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Abstract – Local fractional calculus deals with everywhere continuous but nowhere differentiable functions in fractal space. The local fractional Fourier series is a generalization of Fourier series in fractal space, and the Yang-Fourier transform is a generalization of Fourier transform in fractal space. This letter points out the generalized sampling theorem for fractal signals (local fractional continuous signals) by using the local fractional Fourier series and Yang-Fourier transform techniques based on the local fractional calculus. This result is applied to process the local fractional continuous signals.

Keywords – Generalized sampling theorem; Fractal space; Yang-Fourier transform; Local fractional calculus; Local fractional continuous signals

1. Introduction

As is well-known, a signal is defined as a function. However, some signal functions are fractal curves, which are everywhere continuous but nowhere differentiable. As a result, we cannot employ the classical sampling theorem [1-6], which requires that the defined functions should be differentiable, to process the signals in fractal space.

Recently, local fractional calculus (fractal calculus), which was dealing with fractal functions, had been proposed and developed [7-12]. For these merits, local fractional calculus was successfully applied in the fractal elasticity [9], the fractal release equation [13], the fractal wave equation [15], the Yang-Fourier transforms [14, 17, 21, 24], the discrete Fourier transform [16, 21], the local fractional short time transform [11, 12] and the local fractional wavelet transform [11, 12].

In this letter, we apply the local fractional Fourier series and the Yang-Fourier transform to prove the generalized sampling theorem for fractal signals. The organization of this paper is as follows. In section 2, the preliminary results are shown. The generalized sampling theorem for fractal signals is investigated in section 3. Conclusions are presented in section 4.

2. Preliminaries

In this section we start with local fractional continuity of functions, and we introduce the notions of local fractional calculus, local fractional Fourier series and the Yang-Fourier transforms.

2.1 Local Fractional Continuity of Functions

Definition 1 If there exists \[ |x - x_0| < \delta \] for \( \delta > 0 \) and \( \varepsilon, \delta \in \mathbb{R} \), now \( f(x) \) is called local fractional continuous at \( x = x_0 \), denote by \( \lim_{x \to x_0} f(x) = f(x_0) \). Then \( f(x) \) is called local fractional continuous on the interval \( (a,b) \), denoted by \( f(x) \in C_\alpha (a,b) \).

Definition 2 A function \( f(x) \) is called a non-differentiable function of exponent \( \alpha \), \( 0 < \alpha \leq 1 \), which satisfies Hölder function of exponent \( \alpha \), then for \( x, y \in X \) such that \[ |f(x) - f(y)| \leq C|x - y|\alpha \].

Definition 3 A function \( f(x) \) is called to be continuous of order \( \alpha \), \( 0 < \alpha \leq 1 \), or shortly \( \alpha \) continuous, when we have that \[ f(x) - f(x_0) = o\left((x-x_0)^\alpha\right) \] .

Remark 1. Compared with (2.4), (2.1) is standard definition of local fractional continuity. Here (2.3) is unified local fractional continuity.

2.2 Local Fractional Derivatives and Integrals

Definition 4 Let \( f(x) \in C_\alpha (a,b) \). Local fractional derivative of \( f(x) \) of order \( \alpha \) at \( x = x_0 \) is given \[ f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha} \bigg|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha f(x) - f(x_0)}{(x-x_0)^\alpha} \].
where \( \Delta^\alpha (f(x) - f(x_0)) \geq \Gamma(1 + \alpha) \Delta (f(x) - f(x_0)) \).

