A short introduction to Yang-Laplace Transforms in fractal space

Yang Xiaojun

Available at: https://works.bepress.com/yang_xiaojun/26/
A short introduction to Yang-Laplace Transforms in fractal space

Xiao-Jun Yang

Department of Mathematics and Mechanics, China University of Mining and Technology, Xuzhou Campus, Xuzhou, Jiangsu, 221008, P. R. China

Email: dyangxiaojun@163.com

Abstract —The Yang-Laplace transforms [W. P. Zhong, F. Gao, In: Proc. of the 2011 3rd International Conference on Computer Technology and Development, 209-213, ASME, 2011] in fractal space is a generalization of Laplace transforms derived from the local fractional calculus. This letter presents a short introduction to Yang-Laplace transforms in fractal space. At first, we present the theory of local fractional derivative and integral of non-differential functions defined on cantor set. Then the properties and theorems for Yang-Laplace transforms are tabulated, and both the initial value theorem and the final value theorem are investigated. Finally, some applications to the wave equation and the partial differential equation with local fractional derivative are studied in detail.

Keywords —Yang-Laplace transforms; Local fractional calculus; Fractal space; Non-differential functions, Initial and final value theorem; Cantor set

1. Introduction

The classical Laplace transform is a wonderful tool for solving ordinary and partial differential equations and has enjoyed much success in this realm [1]. However, it is unable to solving fractal differential equations with local fractional derivative [2-8]. Recently, a new fractal model for Laplace transform based on local fractional calculus, called local fractional Laplace transform was written down in the following form [9]

\[ L_\alpha \{ f(x) \} = f_s^{\alpha}(s) \]

\[ = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha \left( -s^\alpha x^\alpha \right) f(x) (dx)^\alpha, \quad 0 < \alpha \leq 1 \]

(1.1)

where the latter converges and \( s^\alpha \in \mathbb{R}^\alpha \). And of course, a sufficient condition is

\[ \frac{1}{\Gamma(1+\alpha)} \int_0^\infty f(x) (dx)^\alpha < K < \infty. \]

Its inverse was given by [9]

\[ L^{-1}_\alpha \{ f_s^{\alpha}(s) \} = f(t) \]

\[ = \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} E_\alpha \left( s^\alpha x^\alpha \right) f_s^{\alpha}(s) (ds)^\alpha, \quad x > 0 \]

(1.3)

where \( \text{Re}(s) = \beta > 0 \) and \( i^\alpha \) is imaginary unit. Here local fractional integral was written down [7, 9-19]

\[ _a I_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t) (dt)^\alpha \]

(1.4)

with \( \Delta_j = t_{i+1} - t_j \) and \( \Delta = \max \{ \Delta_1, \Delta_2, \Delta_3, \ldots \} \), where

\[ f^{(\alpha)}(x_0) = \lim_{\alpha \to 0} \frac{\Delta^\alpha \left( f(x) - f(x_0) \right)}{(x-x_0)^\alpha}, \]

(1.5)

for more detail for local fractional calculus, see [7-11]. However, the inversion of local fractional derivative, (1.3) was modified in [12, 13], and the transform is called the Yang-Laplace transforms [14]. In this paper, our attempt is to give an introduction to Yang-Laplace transforms in fractal space, which are derived from local fractional calculus and its applications are studied.

This paper is organized as follows: In section 2, notations and recent results are given. In section 3, the convolution of two functions is studied. In section 4, the boundary value theorems are considered. The Dirac’s distributions are in section 5. Some applications are shown in section 6.

2. Preliminary results

2.1 Notations and recent results

Definition 1 If there exists the relation [12,13]

\[ |f(x) - f(x_0)| < \epsilon^\alpha \]

(2.1)
where \( |x - x_0| < \delta \). for \( \varepsilon, \delta > 0 \) and \( \varepsilon, \delta \in \mathbb{R} \).

Now \( f(x) \) is called local fractional continuous at \( x_0 \), denoted by

\[
\lim_{x \to x_0} f(x) = f(x_0).
\]

Then \( f(x) \) is called local fractional continuous on the interval \((a, b)\), denoted by

\[
f(x) \in C_a(a, b).
\]

**Definition 2** For any \( x \in (a, b) \), there exists

\[
f^{(a)}(x) = D_x^{(a)} f(x),
\]

denoted by

\[
f(x) \in D_x^{(a)}(a, b).
\]

where \( D_x^{(a)} f(x) = f^{(a)}(x) \).

