Fast Yang-Fourier Transforms in fractal space

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Abstract –The Yang-Fourier transform (YFT) in fractal space is a generation of Fourier transform based on the local fractional calculus. The discrete Yang-Fourier transform (DYFT) is a specific kind of the approximation of discrete transform based on the Yang-Fourier transform in fractal space. In the present letter we point out a new fractal model for the algorithm for fast Yang-Fourier transforms of discrete Yang-Fourier transforms. It is shown that the classical fast Fourier transforms is a special example in fractal dimension \( \alpha = 1 \).

Keywords –Yang-Fourier transforms; Fast Yang-Fourier transforms; Discrete Yang-Fourier transforms; Fractal space; Local fractional calculus

1. Introduction

Local fractional calculus (fractal calculus) has become a hot topic in both mathematics and engineering [14-19]. Here we give the definition of local fractional derivative [14-19]

\[
\Delta^\alpha \left( f(x) - f(x_0) \right) = \frac{d^\alpha f(x)}{d x^\alpha} \bigg|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x-x_0)^\alpha},
\]

(1.1)

with \( \Delta^\alpha (f(x) - f(x_0)) \geq \Gamma(1+\alpha) \Delta f(x) \) and the definition of local fractional integral [14-19, 27]

\[
\frac{1}{\Gamma(1+\alpha)} \int_a^b f(t) (dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta \to 0} \sum_{j=0}^{\max \left\{ \Delta t_1, \Delta t_2, \Delta t_3, \ldots \right\}} f(t)(\Delta t)^\alpha
\]

(1.2)

with \( \Delta t_1 = t_{j+1} - t_j \) and \( \Delta t = \max \left\{ \Delta t_1, \Delta t_2, \Delta t_3, \ldots \right\} \), where for \( j = 0, \ldots, N-1 \), \( [t_0, t_1, \ldots, t_N] \) is a partition of the interval \( [a,b] \) and \( t_0 = a, t_N = b \).

Recently, both Yang-Fourier transform (also local fractional Laplace transform) was shown by [14-15, 17, 21-22, 24]

\[
F_{a} \left\{ f \left( x \right) \right\} = F_{a}^{F} \left( \omega \right) \equiv \frac{1}{\Gamma(1+\alpha)} \int_{a}^{\infty} E_{\alpha} \left( -i^\alpha \omega^\alpha x^\alpha \right) f \left( x \right) (dx)^\alpha
\]

(1.3)

and the inverse representation was in the form [14-15, 21-22, 24]

\[
f \left( x \right) = F_{a}^{-1} \left( F_{a}^{F} \left( \omega \right) \right) = \frac{1}{\left( 2\pi \right)^{\alpha}} \int_{-\infty}^{\infty} E_{\alpha} \left( i^\alpha \omega^\alpha x^\alpha \right) f_{a}^{F} \left( \omega \right) (d\omega)^\alpha
\]

(1.4)

Furthermore, both Yang-Laplace transform (also local fractional Laplace transform), [14-15, 18, 25, 26]

\[
L_{a} \left\{ f \left( x \right) \right\} = f_{a}^{L} \left( s \right) = \frac{1}{\Gamma(1+\alpha)} \int_{0}^{\infty} E_{\alpha} \left( -s^\alpha x^\alpha \right) f \left( x \right) (dx)^\alpha, 0 < \alpha \leq 1
\]

(1.5)

and inversion [14-15, 25, 26]

\[
L_{a}^{-1} \left( f_{a}^{L} \left( s \right) \right) = f \left( t \right) = \frac{1}{\left( 2\pi \right)^{\alpha}} \int_{-\infty}^{\infty} E_{\alpha} \left( s^\alpha x^\alpha \right) f_{a}^{L} \left( s \right) (ds)^\alpha,
\]

(1.6)

were introduced. Moreover, the discrete Yang-Fourier transform (shortly called DYFT) was given in the form [20, 23]

\[
F \left( k \right) = \sum_{n=0}^{N-1} f \left( n \right) W_{N, \alpha}^{-nk}
\]

(1.7)

and inversion was read as [20, 23]

\[
f \left( n \right) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^{\alpha}} \sum_{k=0}^{N-1} F \left( k \right) W_{N, \alpha}^{nk}
\]

(1.8)

with \( W_{N, \alpha}^{nk} = E_{\alpha} \left( i^\alpha n^\alpha k^\alpha \left( 2\pi \right)^\alpha \right) \). Here, aim of this letter is to suggest a new model for the fast Yang-Fourier transforms based on the discrete Yang-Fourier transforms.

This letter is organized as follows: In section 2, the fast Yang-Fourier transform of discrete Yang-Fourier transform is given. In section 3, the fast Yang-Fourier transform of inverse discrete Yang-Fourier transform is considered. Conclusions are shown in section 4.