For any \( x \in (a, b) \), there exists
\[
f^{(a)}(x) = D_x^{(a)} f(x),
\]
denoted by
\[
f(x) \in D_x^{(a)}(a, b).
\]
Local fractional derivative of high order is derived as
\[
f^{(k+1\alpha)}(x) = D_x^{(a)} ... D_x^{(a)} f(x) \quad (2.6)
\]
and local fractional partial derivative of high order
\[
\frac{\partial^{\alpha} f(x)}{\partial x^{\alpha}} = \frac{\partial^{\alpha}}{\partial x^{\alpha}} ... \frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x). \quad (2.7)
\]

**Definition 5** Let \( f(x) \in C_{x}(a, b) \). Local fractional integral of \( f(x) \) of order \( \alpha \) in the interval \([a, b]\) is given\[11,12, 14, 15, 18, 19, 25,26,27\]
\[
\int_a^b f(x) \, dx = \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(t) \, dt^{\alpha}, \quad (2.8)
\]
where \( \Delta = \max \{\Delta_1, \Delta_2, \Delta_3, ...\} \) and \([t_1, t_2, ...] \), \( j = 0, ..., N-1, t_0 = a, t_N = b \), is a partition of the interval \([a, b]\). For convenience, we assume that
\[
\int_a^b f(x) \, dx = 0 \quad \text{if} \quad a = b \quad \text{and} \quad \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \quad \text{if} \quad a < b.
\]
For any \( x \in (a, b) \), we get
\[
\int_a^x f(x) \, dx, \quad (2.9)
\]
denoted by
\[
f(x) \in I_{x}^{(a)}(a, b).
\]

**Remark 2** If \( f(x) \in D_x^{(a)}(a, b) \), or \( I_x^{(a)}(a, b) \), we have that
\[
f(x) \in C_{x}(a, b). \quad (2.10)
\]

### 2.3 Special Functions in Fractal Space

**Definition 6** The Mittag-Leffler function in fractal space is defined by\[11, 12\]
\[
E_{\alpha}(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+k\alpha)}, \quad x \in \mathbb{R} \text{ and } 0 < \alpha \leq 1. \quad (2.11)
\]

**Definition 7** The sine function in fractal space is by the expression\[11,12\]
\[
sin \alpha \, x^\alpha = \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k x^{2k}}{\Gamma(1+2k\alpha)}, \quad x \in \mathbb{R} \text{ and } 0 < \alpha \leq 1. \quad (2.12)
\]

**Definition 8** The cosine function in fractal space is given\[11,12\]
\[
\cos \alpha \, x^\alpha = \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k x^{2k}}{\Gamma(1+2k\alpha)}, \quad x \in \mathbb{R} \text{ and } 0 < \alpha \leq 1. \quad (2.13)
\]