**Definition 3** For any \( x \in (a, b) \), there exists

\[
\int_a^{(a)} f(x) \in I_x^{(a)}(a, b).
\]

where \( I_x^{(a)} f(x) = \frac{1}{\Gamma(1 + \alpha)} \int_0^{x} f(t)(dt)^\alpha \),

for \( x, x_0 \in (a, b) \).

**Remark 1.** Suppose that \( f(x) \in D_x^{(a)}(a, b) \), or \( \int_a^{(a)} f(x) \in C_a(a, b) \), we have the following relation

\[
f(x) \in C_a(a, b).
\]

Suppose that \( f(x), g(x) \in D_a(a, b) \), the following differentiation rules are valid [7, 8, 12, 13]:

\[
\frac{d^\alpha}{dx^\alpha} \left( f(x) \pm g(x) \right) = \frac{d^\alpha}{dx^\alpha} f(x) \pm \frac{d^\alpha}{dx^\alpha} g(x);
\]

\[
\frac{d^\alpha}{dx^\alpha} \left( f(x) g(x) \right) = \frac{d^\alpha}{dx^\alpha} f(x) + f(x) \frac{d^\alpha}{dx^\alpha} g(x);
\]

\[
\frac{d^\alpha}{dx^\alpha} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{d^\alpha}{dx^\alpha} f(x) - f(x) \frac{d^\alpha}{dx^\alpha} g(x)}{g(x)^2};
\]

\[
\frac{d^\alpha}{dx^\alpha} \left( C f(x) \right) = C \frac{d^\alpha}{dx^\alpha} f(x), \quad C \text{ is a constant};
\]

If \( y(x) = (f \circ u)(x) \) where \( u(x) = g(x) \), then

\[
\frac{d^\alpha}{dx^\alpha} y(x) = f^{(a)} \left( g(x) \right) \left( g^{(a)}(x) \right) \alpha.
\]

**Theorem 1** [7, 10, 12, 13]

Suppose that \( f(x), g(x) \in C_a[a, b] \), then

\[
\int_b^{(a)} \left[ f(x) \pm g(x) \right] = \int_b^{(a)} f(x) \pm \int_b^{(a)} g(x).
\]

\[
\int_a^{(a)} \left( f(x) \right) = \int_a^{(a)} f(x), \quad \int_a^{(a)} \left( g(x) \right) = \int_a^{(a)} g(x).
\]

\[
\text{Re} \left( s^\alpha \right) = \beta > 0.
\]
Hence it is convenient to understand these forms.

### 2.2 Some results

Suppose that

\[ L_\alpha \{ f(x) \} = f^{L_\alpha}_x(s) \text{ and } L_\alpha \{ g(x) \} = f^{L_\alpha}_x(s). \]

the following formulas are valid [12, 13]:

\[
L_\alpha \{ af(x) + bg(x) \} = aL_\alpha f^{L_\alpha}_x(s) + bL_\alpha g^{L_\alpha}_x(s). \tag{2.22}
\]

\[
L_\alpha \{ f(x - c) \} = f^{L_\alpha}(s - c). \tag{2.23}
\]

\[
L_\alpha \{ f(ax) \} = \frac{1}{a^{\alpha}} f^{L_\alpha}(s/a), a > 0. \tag{2.25}
\]

\[
L_\alpha \{ f^{(\alpha)}(x) \} = s^{\alpha - 1} L_\alpha \{ f(x) \} - f(0). \tag{2.26}
\]

\[
L_\alpha \{ f^{(k\alpha)}(x) \} = s^{\alpha - k} L_\alpha \{ f(x) \} - s^{k-1} f^{(k-1)}(0). \tag{2.27}
\]

\[
L_\alpha \{ f^{(1\alpha)}(x) \} = \frac{d^{\alpha}}{ds^{\alpha}} f^{L_\alpha}_x(s). \tag{2.29}
\]

\[
L_\alpha \{ af^{L_\alpha}_x(s) + bg^{L_\alpha}_x(s) \} = af^{(1\alpha)}(t) + b g^{(1\alpha)}(t). \tag{2.30}
\]

\[
L_\alpha \{ f^{(1\alpha)}(s - c) \} = f(t) E_\alpha (c^{t\alpha}). \tag{2.31}
\]

\[
L_\alpha \{ f^{(1\alpha)}(s) \} E_\alpha (c^{\alpha} s^\alpha) = f(t - c). \tag{2.32}
\]