2. Fast Yang-Fourier transform of discrete Yang-Fourier transform

In this section we start with the fast Yang-Fourier transform of Yang-Fourier transform. The relations
\[ [F_N]^{\alpha}_{n,k+1} = \frac{1}{N^{\alpha}} W_{N,\alpha}^{-(k+1)n} \]
\[ = \frac{1}{N^{\alpha}} W_{N,\alpha}^{-ln} W_{N,\alpha}^{-n} = [F_N]^{\alpha}_{n,k} W_{N,\alpha}^{-n} \quad (2.1) \]
\[
[F_N]^{\alpha}_{n,k+1} = \frac{1}{N^{\alpha}} W_{N,\alpha}^{-n} = [F_N]^{\alpha}_{n,k} W_{N,\alpha}^{-n} \quad (2.2) 
\]

are the component formulas for the Yang-Fourier transform.

Suppose that \( \{V_0, V_1, V_2, \ldots, V_{N-1}\} \) is the \( N \)th order discrete Yang-Fourier transforms of \( \{V_0, V_1, V_2, \ldots, V_{N-1}\} \).

Starting with the component formulas for the discrete Yang-Fourier transform, we obtain that, for \( n = 0, 1, 2, \ldots, N-1 \),

\[ V_n = \sum_{k=0}^{N-1} W_{N,\alpha}^{-n} v_k \]
\[ = \sum_{k=0}^{N-1} W_{N,\alpha}^{-n} v_k + \sum_{k=0}^{N-1} W_{N,\alpha}^{-n} v_k \]
\[ = \frac{1}{2^n} \left( \sum_{j=0}^{M-1} W_{2M,\alpha^{-n}} v_{2j} + \sum_{j=0}^{M-1} W_{2M,\alpha^{-n}} v_{2j+1} \right) \]
\[ = \frac{1}{2^n} \left( \sum_{j=0}^{M-1} W_{2M,\alpha^{-n}} v_{2j} + \sum_{j=0}^{M-1} W_{2M,\alpha^{-n}} v_{2j+1} \right) \]

and we have the following relation

\[ [F_{N,M}]^{\alpha} = \frac{1}{2^n} \left( [F_{N,M-1}]^{\alpha} + \sum_{j=0}^{n} [F_{N,M-1}]^{\alpha} \right) \quad (2.3) \]

where \( V \) is the sequence vector corresponding to \( \{V_0, V_1, V_2, \ldots, V_{N-1}\} \), \( V_E \) is the \( M \)th order sequence of even-index \( v_k \)'s \( \{V_0, V_2, \ldots, V_{N-2}\} \) and \( V_O \) is the \( M \)th order sequence of odd-index \( v_k \)'s \( \{V_1, V_3, \ldots, V_{N-1}\} \).

Here we can deduce that

\[ W_{M,\alpha}^{-M} = E_{\alpha} \left( -i \left( \frac{2\pi}{M} \right)^{\alpha} (M+1)^{\alpha} \right) \]
\[ = E_{\alpha} \left( -i \left( \frac{2\pi}{M} \right)^{\alpha} M^{\alpha} \right) \]
\[ = W_{M,\alpha}^{-1} \quad (2.4) \]

and

\[ W_{M,\alpha}^{\frac{M+l}{2}} = E_{\alpha} \left( -i \left( \frac{\pi}{M} \right)^{\alpha} (M+1)^{\alpha} \right) \]
\[ = E_{\alpha} \left( -i \left( \frac{\pi}{M} \right)^{\alpha} l^{\alpha} \right) \]
\[ = W_{M,\alpha}^{-\frac{l}{2}} \quad (2.5) \]

Hence for \( l = 0, 1, 2, \ldots, m-1 \),

\[ V_l = \frac{1}{2^n} \left( \sum_{j=0}^{M-1} W_{M,\alpha}^{-\theta} v_{2j} - \sum_{j=0}^{M-1} W_{M,\alpha}^{-\theta} v_{2j+1} \right) \quad (2.6) \]
\[ = \frac{1}{2^n} \left( [F_{M\alpha}]^{\alpha} - W_{M,\alpha}^{-\theta} \right) \quad (2.7) \]

Here, formulas (2.6) and (2.7) contain common elements that can be computed once for each \( l \) and then used to compute both \( V_l \) and \( V_{M+l} \). Hence we can obtain the total number of computations to find all the \( V_l \)'s.

That is to say, this process of increasing levels to our algorithm can be continued to the \( K \)th level provided to \( N = 2^k N_0 \) for some integer \( N_0 \). Moreover, that integer, \( N_0 = 2^{-K} N \) will also be the order of the discrete Yang-Fourier transforms and inverse discrete Yang-Fourier transforms. If \( N = 2^k \), it is this final \( K \)th level algorithm, fully implemented and refined, that is called a fast Yang-Fourier transform of the discrete Yang-Fourier transforms.

