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Abstract

It is suggest that a new fractal model for the Yang-Fourier transforms of discrete approximation based on local fractional calculus and the Discrete Yang-Fourier transforms are investigated in detail.

Key words: local fractional calculus, fractal, Yang Fourier transforms, discrete approximation, discrete Yang-Fourier transforms

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1 Introduction

Fractional Fourier transform becomes a hot topic in both mathematics and engineering. There are many definitions of fractional Fourier transforms [1-5]. Hereby we write down the Yang-Fourier transforms [3-5]

\[ F_{\alpha} f(x) = \int_{-\infty}^{\infty} E_{\alpha}(\omega) f(x)(dx)^{\alpha}, \quad (1.1) \]

and its inverse, denoted by [3,4]

\[ f(x) = F_{\alpha}^{-1}(f_{\alpha}(\omega)) = \frac{1}{(2\pi)^{\alpha}} \int_{-\infty}^{\infty} E_{\alpha}(\omega) f_{\alpha}(\omega)(d\omega)^{\alpha}, \quad (1.2) \]

where local fractional integral of \( f(t) \) is denoted by [3-8]

\[ \int_{x}^{y} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_{a}^{b} f(t)(dt)^{\alpha}, \quad (1.3) \]

with \( \Delta t_j = t_{j+1} - t_j \) and \( \Delta t = \max \{ \Delta t_1, \Delta t_2, \ldots \} \), where for \( j = 0, \ldots, N - 1 \), \( [t_j, t_{j+1}] \) is a partition of the interval \([a, b]\) and \( t_0 = a, t_N = b \). Here, for \( |x - x_0| < \delta \) with \( \delta > 0 \), there exists any \( x \) such that

\[ |f(x) - f(x_0)| < \epsilon^{\alpha}. \quad (1.4) \]

Now \( f(x) \) is called local fractional continuous at \( x = x_0 \) and we have [5]

\[ \lim_{x \to x_0} f(x) = f(x_0). \quad (1.5) \]

Suppose that \( \{f_0, f_1, \ldots, f_{N-1}\} \) is an \( \mathcal{N}_n \) order regular sampling with spacing \( \Delta x \) some piecewise local fractional continuous function over mammal window \([0, L] \). In the present paper, our arms are to get some assurance that local fractional integral of \( f \) can be reasonably...
approximated by the corresponding integration of \( \tilde{f} \) and we will get the discrete Yang-Fourier transforms.

### 2 A fractal model for the Yang-Fourier transforms of discrete approximation

Now we determine from our data,

\[
\frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2^M}}^{\frac{2N-1}{2^M}} \tilde{f}(t) \phi(t)(dt)^\alpha \approx \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2^M}}^{\frac{2N-1}{2^M}} f(t) \phi(t)(dt)^\alpha
\]

for any local fractional continuous function on the natural widow. This sampling can be used to complete a corresponding sum approximation for the integration,

\[
\frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2^M}}^{\frac{2N-1}{2^M}} f(t) \phi(t)(dt)^\alpha \approx \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f(k\Delta t) \phi(k\Delta t)(\Delta t)^\alpha
\]

\[
= \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \phi(k\Delta t)(\Delta t)^\alpha.
\]

Notice, however, that

\[
\frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \phi(k\Delta t)(\Delta t)^\alpha
\]

\[
= \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \left[ \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2^M}}^{\frac{2N-1}{2^M}} \phi(t) \delta_{k,M}(t)(dt)^\alpha \right] (\Delta t)^\alpha
\]

\[
= \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2^M}}^{\frac{2N-1}{2^M}} \left[ \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \delta_{k,M}(t)(\Delta t)^\alpha \right] \phi(t)(dt)^\alpha.
\]

where

\[
\frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2^M}}^{\frac{2N-1}{2^M}} \phi(t) \delta_{k,M}(t)(dt)^\alpha = \phi(k\Delta t), \quad \text{for} \quad k = 0,1,\cdots,N-1
\]

