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A Novel Approach to Processing Fractal Dynamical Systems Using the Yang-Fourier Transforms

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Abstract – In the present paper, local fractional continuous non-differentiable functions in fractal space are investigated, and the control method for processing dynamic systems in fractal space are proposed using the Yang-Fourier transform based on the local fractional calculus. Two illustrative paradigms for control problems in fractal space are given to elaborate the accuracy and reliable results.

Keywords – Non-differentiable functions; Fractal dynamical systems; Yang-Fourier transforms; Fractal space; Local fractional calculus

1. Introduction

As is well-known, Fourier analysis is one of the most frequently used tools in control processing. A system is defined as a function or an equation. However, some dynamical systems functions are fractal curves, which are everywhere continuous but nowhere differentiable [1-9]. As a result, we cannot employ the classical Fourier analysis, which requires that the defined functions should be differentiable, to describe the dynamical states in fractal space.

Recently, local fractional calculus (fractal calculus), which was dealing with fractal functions, had been proposed and developed in[1-10]. For these merits, local fractional calculus was successfully applied in the fractal elasticity [3, the fractal release equation [7], fractal wave equation [9], local fractional Laplace equations[21], fractal boundary problems [21], fractal signals[24,17], the generalized Newton iteration method[20], the Yang-Laplace transforms [9, 25], the local fractional Fourier and integrals transforms [14], the Yang-Fourier transforms[8,17], the discrete Fourier transform [10,18], the fast Yang-Fourier transforms in fractal space [23], the local fractional Mellin transform in fractal space [19], the local fractional Stieltjes transform in fractal space [16], the local fractional Z transform in fractal space [22], the local fractional short time transform [5,6] and the local fractional wavelet transform [5,6].

In this paper, we apply the Yang-Fourier transform to deal with the fractal dynamical systems with local fractional derivative. The organization of this paper is as follows. In section 2, the preliminary results are presented. The fractal dynamical systems are investigated in section 3. Two examples of fractal dynamical systems are studied in section 4. Conclusions are in section 5.

2. Preliminaries

2.1. Local fractional continuity

Definition 1 If there is [5-9]

$$\left| f(x) - f(x_0) \right| < \varepsilon^a$$

(2.1)

with \( |x - x_0| < \delta \) for \( \varepsilon, \delta > 0 \) and \( \varepsilon, \delta \in \mathbb{R} \).

Now \( f(x) \) is called local fractional continuous at \( x = x_0 \), denote by

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Then \( f(x) \) is called local fractional continuous on the interval \((a, b)\), denoted by

$$f(x) \in C_a(a, b).$$

(2.2)

Definition 2 A function \( f(x) \) is called a non-differentiable function of exponent \( \alpha \), \( 0 < \alpha \leq 1 \), which satisfy Hölder function of exponent \( \alpha \), then for \( x, y \in X \) such that [6-7]

$$\left| f(x) - f(y) \right| \leq C|x - y|^\alpha.$$  

(2.3)

Definition 3 A function \( f(x) \) is called to be continuous of order \( \alpha \), \( 0 < \alpha \leq 1 \), or shortly \( \alpha \) continuous, when we have the following relation [6-7]

$$f(x) - f(x_0) = o\left(|x - x_0|^\alpha\right).$$

(2.4)

Remark. 1. Compared with (2.4), (2.1) is standard definition of local fractional continuity. Here (2.3) is unified local fractional continuity.
Definition 4 Setting \( f(x) \in C_\alpha(a,b) \), local fractional derivative of \( f(x) \) of order \( \alpha \) at \( x = x_0 \) is defined [5-9]

\[
f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha} \bigg|_{x = x_0} = \lim_{\rightarrow x_0} \frac{\Delta_\alpha^\alpha(f(x) - f(x_0))}{(x - x_0)^\alpha},
\]
where \( \Delta_\alpha^\alpha(f(x) - f(x_0)) \geq \Gamma(1+\alpha) \Delta(f(x) - f(x_0)) \).

