Local fractional partial differential equations with fractal boundary problems

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Abstract – This letter points out the new alternative approaches to processing local fractional partial differential equations with fractal boundary conditions. Applications of the local fractional Fourier series, the Yang-Fourier transforms and the Yang-Laplace transforms to solve of local fractional partial differential equations with fractal boundary conditions are investigated in detail.

Keywords – Local fractional partial differential equations; Fractal boundary conditions; Local fractional Fourier series; Yang-Fourier transforms; Yang-Laplace transforms; Fractional boundary problems

1. Introduction

Local fractional calculus (also called fractal calculus) is a hot topic in both mathematics and engineering [1–17]. There are many definitions of local fractional calculus [1, 7, 9, 10, 11, 12]. Hereby we write down the definitions of local fractional derivative [11, 12]

\[ f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha} \bigg|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha f(x) - f(x_0)}{(x-x_0)^\alpha} \quad (1.1) \]

with \( \Delta^\alpha f(x) = \prod_{l=1}^{\alpha} f(x_l) \) and the definition of local fractional integral [11, 12]

\[ \frac{1}{(1+\alpha) \Gamma(1+\alpha)} \int_a^b f(t) dt^\alpha = \frac{1}{(1+\alpha) \Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{\Delta t} f(t_j) \Delta t_j^\alpha \quad (1.2) \]

with \( \Delta t = t_{j+1} - t_j \) and \( \Delta t = \max \{ \Delta t_1, \Delta t_2, \Delta t_3, \ldots \} \), where for \( j = 0, 1, \ldots, N-1 \), \([t_j, t_{j+1}]\) is a partition of the interval \([a, b]\) and \( t_0 = a, t_N = b \).

Recently, both Yang-Fourier transforms [11, 15]

\[ F_{\alpha} \{ f(x) \} = f_{\alpha}^{F, \alpha}(\omega) := \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} x^\alpha dx^\alpha \quad (1.3) \]

and inverse [11]

\[ f(x) = F_{\alpha}^{-1} \{ f_{\alpha}^{F, \alpha}(\omega) \} := \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} f_{\alpha}^{F, \alpha}(\omega) e^{i\omega x} x^{-\alpha} d\omega \quad (1.4) \]

were reported. Furthermore, both Yang-Laplace transforms [11, 16]

\[ L_{\alpha} \{ f(x) \} = f_{\alpha}^{L, \alpha}(s) := \frac{1}{(1+\alpha) \Gamma(1+\alpha)} \int_0^{\infty} f(x) x^\alpha dx^\alpha \quad (1.5) \]

and inverse [11, 16]

\[ L_{\alpha}^{-1} \{ f_{\alpha}^{L, \alpha}(s) \} = f(x) = \frac{1}{(1+\alpha) \Gamma(1+\alpha)} \int_0^{\infty} f_{\alpha}^{L, \alpha}(s) s^\alpha ds^\alpha \quad (1.6) \]

were introduced. More recently, local fractional Fourier series [11]

\[ f(x) \sim \sum_{k=-\infty}^{\infty} C_k E_{\alpha} \left( \frac{\pi^\alpha i^\alpha (kx)^\alpha}{l^\alpha} \right) \quad (1.7) \]

where the Fourier coefficients is [11]

\[ C_k = \frac{1}{(2l)^\alpha} \int_{-l}^{l} f(x) E_{\alpha} \left( \frac{-\pi^\alpha i^\alpha (kx)^\alpha}{l^\alpha} \right) (dx)^\alpha \quad (1.8) \]

where \( k \in \mathbb{Z} \) and fractal imaginary unit \( i^\alpha \) was proposed. In this paper, we suggest their applications to local fractional partial differential equations. This paper is organized as follows. In Section 2, we investigate an application of local fractional Fourier series to solve the local fractional partial differential equations. In Section 3, we present an application of Yang-Fourier transforms to solve the local fractional partial differential equations. In Section 4, we study we present an application of Yang-Laplace transforms to solve the local fractional partial differential equations. Finally, the section 5 is conclusions.

2. Application of local fractional Fourier series to solve to local fractional partial differential equations

Local fractional partial differential equation is given by

\[ \frac{\partial^\alpha u}{\partial t^\alpha} = k^2 \frac{\partial^2 u}{\partial x^2} \quad (2.1) \]

with boundary conditions
\[ u(x,t) = \frac{1}{E_\alpha} f(x) (dx)^\alpha \]
\[ + \sum_{m=0}^{\infty} \int_0^1 f(x) \cos_\alpha \left( \left( \frac{m \pi x}{L} \right)^\alpha \right) (dx)^\alpha \]
\[ E_\alpha \left( -\lambda^{2\alpha} k^{2\alpha} f^2 \left( \frac{m \pi x}{L} \right)^2 \right) \cos_\alpha \left( \left( \frac{m \pi x}{L} \right)^\alpha \right) \]

