_0^1 u(x)(dt)^a, \]

\( x \in [0,1] \) (7.3)

with the initial conditions

\( u(0) = 1 \) and \( u^{(a)}(0) = 1 \).

The Fredholm linear local fractional integro-differential equation can be written as

\[ u^{(2a)} + u^{(a)} = \frac{x^a}{\Gamma(1+a)} \int_0^1 E_a\left(t^a\right)u(t)(dt)^a, \]

\( x \in [0,1] \) (7.4)

with the initial conditions

\( u(0) = 1 \) and \( u^{(a)}(0) = 0 \).

The Volterra nonlinear local fractional integro-differential equation can be written in the form

\[ u^{(2a)} + u(x)u^{(a)} = E_a\left(x^a\right) + \frac{x^a}{\Gamma(1+a)} \int_0^1 u^2(x)(dt)^a, \]

\( x \in [0,1] \) (7.5)

with the initial conditions

\( u(0) = 1 \) and \( u^{(a)}(0) = 1 \).

The Fredholm nonlinear local fractional integro-differential equation can be written as

\[ u^{(2a)} + u(x)u^{(a)} = \frac{x^a}{\Gamma(1+a)} \int_0^1 E_a\left(t^a\right)u^2(t)(dt)^a, \]

\( x \in [0,1] \) (7.6)

with the initial conditions

\( u(0) = 1 \) and \( u^{(a)}(0) = 0 \).

8. Conclusions

In this letter, we study the local fractional calculus and its fractal geometrical explanation. The theory of integral equations in the local fractional calculus is one of the most useful mathematical tools in both pure and applied mathematics in the fractal fields. Here, we investigate the local fractional integral equations, such as local fractional Volterra/ Fredholm integral equations, local fractional nonlinear integral equations, local fractional singular integral equations and local fractional integro-differential equations. It is useful for engineers and scientists to handle some differential equations with local fractional derivative and local fractional integral transforms in fractal domains, which is applied to deal with the fractal problems, i.e., the fractal differential equations, fractal signals and the governing equations in fractal media. Hence, the theory of local fractional equations is of great significance for engineers and scientists to handle the problems and nonlinear behaviors of the fractal mathematics and engineering [40-44].

References
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