For any \( x \in (a,b) \), there exists

\[
f^{(\alpha)}(x) = D_\alpha^{(\alpha)} f(x),
\]
denoted by

\[
f(x) \in D_\alpha^{(\alpha)}(a,b).
\]

Local fractional derivative of order \( \alpha \) is written in the form

\[
f^{(k\alpha)}(x) = D_\alpha^{(\alpha)} \ldots D_\alpha^{(\alpha)} f(x) \] (2.6)

and local fractional partial derivative of order \( \alpha \)

\[
\frac{\partial^{k\alpha} f(x)}{\partial x^{\alpha k}} = \frac{\partial^{\alpha} x}{\partial x^{\alpha}} \ldots \frac{\partial^{\alpha} x}{\partial x^{\alpha}} f(x) \] (2.7)

Definition 5 Setting \( f(x) \in C_\alpha(a,b) \), local fractional integral of \( f(x) \) of order \( \alpha \) in the interval \([a,b]\) is defined [6-9]

\[
_{a}I_\alpha^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_{a}^{x} f(t)(dt)^\alpha, \]

\[
_{a}I_\alpha^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \lim_{\rightarrow x_0} \sum_{j=0}^{N-1} f(t_j)(\Delta_\alpha t_j)^\alpha
\]
where \( \Delta_\alpha t_j = t_{j+1} - t_j \), \( \Delta_\alpha = \max \{\Delta_\alpha t_1, \Delta_\alpha t_2, \ldots\} \) and \([t_0, t_{j+1}]\). For any \( x \in (a,b) \), there exists

\[
_{a}I_\alpha^{(\alpha)} f(x),
\]
denoted by

\[
f(x) \in I_\alpha^{(\alpha)}(a,b).
\]

Remark. 2. If \( f(x) \in D_\alpha^{(\alpha)}(a,b), \) or \( I_\alpha^{(\alpha)}(a,b), \) we have

\[
f(x) \in C_\alpha(a,b).
\]

2.3. Special Functions in Fractal Space

Definition 6 The Mittag-Leffler function in fractal space is defined by [5,6]

\[
E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+k\alpha)}. \quad x \in \mathbb{R} \text{ and } 0 < \alpha \leq 1.
\]

Definition 7 The sine function in fractal space is by the expression [5,6]

\[
\sin_\alpha x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{\Gamma(1+\alpha(2k+1))}, \quad x \in \mathbb{R} \text{ and } 0 < \alpha \leq 1.
\]

Definition 8 The cosine function in fractal space is given [5,6]

\[
\cos_\alpha x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{\Gamma(1+2\alpha k)}, \quad x \in \mathbb{R} \text{ and } 0 < \alpha \leq 1.
\]

The following rules hold [11, 12]:

\[
E_\alpha(x) E_\alpha(y) = E_\alpha((x+y)^\alpha);
\]

\[
E_\alpha(x) E_\alpha(-y) = E_\alpha((x-y)^\alpha);
\]

\[
E_\alpha(i^\alpha x^\alpha) E_\alpha(i^\alpha y^\alpha) = E_\alpha(i^\alpha (x+y)^\alpha);
\]

\[
E_\alpha(i^\alpha x^\alpha) = \cos_\alpha x^\alpha + i^\alpha \sin_\alpha x^\alpha;
\]

\[
\sin_\alpha x^\alpha = \frac{E_\alpha(i^\alpha x^\alpha) - E_\alpha(-i^\alpha x^\alpha)}{2i^\alpha};
\]

\[
\cos_\alpha x^\alpha = E_\alpha(i^\alpha x^\alpha) + E_\alpha(-i^\alpha x^\alpha);
\]

\[
\cos_\alpha (-x)^\alpha = \cos_\alpha x^\alpha;
\]

\[
\sin_\alpha (-x)^\alpha = -\sin_\alpha x^\alpha;
\]

\[
\cos_\alpha^2 x^\alpha + \sin_\alpha^2 x^\alpha = 1;
\]

\[
\sin_\alpha^2 x^\alpha = \frac{1 - \cos_\alpha(2x)^\alpha}{2};
\]

\[
\cos_\alpha^2 x^\alpha = \frac{1 + \cos_\alpha(2x)^\alpha}{2};
\]

\[
\tan_\alpha x^\alpha = \frac{\sin_\alpha(2x)^\alpha}{1 + \cos_\alpha(2x)^\alpha} = \frac{1 - \cos_\alpha(2x)^\alpha}{\sin_\alpha(2x)^\alpha};
\]

\[
\sin_\alpha(2x)^\alpha = 2\sin_\alpha x^\alpha \cos_\alpha x^\alpha;
\]

