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The Discrete Yang-Fourier Transforms in Fractal Space

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Abstract –The Yang-Fourier transform (YFT) in fractal space is a generation of Fourier transform based on the local fractional calculus. The discrete Yang-Fourier transform (DYFT) is a specific kind of the approximation of discrete transform, used in Yang-Fourier transform in fractal space. This paper points out new standard forms of discrete Yang-Fourier transforms (DYFT) of fractal signals, and both properties and theorems are investigated in detail.

Keywords – Fractal, Signal, Discrete, Yang-Fourier transforms

1. Introduction

Fractional Fourier transform has played an important role in both mathematics and engineering [1-5]. Recently, the Yang-Fourier transforms based on the local fractional calculus [4-10] and the discrete Yang-Fourier transforms were introduced [6]. Here we write down the discrete Yang-Fourier transform (DYFT) of

\[ F(n) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f(k) E_{\alpha} \left( -i^n \frac{(2\pi)^\alpha n^\alpha k^\alpha}{N^\alpha} \right) \]  

(1.1)

and its inverse discrete Yang-Fourier transform (IDYFT) [6]

\[ f(k) = \sum_{n=0}^{N-1} F(n) E_{\alpha} \left( i^n \frac{(2\pi)^\alpha n^\alpha k^\alpha}{N^\alpha} \right) \]  

(1.2)

Our attempts of the present paper are to continue to study the discrete Yang-Fourier transform and to suggest the theorems and properties for the discrete Yang-Fourier transform of periodic discrete-time fractal signals.

This paper is organized as follows: In section 2, the standard form of the discrete Yang-Fourier transform is presented. The properties and theorems are proposed in section 3. Conclusions are presented in section 4.

2. A Standard form of the DYFTs

In this section we start with a standard form of the discrete Yang-Fourier transform of fractal signal.

**Definition 1** From (1.1), the discrete Yang-Fourier transform (DYFT) is written in the form

\[ F(k) = \sum_{n=0}^{N-1} f(n) W_{N,\alpha}^{-nk} \]  

(2.1)

with

\[ W_{N,\alpha}^{-nk} = E_{\alpha} \left( -i^n \frac{(2\pi)^\alpha n^\alpha k^\alpha}{N^\alpha} \right) \]  

This is called \( N \)-point discrete Yang-Fourier transform of \( F(n) \), denoted by

\[ f(n) \leftrightarrow F(k). \]  

(2.2)

**Definition 2** From (1.2), the inverse discrete Yang-Fourier transform (IDYFT) is given by

\[ f(n) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} F(k) W_{N,\alpha}^{kn} \]  

(2.3)

with

\[ W_{N,\alpha}^{kn} = E_{\alpha} \left( i^n \frac{(2\pi)^\alpha n^\alpha k^\alpha}{N^\alpha} \right) \]  

Taking into account the relation [4,5]

\[ E_{\alpha} \left( i^n \frac{(2\pi)^\alpha n^\alpha (k+N)^\alpha}{N^\alpha} \right) = E_{\alpha} \left( i^n \frac{(2\pi)^\alpha n^\alpha k^\alpha}{N^\alpha} \right) \]  

(2.4)

for all \( n \in \mathbb{Z} \). That is to say, \( W_{1,\alpha}^{(n+N)} = W_{1,\alpha}^n \) and

\[ W_{N,\alpha}^{(k+N)n} = W_{N,\alpha}^{kn}. \]

Hence, for periodic discrete time signals it is now quite appropriate to call the quantity \( \frac{2\pi}{N} \) the fundamental frequency. In general, \( f(k) \) will of course be complex. As for continuous time signals we call the modulus \( |f(k)| \) of the spectrum. The \( f(k) \) amplitude spectrum and \( \text{arg}_\alpha f(k) \) the phase spectrum of \( F(n) \). The phase spectrum is determined on a multiple of \( (2\pi)^\alpha \), where \( \alpha \) is fractal dimension.