### 3. Convolutions

The convolution of two functions, which satisfy the condition (2.1), is defined symbolically by

\[
f_1(x) * f_2(x) = \int_0^x f_1(t) f_2(x-t)(dt)^x. \tag{2.33}
\]

As further results, the properties of the convolution of the non-differentiable functions for convenience read as:

1. \( f_1(x) * f_2(x) = f_2(x) * f_1(x) \);
2. \( f_1(x) * \{ f_2(x) + f_3(x) \} = f_1(x) * f_2(x) + f_1(x) * f_3(x) \).

#### Theorem 5 (The convolution theorem)

Let \( L_\alpha \{ f_1(x) \} = f^{L_\alpha}_{x_1}(s) \),

and \( L_\alpha \{ f_2(x) \} = f^{L_\alpha}_{x_2}(s) \), then

\[
L_\alpha \{ f_1(x) * f_2(x) \} = f^{L_\alpha}_{x_1}(s) f^{L_\alpha}_{x_2}(s). \tag{2.23}
\]

**Proof.** Taking the definition of the convolution into account, we arrive at the following relation

\[
L_\alpha \{ f_1(x) * f_2(x) \} = \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty E_\alpha (-s^{\alpha} x^{\alpha}) f_1(x) f_2(x)(dx)^x. \tag{2.24}
\]

By using integration by parts in (2.24) we have the following identity

\[
\frac{1}{\Gamma(1 + \alpha)} \int_0^\infty E_\alpha (-s^{\alpha} x^{\alpha}) f_1(t) f_2(x-t)(dt)^x \tag{2.25}
\]

Substituting the formula

\[
\frac{1}{\Gamma(1 + \alpha)} \int_0^\infty E_\alpha (-s^{\alpha} x^{\alpha}) f_1(t) f_2(x-t)(dt)^x \tag{2.26}
\]

into (2.25) implies that

\[
\frac{1}{\Gamma(1 + \alpha)} \int_0^\infty E_\alpha (-s^{\alpha} x^{\alpha}) \tag{2.27}
\]

Rearranging the formula (2.26), the formula (2.25) becomes

\[
f^{L_\alpha}_{x_2}(s) \left[ \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty f_1(t) E_\alpha (-s^{\alpha} t^{\alpha})(dt)^x \right]. \tag{2.28}
\]

Therefore, the proof of this theorem is completed.

**Remark 3.** Let

\[
L_\alpha \{ f_1(x) \} = f^{L_\alpha}_{x_1}(s) \text{ and } L_\alpha \{ f_2(x) \} = f^{L_\alpha}_{x_2}(s). \]

If \( f^{L_\alpha}_{x_1}(s) = f^{L_\alpha}_{x_2}(s) \), then

\[
f_1(x) = f_2(x) \tag{2.27}
\]

at all points \( x > 0 \) where both functions are continuous.

### 4. Boundary value theorems

#### Theorem 6 (The initial value theorem)

Let \( L_\alpha \{ f(x) \} = f^{L_\alpha}_x(s) \) and

\[
\lim_{x \to 0} f(x) = f(0_+), \text{ then } \lim_{x \to 0} f^{L_\alpha}_x(s) = f^{L_\alpha}_x(s). \tag{3.1}
\]

**Proof.** Setting

\[
\lim_{x \to 0} f^{(1\alpha)}(x) E_\alpha (-s^{\alpha} x^{\alpha})(dx)^x = 0, \tag{3.2}
\]

we have the following relation

\[
\lim_{x \to 0} \int_0^x \frac{d^{\alpha}}{dx^{\alpha}} f(x) E_\alpha (-s^{\alpha} x^{\alpha})(dx)^x. \tag{3.3}
\]

Taking the limit
\[
\lim_{s \to \infty} E_\alpha \left(-s^\alpha x^\alpha \right) = 0 \quad (3.4)
\]

in (3.3) implies that

\[
\lim_{s \to \infty} \int_0^x \frac{d^\alpha f(x)}{dx^\alpha} E_\alpha \left(-s^\alpha x^\alpha \right) \alpha (dx)^\alpha = 0. \quad (3.5)
\]