3. Fast Yang-Fourier transform of inverse discrete Yang-Fourier transform

In this section we start with the fast Yang-Fourier transform of inverse Yang-Fourier transform. Similarly, suppose that \( \{V_0^{-1}, V_1^{-1}, \ldots, V_{N-1}^{-1}\} \) is the \( N \)th order discrete Yang-Fourier transforms of \( \{V_0^{-1}, V_1^{-1}, \ldots, V_{N-1}^{-1}\} \), starting with the component formulas for the inverse discrete
Yang-Fourier transform, we obtain that, for 
\( n=0,1,2,\ldots,N-1 \),
\[
V_n^{-1} = \frac{1}{\Gamma(1+\alpha) N^\alpha} \sum_{k=0}^{N-1} W_{N,\alpha}^{[k]} V_k^{-1}
\]
\[
= \frac{1}{\Gamma(1+\alpha) N^\alpha} \sum_{k=0}^{N-1} W_{N,\alpha}^{[k]} V_k^{-1} + \sum_{k=0}^{N-1} W_{N,\alpha}^{[k]} V_k^{-1}
\]
\[
= \frac{1}{\Gamma(1+\alpha) (2M)^\alpha} \left( \sum_{j=0}^{M-1} W_{M,\alpha}^{[n]} V_{j}^{-1} + \sum_{j=0}^{M-1} W_{M,\alpha}^{[n]} V_{j+l}^{-1} \right)
\]
\[
= \frac{1}{\Gamma(1+\alpha) (2M)^\alpha} \left( \sum_{j=0}^{M-1} W_{M,\alpha}^{[n]} V_{j}^{-1} + \sum_{j=0}^{M-1} W_{M,\alpha}^{[n]} V_{j+l}^{-1} \right)
\]
and we have the following relation
\[
[F_N^V]_{\alpha} = \frac{1}{\Gamma(1+\alpha) (2M)^\alpha} \left( \sum_{j=0}^{M-1} W_{M,\alpha}^{[n]} V_{j}^{-1} + \sum_{j=0}^{M-1} W_{M,\alpha}^{[n]} V_{j+l}^{-1} \right)
\]
where \( V^{-1} \) is the sequence vector corresponding to \( \{ V_0^{-1}, V_1^{-1}, V_2^{-1}, \ldots, V_{N-1}^{-1} \} \), \( V_E^{-1} \) is the \( M - th \) order sequence of even-index \( v_k^{-1} \), \( V_O^{-1} \) is the \( M - th \) order sequence of odd-index \( v_k^{-1} \).

Here we can deduce that
\[
W_{M,\alpha}^{[n]} = E_{\alpha} \left( i^\alpha \left( \frac{2\pi}{M} \right)^\alpha (M+l)^\alpha \right)
\]
\[
= E_{\alpha} \left( i^\alpha \left( \frac{2\pi}{M} \right)^\alpha \right)
\]
\[
= W_{M,\alpha}^{[l/2]}
\]
Hence for \( l = 0,1,2,\ldots,m-1 \),
\[
V_L^{-1} = \frac{1}{\Gamma(1+\alpha) (2M)^\alpha} \left( \sum_{j=0}^{M-1} W_{M,\alpha}^{[n]} V_{j}^{-1} + \sum_{j=0}^{M-1} W_{M,\alpha}^{[n]} V_{j+l}^{-1} \right)
\]
\[
= \frac{1}{\Gamma(1+\alpha) (2M)^\alpha} \left( F_{M,\alpha}^{-1} + \sum_{j=0}^{M-1} W_{M,\alpha}^{[n]} V_{j+l}^{-1} \right)
\]
and
\[
V_{M+l}^{-1} = \frac{1}{\Gamma(1+\alpha) (2M)^\alpha} \left( \sum_{j=0}^{M-1} W_{M,\alpha}^{[n]} V_{j}^{-1} - \sum_{j=0}^{M-1} W_{M,\alpha}^{[n]} V_{j+l}^{-1} \right)
\]
\[
= \frac{1}{\Gamma(1+\alpha) (2M)^\alpha} \left( F_{M,\alpha}^{-1} - \sum_{j=0}^{M-1} W_{M,\alpha}^{[n]} V_{j+l}^{-1} \right)
\]
It is shown that, formulas (2.12) and (2.13) contain common elements that can also be computed once for each \( l \) and then used to compute both \( V_L^{-1} \) and \( V_{M+l}^{-1} \). These can also yield the total number of computations to find all the \( V_n^{-1} \)'s. That is to say, this process of increasing levels to our algorithm of inverse discrete Yang-Fourier transforms is similar to that of the discrete Yang-Fourier transforms. Taking into account the relation \( N = 2^K \), it is also this final \( K^{th} \) level algorithm, fully implemented and refined, that is called a fast Yang-Fourier transform of the inverse discrete Yang-Fourier transforms.

3. Conclusions

In the present letter we suggest the fast algorithm for the discrete Yang-Fourier transform (DYFT), which is a specific kind of the approximation of discrete transform based on the Yang-Fourier transform in fractal space[20, 23]. Here, we call the fast Yang-Fourier transform. Moreover, it is shown that the classical fast Fourier transforms is a special example in fractal dimension \( \alpha = 1 \). Based on the fast Yang-Fourier transform, we may structure a new algorithm for the generalized Fourier transforms in fractal space.

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