The following rules hold\[11, 12\]:
\[
E_{\alpha}(x^\alpha) E_{\alpha}(y^\alpha) = E_{\alpha}((x+y)^\alpha);
\]
\[
E_{\alpha}(x^\alpha) E_{\alpha}(-y^\alpha) = E_{\alpha}((x-y)^\alpha);
\]
\[
E_{\alpha}(i^\alpha x^\alpha) E_{\alpha}(i^\alpha y^\alpha) = E_{\alpha}(i^\alpha(x+y)^\alpha);
\]
\[
E_{\alpha}(i^\alpha x^\alpha) = \cos \alpha \, x^\alpha + i^\alpha \sin \alpha \, x^\alpha;
\]
\[
\sin \alpha \, x^\alpha = \frac{E_{\alpha}(i^\alpha x^\alpha) - E_{\alpha}(-i^\alpha x^\alpha)}{2i^\alpha};
\]
\[
\cos \alpha \, x^\alpha = \frac{E_{\alpha}(i^\alpha x^\alpha) + E_{\alpha}(-i^\alpha x^\alpha)}{2};
\]
\[
\tan \alpha \, x^\alpha = \frac{\sin \alpha(2x^\alpha)}{1 + \cos \alpha(2x^\alpha)} = \frac{1 - \cos \alpha(2x^\alpha)}{\sin \alpha(2x^\alpha)};
\]
\[
\sin \alpha(2x^\alpha) = 2 \sin \alpha \, x^\alpha \cos \alpha \, x^\alpha;
\]
\[
\cos \alpha(2x^\alpha) = \cos^2 \alpha \, x^\alpha - \sin^2 \alpha \, x^\alpha;
\]
\[
\tan \alpha \, (x+y)^\alpha = \frac{\tan \alpha \, x^\alpha + \tan \alpha \, y^\alpha}{1 + \tan \alpha \, x^\alpha \tan \alpha \, y^\alpha};
\]
\[
\cos \alpha \, x^\alpha - \cos \alpha \, y^\alpha = -2 \sin \alpha \, \left(\frac{x+y}{2}\right)^\alpha \sin \alpha \, \left(\frac{x-y}{2}\right)^\alpha;
\]
\[
\sin_\alpha x^\alpha + \sin_\alpha y^\alpha = 2\sin_\alpha \left( \frac{x + y}{2} \right)^\alpha \cos_\alpha \left( \frac{x - y}{2} \right)^\alpha;
\]
\[
\sin_\alpha x^\alpha - \sin_\alpha y^\alpha = 2\cos_\alpha \left( \frac{x + y}{2} \right)^\alpha \sin_\alpha \left( \frac{x - y}{2} \right)^\alpha;
\]
\[
\cos_\alpha \left( x + y \right)^\alpha = \cos_\alpha x^\alpha \cos_\alpha y^\alpha - \sin_\alpha x^\alpha \sin_\alpha y^\alpha;
\]
\[
\cos_\alpha \left( x - y \right)^\alpha = \cos_\alpha x^\alpha \cos_\alpha y^\alpha + \sin_\alpha x^\alpha \sin_\alpha y^\alpha;
\]
\[
\sin_\alpha \left( x + y \right)^\alpha = \sin_\alpha x^\alpha \cos_\alpha y^\alpha + \cos_\alpha x^\alpha \sin_\alpha y^\alpha;
\]
\[
\cos_\alpha x^\alpha \cos_\alpha y^\alpha = \cos_\alpha \left( x + y \right)^\alpha + \cos_\alpha \left( x - y \right)^\alpha;
\]
\[
\sin_\alpha x^\alpha \sin_\alpha y^\alpha = -\frac{\cos_\alpha \left( x + y \right)^\alpha - \cos_\alpha \left( x - y \right)^\alpha}{2};
\]
\[
\sin_\alpha x^\alpha \cos_\alpha y^\alpha = \frac{\sin_\alpha \left( x + y \right)^\alpha + \sin_\alpha \left( x - y \right)^\alpha}{2};
\]
\[
\sin_\alpha (m x)^\alpha = \frac{\cos_\alpha \left( (m-n)x \right)^\alpha - \cos_\alpha \left( (m+n)x \right)^\alpha}{2};
\]
\[
\cos_\alpha (m x)^\alpha \sin_\alpha (n x)^\alpha = \frac{\sin_\alpha \left( (m+n)x \right)^\alpha - \sin_\alpha \left( (m-n)x \right)^\alpha}{2};
\]
\[
E_\alpha \left( i^n (n x)^\alpha \right) = \left( \cos_\alpha \left( nx \right)^\alpha + i^n \sin_\alpha \left( nx \right)^\alpha \right)^\alpha;
\]
\[
\sum_{k=1}^{n} \sin_\alpha (k x)^\alpha = \frac{\sin_\alpha \left( \frac{n x}{2} \right)^\alpha}{\sin_\alpha \left( \frac{x}{2} \right)^\alpha} \sin_\alpha \left( \frac{(n+1) x}{2} \right)^\alpha;
\]
\[
\sin_\alpha \left( \frac{x}{2} \right)^\alpha \neq 0;
\]
\[
\sum_{k=1}^{n} \cos_\alpha (k x)^\alpha = \frac{\sin_\alpha \left( \frac{n x}{2} \right)^\alpha}{\sin_\alpha \left( \frac{x}{2} \right)^\alpha} \cos_\alpha \left( \frac{(n+1) x}{2} \right)^\alpha;
\]
\[
\sin_\alpha \left( \frac{x}{2} \right)^\alpha \neq 0;
\]
\[
\frac{1}{2} + \sum_{k=1}^{n} \cos_\alpha (k x)^\alpha = \frac{\sin_\alpha \left( \frac{(2n+1) x}{2} \right)^\alpha}{2 \sin_\alpha \left( \frac{x}{2} \right)^\alpha};
\]
\[
\sin_\alpha \left( \frac{x}{2} \right)^\alpha \neq 0
\]