Using relations (2.17) and (2.26), we arrive at the following relation

\[
L_\alpha \left\{ f^{\alpha}(x) \right\} = s^\alpha f_{s}^{L_\alpha}(s) - f(0) \quad (3.6)
\]

which yields

\[
\frac{1}{\Gamma(1+\alpha)} \int_0^x \frac{d^\alpha f(x)}{dx^\alpha} E_\alpha \left(-s^\alpha x^\alpha \right) \alpha (dx)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{d^\alpha f(x)}{dx^\alpha} E_\alpha \left(-s^\alpha x^\alpha \right) \alpha (dx)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{d^\alpha f(x)}{dx^\alpha} E_\alpha \left(-s^\alpha x^\alpha \right) \alpha (dx)^\alpha. \quad (3.7)
\]

Taking

\[
\frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{d^\alpha f(x)}{dx^\alpha} E_\alpha \left(-s^\alpha x^\alpha \right) \alpha (dx)^\alpha = f(0^+) - f(0^-)
\]

into account, the formula (3.7) becomes

\[
\frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{d^\alpha f(x)}{dx^\alpha} E_\alpha \left(-s^\alpha x^\alpha \right) \alpha (dx)^\alpha = f(0^+) - f(0^-) + \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{d^\alpha f(x)}{dx^\alpha} E_\alpha \left(-s^\alpha x^\alpha \right) \alpha (dx)^\alpha. \quad (3.8)
\]

Successively, the following equality

\[
\lim_{s \to \infty} s^\alpha f_{s}^{L_\alpha}(s) = \lim_{s \to \infty} L_\alpha \left\{ f^{\alpha}(x) \right\} + f(0^-) \quad (3.10)
\]

holds. Taking the limit in (3.10) as \( s \to \infty \), there is the following relation

\[
\lim_{s \to \infty} f(0^+) + \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{d^\alpha f(x)}{dx^\alpha} E_\alpha \left(-s^\alpha x^\alpha \right) \alpha (dx)^\alpha = \lim_{s \to \infty} f(x). \quad (3.11)
\]

From the formulas (3.10) and (3.11), we derive the following identity

\[
\lim_{s \to \infty} s^\alpha f_{s}^{L_\alpha}(s) = f(0^+) = \lim_{s \to \infty} f(x). \quad (3.12)
\]

Therefore, we derive the result.

**Theorem 7 (The final value theorem)**

Suppose \( L_\alpha \left\{ f(x) \right\} = f_{s}^{L_\alpha}(s) \) and

\[
L_\alpha \left\{ f^{(n)}(x) \right\} = s^\alpha f_{s}^{L_\alpha}(s) - f(0^-) \quad (3.13)
\]

Prove: By using the formula

\[
L_\alpha \left\{ f^{(n)}(x) \right\} = s^\alpha f_{s}^{L_\alpha}(s) - f(0^-) \quad (3.14)
\]

we arrive at the following relation

\[
\lim_{s \to \infty} L_\alpha \left\{ f^{(n)}(x) \right\} = \lim_{s \to \infty} s^\alpha f_{s}^{L_\alpha}(s) - f(0^-) \quad (3.15)
\]

Taking the limit

\[
\lim_{s \to \infty} E_\alpha \left(-s^\alpha x^\alpha \right) = 1^\alpha, \quad (3.16)
\]

we have the following result

\[
\lim_{s \to \infty} s^\alpha f_{s}^{L_\alpha}(s) - f(0^-) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{d^\alpha f(x)}{dx^\alpha} \left(E_\alpha \left(-s^\alpha x^\alpha \right)\right) (dx)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{d^\alpha f(x)}{dx^\alpha} \alpha (dx)^\alpha. \quad (3.17)
\]

Taking the integration in the third term of (3.17), this implies that

\[
\lim_{s \to \infty} f(x) - f(0^-) = \lim_{s \to \infty} f(x) - f(0^-). \quad (3.18)
\]

Now the following equality

\[
\lim_{s \to \infty} f(x) = \lim_{s \to \infty} s^\alpha f_{s}^{L_\alpha}(s) \quad (3.19)
\]

is derived from (3.18). Therefore the proof of this theorem is completed.