\[
\cos_\alpha(2x)^\alpha = \cos_\alpha^2 x^\alpha - \sin_\alpha^2 x^\alpha;
\]

\[
\tan_\alpha(2y)^\alpha = \frac{2 \tan_\alpha y^\alpha}{1 + \tan_\alpha^2 y^\alpha};
\]

\[
\sin_\alpha(2x)^\alpha = \frac{2 \tan_\alpha x^\alpha}{1 + \tan_\alpha^2 x^\alpha};
\]

\[
\cos_\alpha(2x)^\alpha = \frac{1 - \tan_\alpha^2 x^\alpha}{1 + \tan_\alpha^2 x^\alpha};
\]

\[
\tan_\alpha(x+y)^\alpha = \frac{\tan_\alpha x^\alpha + \tan_\alpha y^\alpha}{1 + \tan_\alpha x^\alpha \tan_\alpha y^\alpha};
\]
\[\cos_{\alpha} x^a + \cos_{\alpha} y^a = 2 \cos_{\alpha} \left(\frac{x+y}{2}\right) \cos_{\alpha} \left(\frac{x-y}{2}\right);\]

\[\cos_{\alpha} x^a - \cos_{\alpha} y^a = -2 \sin_{\alpha} \left(\frac{x+y}{2}\right) \sin_{\alpha} \left(\frac{x-y}{2}\right);\]

\[\sin_{\alpha} x^a + \sin_{\alpha} y^a = 2 \sin_{\alpha} \left(\frac{x+y}{2}\right) \cos_{\alpha} \left(\frac{x-y}{2}\right);\]

\[\sin_{\alpha} x^a - \sin_{\alpha} y^a = 2 \cos_{\alpha} \left(\frac{x+y}{2}\right) \sin_{\alpha} \left(\frac{x-y}{2}\right);\]

2.4. Yang-Fourier Transform in Fractal Space

Definition 9 Suppose that \(f(x) \in C_{\alpha} (-\infty, \infty)\), the Yang-Fourier transform, denoted by \(\mathcal{F}_{\alpha} \{f(x)\} \equiv f_{\alpha}^F(\omega)\), is written in the form \([8, 17]\)

\[\mathcal{F}_{\alpha} \{f(x)\} = f_{\alpha}^F(\omega) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} E_{\alpha} (-\omega^a x^a) f(x) (dx)^a,\]  

where the latter converges.

And of course, a sufficient condition for convergence is

\[\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \left| f(x) \right| (dx)^a < K < \infty.\]

Definition 10 If \(\mathcal{F}_{\alpha} \{f(x)\} \equiv f_{\alpha}^F(\omega)\), its inversion formula is written in the form \([8, 17]\)

\[f(x) = \mathcal{F}_{\alpha}^{-1} \{f_{\alpha}^F(\omega)\} = \frac{1}{(2\pi)^\frac{\alpha}{2}} \int_{-\infty}^{\infty} E_{\alpha} (\omega^a x^a) f_{\alpha}^F(\omega) (d\omega)^a,\]

Suppose that

\[\mathcal{F}_{\alpha} \{f(x)\} = f_{\alpha}^F(\omega) \text{ and } \mathcal{F}_{\alpha} \{g(x)\} = g_{\alpha}^F(\omega),\]

the following formulas are valid \([8, 17]\): \(\mathcal{F}_{\alpha} \{af(x) + bg(x)\} = af_{\alpha}^F(\omega) + bg_{\alpha}^F(\omega)\) (2.15)

If \(\lim_{\|x\| \to \infty} f(x) = 0\), then

\[\mathcal{F}_{\alpha} \left\{f^{(\alpha)}(x)\right\} = i^a \omega^a f_{\alpha}^F(\omega);\]

If \(\lim_{\|x\| \to \infty} f(x) = \ldots = f^{(n-1)}(x) = 0\), then

\[\mathcal{F}_{\alpha} \left\{f^{(n)}(x)\right\} = i^n \omega^a f_{\alpha}^F(\omega).\]

3. Fractal Dynamical Systems

In this section we start with the dynamical systems of differential equations from fractional derivative to local fractional derivative.