**Remark 3.** \( i^n \) is fractal imaginary unit, for more details, see [11, 12].

**2.4 Local fractional Fourier series**

**Definition 9** Suppose that 
\( f(x) \in C_\alpha (\infty, \infty) \) and \( f(x) \) be \( 2L \)-periodic. For \( k \in \mathbb{Z} \), local fractional Fourier series of \( f(x) \) is defined [11, 12, 17]
\[
f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos_\alpha \left( \frac{\pi \alpha}{l^\alpha} \right) + b_k \sin_\alpha \left( \frac{\pi \alpha}{l^\alpha} \right) \right),
\]
(2.14)
where
\[
a_k = \frac{1}{l^\alpha} \int_{-L}^{L} f(x) \cos_\alpha \left( \frac{\pi \alpha}{l^\alpha} \right) (dx)^\alpha
\]
(2.15)
and
\[
b_k = \frac{1}{l^\alpha} \int_{-L}^{L} f(x) \sin_\alpha \left( \frac{\pi \alpha}{l^\alpha} \right) (dx)^\alpha
\]
(2.16)
are the local fractional Fourier coefficients.

For local fractional Fourier series (2.14), the weights of the fractional trigonometric functions are calculated as
\[
a_k = \frac{1}{l^\alpha} \int_{-L}^{L} f(x) \cos \left( \frac{\pi}{l^\alpha} \right) (dx)^\alpha
\]
(2.17)
and
\[
b_k = \frac{1}{l^\alpha} \int_{-L}^{L} f(x) \sin \left( \frac{\pi}{l^\alpha} \right) (dx)^\alpha
\]
(2.18)

**Definition 10** Suppose that \( f(x) \in C_\alpha (\infty, \infty) \) and \( f(x) \) be \( 2L \)-periodic. For \( k \in \mathbb{Z} \), complex generalized Mittag-Leffler form of local fractional Fourier series of \( f(x) \) is defined [46, 47]
\[
f(x) = \sum_{k=\infty}^{\infty} C_k E_\alpha \left( \frac{\pi \alpha}{l^\alpha} \left( \frac{\pi \alpha}{l^\alpha} \right)^\alpha \right),
\]
(2.19)
where the local fractional Fourier coefficients is
\[
C_k = \frac{1}{(2L)^\alpha} \int_{-L}^{L} f(x) E_\alpha \left( -\frac{\pi \alpha}{l^\alpha} \left( \frac{\pi \alpha}{l^\alpha} \right)^\alpha \right) (dx)^\alpha
\]
(2.20)
with \( k \in \mathbb{Z} \).

The above generalized forms of local fractional series are valid and are also derived from the generalized Hilbert space [11, 17].

For local fractional Fourier series (2.19), the weights of the Mittag-Leffler functions are written in the form
\[
C_k = \frac{1}{(2L)^\alpha} \int_{-L}^{L} f(x) E_\alpha \left( -\frac{\pi \alpha}{l^\alpha} \left( \frac{\pi \alpha}{l^\alpha} \right)^\alpha \right) (dx)^\alpha
\]
(2.21)
\[
\int_{-L}^{L} E_\alpha \left( -\frac{\pi \alpha}{l^\alpha} \right) (dx)^\alpha
\]
Above is generalized to calculate local fractional Fourier series.

### 2.5 The Yang-Fourier transform in fractal space

**Definition 11** Suppose that \( f(x) \in C_\alpha (-\infty, \infty) \), the Yang-Fourier transform, denoted by \( F_\alpha \{ f(x) \} = f_{-\alpha}(\omega) \), is written in the form \([14, 17, 20]\)

\[
F_\alpha \{ f(x) \} = f_{\alpha}(\omega),
\]

where the latter converges.