**5. The Dirac’s distributions**

The Dirac’s distribution, or a generalized function, \( \delta_\alpha(x) \) of fractional order \( \alpha, 0 < \alpha \leq 1 \), is defined by the equality[9, 12, 13]

\[
\frac{1}{\Gamma(1+\alpha)} \int_0^x \delta_\alpha(x) f(x) (dx)^\alpha = f(0). \quad (4.1)
\]

As a direct result, we have the following relation

\[
\frac{1}{\Gamma(1+\alpha)} \int_0^x \delta_\alpha(x-a) f(x) (dx)^\alpha = f(a) \quad (4.2)
\]

and

\[
\frac{1}{\Gamma(1+\alpha)} \int_0^\infty \delta_\alpha(x) (dx)^\alpha = 1. \quad (4.3)
\]
\textbf{Theorem 8} Define the function
\[ \delta_a(x, \varepsilon) = \begin{cases} \Delta, & x \in \mathbb{R} \\ \frac{1}{\Gamma(1+\alpha)} x^\alpha, & 0 < x \leq \varepsilon, 0 < \alpha \leq 1 \end{cases} \] then we have the limit
\[ \lim_{\varepsilon \to 0} \delta_a(x, \varepsilon) = \delta_a(x). \] 

\textit{Proof.} By using Dirac’s distribution we deduce the relation
\[ \lim_{\varepsilon \to 0} \frac{1}{\Gamma(1+\alpha)} \int_0^\varepsilon \delta_a(x, \varepsilon) f(x) (dx)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \delta_a(x) f(x) (dx)^\alpha. \] Taking mean value theorem in (4.6), we arrive at the following result
\[ \lim_{\varepsilon \to 0} \delta_a(x, \varepsilon) = \delta_a(x). \] Therefore
\[ \lim_{\varepsilon \to 0} \delta_a(x, \varepsilon) = \delta_a(x). \] 

\section{6. Applications}

\subsection{6.1 Application to fractional wave equation with local fractional derivative}
We consider the wave equation in fractal space, which reads:
\[ \frac{d^{2a}u}{dt^{2a}} + u = 0, t > 0, 0 < \alpha \leq 1 \] with boundary conditions, which are
\[ \frac{d^\alpha u}{dt^\alpha} |_{t=0} = 0 \text{ and } u(t) |_{t=0} = 1. \] Taking the Yang-Laplace transforms in (3.1) yields
\[ s^{2a} \tilde{u}(s) - s^a u(0) - \frac{d^\alpha u}{dt^\alpha} |_{t=0} = -\tilde{u}(s). \] From (3.3) we deduce that
\[ \tilde{u}(s) = \frac{s^a}{s^{2a} + 1}. \] Furthermore
\[ u(t) = \cos^\alpha t^a. \] with \[ \cos^\alpha t^a = \sum_{n=0}^\infty (-1)^k \frac{t^{2na}}{\Gamma(1+2na)}. \]

\textbf{Remark 4.} The above wave equation is considered in case of [14].

\subsection{6.2 Application to local fractional partial differential equation}
The local fractional partial differential equation is given by[17]
\[ \frac{\partial^{2a}u}{\partial x^{2a}} = \frac{\partial^2 u}{\partial t^2}, x > 0, t > 0 \] with the boundary conditions
\[ \frac{\partial^\alpha u}{\partial t^\alpha} (x, 0^+) = 0, x > 0 \]
\[ u(0, t) = f(t)/(f(0) = 0) \]
\[ u(0, t) = 0, t > 0 \]
\[ \lim_{s \to x} u(x, t) = 0. \] The transformed equation becomes
\[ s^{2a} \tilde{u}(s, x) - s^a \tilde{u}(x, 0^+) - \frac{\partial^\alpha \tilde{u}}{\partial t^\alpha}, \] which reads:
\[ \tilde{u}(s, x) = \frac{d^{2a} \tilde{u}(x, s)}{dt^{2a}}. \] Hence we get
\[ \tilde{u}(x, s) = C_1 E_\alpha \left(s^a x^a\right) + C_2 E_\alpha \left(-s^a x^a\right). \] By condition, \[ \tilde{u}(0, t) = L_0 \left[f(t)\right] = \tilde{f}(s) \] and
\[ \lim_{s \to x} \tilde{u}(x, t) = 0, \text{ so that} \]
\[ C_2 = \tilde{f}(s), C_1 = 0. \] and
\[ \tilde{u}(x, s) = \tilde{f}(s) E_\alpha \left(-s^a x^a\right). \] Hence we get
\[ u(x, s) = u_\alpha(t) f(t - x) \] where \[ u_\alpha(t) = \begin{cases} 1, t \geq x \\ 0, t < x. \end{cases} \]

\section{References}