### 3 Application of Yang-Fourier transforms to solve to local fractional partial differential equations

**Example 1 (Local fractional Laplace equation)**

The local fractional Laplace equations is given by

\[ \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = 0, \quad x \in \mathbb{R}, \quad y > 0 \]

\[ u(x,0) = f(x), \quad x \in \mathbb{R} \]

\[ u \text{ bounded for } y > 0 \]

Taking the Yang-Fourier transforms yields

\[ \frac{d^{2\alpha} \tilde{u}(y,\omega)}{dy^{2\alpha}} = -\omega^{2\alpha} \tilde{u}(y,\omega) = 0. \]

Its general solution is

\[ \tilde{u}(y,\omega) = C_1 E_\alpha \left( i^\alpha \omega^\alpha y^\alpha \right) + C_2 E_\alpha \left( -i^\alpha \omega^\alpha y^\alpha \right). \]

Taking into boundary condition, we get

\[ \tilde{u}(y,\omega) = \tilde{f}(\omega) E_\alpha \left( \omega^\alpha \left| y^\alpha \right. \right). \]

Taking

\[ \frac{1}{(2\pi)^\alpha} \int_0^\infty E_\alpha \left( \omega^\alpha \left| y^\alpha \right. \right) E_\alpha \left( -i^\alpha \omega^\alpha x^\alpha \right) (d\omega)^\alpha \]

\[ = \left( \frac{y}{2\pi} \right)^\alpha \frac{2\omega^\alpha}{y^{2\alpha} + \omega^{2\alpha}} \]

into account, we have

\[ u(y,\omega) = \frac{1}{(2\pi)^\alpha} \int_0^\infty \tilde{u}(y,\omega) E_\alpha \left( i^\alpha \omega^\alpha x^\alpha \right) (d\omega)^\alpha \]

\[ = \frac{1}{(2\pi)^\alpha} \int_0^\infty \tilde{f}(\omega) E_\alpha \left( \omega^\alpha \left| y^\alpha \right. \right) E_\alpha \left( i^\alpha \omega^\alpha x^\alpha \right) (d\omega)^\alpha \]

\[ = \left( \frac{y}{2\pi} \right)^\alpha \frac{2}{\Gamma(1+\alpha)} \int_0^\infty \frac{f(t)}{(x-t)^{2\alpha} + y^{2\alpha}} (dx)^\alpha. \]

**Example 2**

The local fractional Laplace equations is given by
Taking the Yang-Fourier transforms yields
\[
\frac{d^2u}{dx^{2\alpha}} = \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, x \in \mathbb{R}, t > 0
\]  
(3.9)

Its general solution is
\[
\tilde{u}(\omega, \alpha) = \frac{1}{(2\pi)^{\alpha}} \int_{-\infty}^{\infty} \tilde{u}(\omega, t) E_{\alpha}(-\omega^{2\alpha} x^{\alpha}) \, d\omega
\]  
(4.9)

By condition, \( \tilde{u}(0, \alpha) = 0, x > 0 \)

Taking into boundary condition, we get
\[
\tilde{u}(0, \alpha) = \tilde{f}(0) E_{\alpha}(-\omega^{2\alpha} x^{\alpha})
\]  
(4.10)

Hence we have the result.

4 Application of Yang-Laplace transforms to solve to local fractional partial differential equations

Example 1

The local fractional partial differential equation is given by
\[
\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, x > 0, t > 0
\]  
(4.11)

with the boundary conditions
\[
\begin{align*}
\lim_{x \to \infty} u(x, t) &= 0, x > 0 \\
\lim_{t \to 0^+} u(x, t) &= 0, t > 0 \\
\lim_{x \to 0} u(x, t) &= 0.
\end{align*}
\]  
(4.12)

Taking the Yang-Laplace transform of (7.49) yields
\[
\frac{d^2u}{dx^{2\alpha}} + s^2 \tilde{u} - \tilde{u}(x, 0^+) = s^\alpha \tilde{u} - 1 - \tilde{f}(s)
\]  
(4.13)

Transforming the boundary conditions gives
\[
\tilde{u}(0, s) = L_n \left[ u(0, t) \right] = 0
\]  
(4.14)

and
\[
\lim_{x \to \infty} \tilde{u}(x, s) = \lim_{x \to \infty} L_n \left[ u(x, t) \right] = \frac{1}{s^\alpha}
\]  
(4.15)

Solution of (7.51) is given by
\[
\tilde{u}(x, s) = C_1 E_{\alpha} \left( \frac{x^{\alpha}}{s^2} \right) + C_2 E_{\alpha} \left( \frac{x^{\alpha}}{s^2} \right) + \frac{1}{s^\alpha}
\]  
(4.16)

Taking the boundary conditions implies that
\[
\tilde{u}(x, s) = \frac{1}{s^\alpha} E_{\alpha} \left( \frac{x^{\alpha}}{s^2} \right) + \frac{1}{s^\alpha}
\]  
(4.17)

Hence we have
\[
\tilde{u}(x, 0) = \tilde{f}(s) E_{\alpha} \left( \frac{x^{\alpha}}{s^2} \right)
\]  
(4.18)

which is integral and its result is the desired solution.