3. Properties and theorems for the DYFT

In this section we take into account the properties and theorems for the DYFT. Here, we start with the existence of the DYFT.
Lemma 1 Suppose that \( F(k) = \sum_{n=0}^{N-1} f(n) W_{N,\alpha}^{-nk} \), then
\[
 f(n) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} F(k) W_{N,\alpha}^{nk}.
\] (3.1)

**Proof.** The theorem is proved, see [6].

Lemma 2 Suppose that \( f(n) \) be periodic discrete time signals with period \( N \), then [6]
\[
\sum_{n=0}^{N-1} f(n) = \sum_{n=-j}^{j-N+1} f(n).
\] (3.2)

**Lemma 3** Suppose that \( f_1(n) \leftrightarrow F_1(k) \) and \( f_2(n) \leftrightarrow F_2(k) \), then [6]
\[
af_1(n) + bf_2(n) \leftrightarrow aF_1(k) + bF_2(k).
\] (3.3)

For more detail of lemmas 2 and 3, see [6].

**Corollary 4**
\[
F(n) \leftrightarrow N^\alpha \Gamma(1+\alpha) f(-k).
\] (3.4)

**Proof.** The expression for the inverse DYFT reads as follows:
\[
F(n) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} F(k) W_{N,\alpha}^{nk}.
\] (3.5)

From this it follows, by interchanging the variables \( n \) and \( k \), that
\[
F(k) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{n=0}^{N-1} f(n) W_{N,\alpha}^{nk}.
\] (3.6)

and furthermore
\[
\sum_{n=0}^{N-1} F(n) W_{N,\alpha}^{-nk} = N^\alpha \Gamma(1+\alpha) \left[ \frac{1}{N^\alpha} \frac{1}{\Gamma(1+\alpha)} \sum_{n=0}^{N-1} F(n) W_{N,\alpha}^{-nk} \right]
\] (3.7)

Corollary 5 (Time reversal rule for DYFT)
\[
f(-n) \leftrightarrow F(-k).
\] (3.8)

**Proof.** The DYFT of the signal \( f(-n) \) is equal to
\[
\sum_{n=0}^{N-1} f(-n) W_{N,\alpha}^{-nk} = \sum_{n=0}^{N-1} f(N-n) W_{N,\alpha}^{(N-n)k}
\] (3.9)
\[
= \sum_{n=0}^{N-1} f(n) W_{N,\alpha}^{nk} = F(-k).
\]

Hence, we deduce the result.

**Corollary 6** (Conjugation rule for DYFT)
\[
f(n) \leftrightarrow F(-k).
\] (3.10)

**Proof.** The discrete spectrum of the complex conjugate \( f(n) \) of \( f(n) \) a signal can be found by a direct calculation
\[
\sum_{n=0}^{N-1} f(n) W_{N,\alpha}^{-nk} = \sum_{n=0}^{N-1} f(n) W_{N,\alpha}^{nk} = F(-k).
\] (3.11)

This completes the proof.

**Corollary 7** (Shift in the \( n \)-domain rule for DYFT)
\[
f(n-l) \leftrightarrow E_\alpha \left( -i^{\alpha} \frac{2\pi}{\gamma^\alpha} k^n / N^\alpha \right) F(k).
\] (3.12)

**Proof.** Take the DYFT of \( f(n-l) \), then we get
\[
\sum_{n=0}^{N-1} f(n-l) W_{N,\alpha}^{-nk} = \sum_{n=0}^{N-1} f(n) W_{N,\alpha}^{-(n+l)k}.
\] (3.13)

Hence there is the result.

**Corollary 8** (Shift in the \( k \)-domain rule for DYFT)
\[
f(n) \leftrightarrow E_\alpha \left( i^{\alpha} \frac{2\pi}{\gamma^\alpha} k^n / N^\alpha \right) F(k-l).
\] (3.14)

**Proof.** Take the inverse DYFT of \( f(n-l) \), then we have
\[
\sum_{n=0}^{N-1} F(k-l) W_{N,\alpha}^{nk} = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} F(k) W_{N,\alpha}^{(k+l)n}.
\] (3.15)

Hence we get the result.