At first, we introduce some of the most important methods of solving linear fractional differential equations with fractional derivative:

\[\sum_{i=1}^{n} A_{m-i} \frac{\partial^a y(t)}{\partial t^a} = \sum_{i=1}^{m} B_{m-i} \frac{\partial^a x(t)}{\partial t^a}.\]
where both $\frac{\partial^{\alpha} x(t)}{\partial t^{\alpha}}$ and $\frac{\partial^{\alpha} y(t)}{\partial t^{\alpha}}$ represent a fractional derivative[11-13], and both $x(t)$ and $y(t)$ are continuous.

Moreover, we will introduce some of the most important methods of solving linear local fractional differential equations with local fractional derivative:

$$\sum_{i=1}^{n} A_{n-i} \frac{\partial^{\alpha} y(t)}{\partial t^{\alpha}} = \sum_{i=1}^{n} B_{n-i} \frac{\partial^{\alpha} x(t)}{\partial t^{\alpha}}, \hspace{1cm} (3.2)$$

where both $\frac{\partial^{\alpha} x(t)}{\partial t^{\alpha}}$ and $\frac{\partial^{\alpha} y(t)}{\partial t^{\alpha}}$ represent a local fractional derivative and both $x(t)$ and $y(t)$ are local fractional continuous. Mostly we will consider an initial-value problem composed of the equation (3.2) and appropriate initial conditions in fractal space, which depend on a construction of all operators

$$\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}} \text{ for } k \in \{1, 2, \ldots, m, \ldots, n\}. \hspace{1cm} (3.3)$$

4. Two Illustrative Examples

In this section, we give the usual cases. Suppose we consider fractal systems described by the fractional differential equations with local fractional derivative.

Example 1. A fractal system is described by the differential equation with local fractional derivative

$$a_0 \frac{\partial^{\alpha} y(t)}{\partial t^{\alpha}} + a_1 y(t) = b_0 \frac{\partial^{\alpha} x(t)}{\partial t^{\alpha}} + b_1 x(t). \hspace{1cm} (4.1)$$

Taking the Yang-Fourier transform, we have

$$\left(i^\alpha \omega^\alpha a_0 + a_1\right) y_{\omega, \alpha}(\omega) = \left(i^\alpha \omega^\alpha b_0 + b_1\right) x_{\omega, \alpha}(\omega). \hspace{1cm} (4.2)$$

Therefore, the transform function is written in the from

$$y_{\omega, \alpha}(\omega) = \left(i^\alpha \omega^\alpha b_0 + b_1\right) x_{\omega, \alpha}(\omega). \hspace{1cm} (4.3)$$

Example 2. A fractal system is described by the differential equation with local fractional derivative

$$a_0 \frac{\partial^{2\alpha} y(t)}{\partial t^{2\alpha}} + a_1 \frac{\partial^{\alpha} y(t)}{\partial t^{\alpha}} + a_2 y(t) = b_0 \frac{\partial^{\alpha} x(t)}{\partial t^{\alpha}} + b_1 x(t). \hspace{1cm} (4.4)$$

Take the Yang-Fourier transform,

$$\left(i^\alpha \omega^\alpha a_0 + i^\alpha \omega^\alpha a_1 + a_2\right) y_{\omega, \alpha}(\omega) = \left(i^\alpha \omega^\alpha b_0 + b_1\right) x_{\omega, \alpha}(\omega). \hspace{1cm} (4.5)$$

then the transform function is given

$$y_{\omega, \alpha}(\omega) = -\omega^{2\alpha} a_0 + i^\alpha \omega^\alpha a_1 + a_2. \hspace{1cm} (4.6)$$

5. Conclusions

Local fractional derivative and integrals are revealed as one of useful tools to deal with everywhere continuous but nowhere differentiable functions in fractal areas ranging from fundamental science to engineering. In this paper, we show that the fractal dynamical systems are derived from local fractional derivative, which are proposed using the Yang-Fourier transform based on the local fractional calculus, and give two paradigms for control problems to elaborate accuracy and reliable results. Our method can be applied to other academic fields [26-33].

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References


**Vitae**

Yang Xiao-Jun was born in 1981. He worked as a scientist and engineer in CUMT. His research interest includes Fractal mathematics (Geometry, applied mathematics and functional analysis), fractal Mechanics (fractal elasticity and fractal fracture mechanics, fractal rock mechanics and fractional continuous mechanics in fractal media), fractional calculus and its applications, fractional differential equation, local fractional integral equation, local fractional differential equation, local fractional integral transforms, local fractional calculus and its applications and local fractional functional analysis and its applications.

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