And of course, a sufficient condition for convergence is

\[
\left| \frac{1}{\Gamma(1+\alpha)} \int_\infty^{-\infty} E_\infty \left( -\theta^{\alpha} \theta x^\alpha \right) f(x) (dx)^\alpha \right| \leq \frac{1}{\Gamma(1+\alpha)} \int_\infty^{-\infty} f(x) (dx)^\alpha < K < \infty.
\]  

**Proof.** By the Yang-Fourier transform inversion formula, we have

\[
f(x) = \frac{1}{(2\pi)^\alpha} \int_\infty^{-\infty} \tilde{f}(\omega) E_\alpha \left( -\theta^{\alpha} \theta x^\alpha \right) (d\omega)^\alpha.
\]

We introduce a function \( \tilde{g}(\omega) \) as follows:

\[
\tilde{g}(\omega) = \frac{c^\alpha}{\pi^\alpha} \tilde{f}(\omega), |\omega| < c.
\]

This can be considered as a restriction to the interval \((-c, c)\) of a \(2c\) periodic function with local fractional Fourier series

\[
\tilde{g}(\omega) \sim \sum_{n=-\infty}^{\infty} C_{c, \omega} E_\alpha \left( \omega c \right) (\omega)^\alpha.
\]

In the same manner, we have

\[
\tilde{h}(\omega) \sim \sum_{n=-\infty}^{\infty} C_{c, \omega} E_\alpha \left( \omega c \right) (\omega)^\alpha.
\]

We also take into account the function \( h \) given by

\[
\tilde{h}(\omega) = E_\alpha \left( -\theta^{\alpha} \theta x^\alpha \right), |\omega| < c.
\]

### 3. Generalized sampling theorem for fractal signals

Suppose that \( f(x) \in C_\alpha (-\infty, \infty) \), that \( f \in L_\alpha (\mathbb{R}) \) and that \( \tilde{f}(\omega) = 0 \) for \(|\omega| > c\), then for \( 0 < \alpha \leq 1 \) we have

\[
f(x) = \sum_{n=-\infty}^{\infty} f \left( \frac{n\pi}{c} \right) \sin \left( \frac{\pi x^\alpha}{c} \right) - \pi^\alpha \frac{x^\alpha}{c} \tilde{f}(\omega) \left( -\theta^{\alpha} \theta x^\alpha \right) (d\omega)^\alpha.
\]

The sum is uniformly convergent.

\[
f(x) = \sum_{n=-\infty}^{\infty} \frac{c^\alpha}{\pi^\alpha} \tilde{f}(\omega) E_\alpha \left( -\theta^{\alpha} \theta x^\alpha \right) (d\omega)^\alpha.
\]
\[ \sum_{n=-\infty}^{\infty} f\left( \frac{\pi n}{c} \right) \sin_{\alpha} \left( \frac{c \pi n}{c} \alpha \right) = \sum_{n=-\infty}^{\infty} f\left( \frac{\pi n}{c} \right) \frac{\sin_{\alpha} \left( \frac{c \pi n}{c} \alpha \right)}{c^\alpha x^\alpha - n^\alpha \pi^\alpha}. \]

Indeed, the convergence of symmetric partial sums is uniform, because estimates of the remainder are uniform.

4. Conclusions

In the present letter we suggest a generalized sampling theorem for fractal signals based on local fractional calculus as follows:

Suppose that \( f(x) \in C_{\alpha}(-\infty, \infty) \), that

\[ f \in L_{1,\alpha}(\mathbb{R}) \] and that \( \tilde{f}(\omega) = 0 \) for \( |\omega| > c \),

then for \( 0 < \alpha \leq 1 \) we have

\[ f(x) = \sum_{n=-\infty}^{\infty} f\left( \frac{\pi n}{c} \right) \frac{\sin_{\alpha} \left( \frac{c \pi n}{c} \alpha \right)}{c^\alpha x^\alpha - n^\alpha \pi^\alpha}, \]

where the sum is uniformly convergent. When we take into account the fractal dimension \( \alpha = 1 \), we induce to the classical result [1-6], which is cannot employ it to handle the local fractional continuous signals.

References