**Definition 3** (Cyclical convolution)

The cyclical convolution product of two periodic discrete time signals \( f(n) \) and \( g(n) \) with periodic \( N \) is the fractal discrete time signal \( (f * g)(n) \) defined by
\[
(f * g)(n) = \sum_{l=0}^{N-1} f(l) g(n-l).
\] (3.16)

**Theorem 9** (Convolution in the \( n \)-domain rule for DYFT)

Let \( f(n) \) and \( g(n) \) be periodic discrete time signals with period \( N \). Suppose that \( f(n) \leftrightarrow F(k) \) and \( g(n) \leftrightarrow G(k) \), then
\[(f * g)(n) \leftrightarrow F(k)G(k). \] (3.16)

**Proof.** Taking into account the DYFT of \(g(n-l)\) yields that
\[g(n-l) \leftrightarrow G(k)W_{N,\alpha}^{-lk}. \] (3.17)

Furthermore, we obtain that
\[
(f * g)(n) = \sum_{l=0}^{N-1} f(l)g(n-l)
= \sum_{n=0}^{N-1} f(l)G(k)W_{N,\alpha}^{-lk}
= G(k)F(k)
\]

Hence the proof of the theorem is completed.

**Theorem 10 (Convolution in the \(k\)-domain rule for DYFT)**

Let \(f(n)\) and \(g(n)\) be periodic discrete time signals with period \(N\). Suppose that \(f(n) \leftrightarrow F(k)\) and \(g(n) \leftrightarrow G(k)\), then
\[
\frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} f(n)g(n) \leftrightarrow (F*G)(k).
\] (3.19)

**Proof.** Using the inverse DYFT of \(g(n)\) implies that
\[g(n)W_{N,\alpha}^{ln} \leftrightarrow G(n-k). \] (3.20)

Furthermore, we get
\[
(F*G)(k) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{l=0}^{N-1} F(l)G(k-l)W_{N,\alpha}^{ln}
= \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{l=0}^{N-1} F(l)g(n)W_{N,\alpha}^{ln}
= \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} f(n)g(n).
\]

Hereby, this is finished.

**Theorem 11(Paserval theorem for DYFT)**

Let \(f(n)\) and \(g(n)\) be periodic discrete time signals with period \(N\). Suppose that \(f(n) \leftrightarrow F(k)\) and \(g(n) \leftrightarrow G(k)\), then
\[
\sum_{n=0}^{N-1} |f(n)|^2 = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} |G(k)|^2.
\] (3.21)

**Proof.** By using the inverse DYFT, we have that
\[
\sum_{n=0}^{N-1} f(n)g(n) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} F(k)G(k).
\]

Here we deduce the result.

**Corollary 12** Let \(f(n)\) and \(g(n)\) be periodic discrete time signals with period \(N\). Suppose that \(f(n) \leftrightarrow F(k)\), then
\[
\sum_{n=0}^{N-1} |g(n)|^2 = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} |G(k)|^2.
\] (3.22)

**Proof.** Replacing \(g(n)\) by \(f(n)\) in Theorem 11 directly deduces the result.

**Theorem 13** Let \(f(n)\) and \(g(n)\) be periodic discrete time signals with period \(N\). Suppose that \(f(n) \leftrightarrow F(k)\) and \(g(n) \leftrightarrow G(k)\), then
\[
\sum_{n=0}^{N-1} f(n)g(n) = \sum_{k=0}^{N-1} F(k)g(k).
\] (3.23)

**Proof.** Taking into account the relation
\[G(n) \leftrightarrow N^\alpha \Gamma(1+\alpha) f(-k). \] (3.24)

implies that
\[
\sum_{n=0}^{N-1} f(n)g(n)
= \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} F(k)N^\alpha \Gamma(1+\alpha) f(-k)
= \sum_{k=0}^{N-1} F(k) f(k).
\]

Hence we deduce the result.

**4. Conclusions**

The discrete Yang-Fourier transform (DYFT) in fractal space is not only a specific kind of the approximation of discrete transform, used in Yang-Fourier transform in fractal space, but also a transform for fractal Fourier analysis of finite-domain discrete-time functions, which are local fractional continuous functions.

This paper points out the standard form of discrete YFT (shortly called DYFT):
\[
F(k) = \sum_{n=0}^{N-1} f(n)W_{N,\alpha}^{-nk}
\] (4.1)

and
\[
f(n) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{k=0}^{N-1} F(k)W_{N,\alpha}^{kn}
\] (4.2)
with $W_{N,a}^{kn} = E_{a} \left( \frac{i^{n} n^{a} k^{a} (2\pi)^{a}}{N^{a}} \right)$. As well, time reversal rule, shift rule, conjugation theorem, convolution theorem and Paserval theorem of DYFT are discussed.

References