Example 2

The local fractional partial differential equation is given by
\[
\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = 0, x > 0, t > 0
\]  
(4.19)

with the boundary conditions
\[
\begin{align*}
\lim_{x \to \infty} u(x, t) &= 0, x > 0 \\
\lim_{t \to 0^+} u(x, t) &= 0, t > 0 \\
\lim_{x \to 0} u(x, t) &= 0.
\end{align*}
\]  
(4.20)

The transformed equation becomes
\[
\frac{d^2u}{dx^{2\alpha}} + s^2 \tilde{u} - \tilde{u}(x, 0^+) = \frac{d^{2\alpha} u}{dt^{2\alpha}}
\]  
(4.21)

that is,
\[
s^{2\alpha} \tilde{u}(x, s) = \frac{d^{2\alpha} u(x, s)}{dt^{2\alpha}}
\]  
(4.22)

Hence we get
\[
\tilde{u}(x, s) = \frac{1}{(2\pi)^{\alpha}} \int_{-\infty}^{\infty} \tilde{u}(x, t) E_{\alpha} \left( \frac{s^2 x^{\alpha}}{s^{2\alpha}} \right) \, ds
\]  
(4.23)

Example 3

The local fractional partial differential equation is given by
\[
n(x, t) = f(t), 0 < t < x
\]  
(4.24)
\[
\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = a^{2\alpha} \frac{\partial^{2\alpha} u}{\partial \tau^{2\alpha}}, \quad x > 0, t > 0
\]  
(4.16)

with the boundary conditions
\[
\begin{align*}
\frac{\partial^\alpha}{\partial \tau^\alpha} u(x, 0^+) &= 0, \quad x > 0 \\
u(0, t) &= f(t) \left( f(0) = 0 \right) \\
u(0, t) &= 0, \quad t > 0 \\
\lim_{\tau \to \infty} u(x, t) &= 0.
\end{align*}
\]  
(4.17)

The transformed equation becomes
\[
s^{2\alpha} \tilde{u}(x, s) - s^\alpha \frac{\partial^\alpha \tilde{u}(x, 0^+)}{\partial t^\alpha} = a^{2\alpha} \frac{\partial^{2\alpha} \tilde{u}(x, s)}{\partial t^{2\alpha}},
\]  
(4.18)

that is,
\[
s^{2\alpha} \tilde{u}(x, s) = a^{2\alpha} \frac{\partial^{2\alpha} \tilde{u}(x, s)}{\partial t^{2\alpha}}.
\]  
(4.19)

Hence we get
\[
\tilde{u}(x, s) = C_1 E_a \left( \frac{s}{a} \right) x^\alpha + C_2 E_a \left( -\frac{s}{a} \right) x^\alpha.
\]  
(4.20)

By condition,
\[
\tilde{u}(0, t) = L_a \left[ u(0, t) \right] = f(s) \quad \text{and}
\]
\[
\lim_{\tau \to \infty} u(x, t) = 0, \quad \text{so that } C_2 = f(s), C_1 = 0, \quad \text{and}
\]
\[
\tilde{u}(x, s) = f(s) E_a \left( \frac{s}{a} \right) x^\alpha.
\]  
(4.21)

Hence we get
\[
u(x, s) = \frac{u_\alpha(t)}{a} f \left( \frac{t - \frac{x}{a}}{a} \right)
\]  
(4.22)

where
\[
\alpha \leq \frac{x}{a}, \quad 0 < t < \frac{x}{a}
\]

5 Conclusions

This paper provides efficient approaches to nonlinear fractional partial differential equations with local fractional derivative in fractal boundary conditions. The boundary conditions are local fractional integral and local fractional continuous, hence we call their fractal boundary conditions. The solutions to nonlinear fractional partial differential equations with fractal boundary conditions are discussed. Both Fourier-Jumarie transforms and Laplace-Jumarie transforms, introduced by Jumarie based on the Modified Riemann–Liouville derivative and Jumarie integral [18, 19], are different from the Yang-Fourier transforms and Yang-Laplace transforms based on local fractional calculus. The differences of them see [11, 12, 18]. This note is taken into account application of local fractional Fourier series, Yang-Fourier transforms and Yang-Laplace transforms based on the generalized Hilbert space [11], but not the classical Hilbert space.

References