So,

\[
\frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2^M}}^{\frac{2N-1}{2^M}} f(t) \phi(t)(dt)^\alpha
\]

\[
= \frac{1}{\Gamma(1+\alpha)} \int_{-\frac{1}{2^M}}^{\frac{2N-1}{2^M}} \left[ \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \delta_{k,M}(t)(\Delta t)^\alpha \right] \phi(t)(dt)^\alpha.
\]

Suggesting that, with the natural window, we use

\[
\tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} \tilde{f}_k \delta_{k,M}(t),
\]

where \( \tilde{f}_k = f_k (\Delta t)^\alpha \) for \( k = 0,1,\cdots,N-1. \)

Now there are two natural choices: Either \( \tilde{f} \) define to be 0 outside the nature window, or define \( \tilde{f} \) to be periodic with period \( T \) equalling the length of the natural window,

\[
T = N\Delta t.
\]

Combining with our definition of \( \tilde{f} \) on the natural window, the first choice would be give
\[ \tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} \tilde{f}_k \delta_{k\Delta t}(t), \quad (2.8) \]

while the second choice would be given

\[ \tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=-\infty}^{\infty} \tilde{f}_k \delta_{k\Delta t}(t) \quad (2.9) \]

with \( \tilde{f}_{k+N} = \tilde{f}_k \).

Clearly, the latter is the more clear choice. That is to say, suppose that \( \{f_0, f_1, \cdots, f_{N-1}\} \) is the \( N \) order regular sampling with spacing \( \Delta t \) of some function \( f \). The corresponding discrete approximation of \( f \) is the periodic, regular array

\[ \tilde{f}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{k=-\infty}^{\infty} \tilde{f}_k \delta_{k\Delta t}(t) \quad (2.10) \]

with spacing \( \Delta t \) index period \( N \), and its coefficients

\[ \tilde{f}_k = \begin{cases} f_k (\Delta x)^\alpha, & \text{if } k = 0, 1, \cdots, N-1. \\ f_{k+N}, & \text{in general.} \end{cases} \quad (2.11) \]

From the Yang-Fourier transform theory, we then know

\[ F_{\alpha} \{ f(x) \} = f_{\alpha}^{F,\alpha} (\omega) \]

is a local fractional continuous and is given by

\[ f_{\alpha}^{F,\alpha} (\omega) \]

\[ = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) E_{\alpha} \left(-i \omega^\alpha t^\alpha \right) (dt)^\alpha \]

\[ = \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2} \Delta t}^{2N-1 \Delta t} f(t) E_{\alpha} \left(-i \omega^\alpha t^\alpha \right) (dt)^\alpha \]

\[ \approx \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2} \Delta t}^{\frac{1}{2} \Delta t} \tilde{f}(t) E_{\alpha} \left(-i \omega^\alpha t^\alpha \right) (dt)^\alpha \]

\[ = \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2} \Delta t}^{\frac{1}{2} \Delta t} \left( \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k \delta_{k\Delta t}(t) \right) E_{\alpha} \left(-i \omega^\alpha t^\alpha \right) (dt)^\alpha \]

\[ = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k (\Delta t)^\alpha E_{\alpha} \left(-i \omega^\alpha k^\alpha (\Delta t)^\alpha \right) \]

\[ = \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k (\Delta t)^\alpha E_{\alpha} \left(-i \omega^\alpha k^\alpha (\Delta t)^\alpha \right) \quad (2.12) \]

So, approximation of the formula

\[ \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) E_{\alpha} \left(-i \omega^\alpha t^\alpha \right) (dt)^\alpha \]

reduces to

\[ f_{\alpha}^{F,\alpha} (\omega) \approx \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k (\Delta t)^\alpha E_{\alpha} \left(-i \omega^\alpha k^\alpha (\Delta t)^\alpha \right) \]

\[ \quad (2.13) \]

with \( T = N \Delta t \).

Taking \( \omega = n \Delta \omega \) and \( \frac{2\pi}{T} = \Delta \omega \) in (2.13) implies that
\[ \phi(n) = f^{F,\alpha}_{\omega}(\omega) \]
\[ \approx \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{N-1} f_k(\Delta t)^\alpha E_a \left( -i^\alpha \omega^\alpha k^\alpha (\Delta t)^\alpha \right) \]
\[ = \frac{1}{\Gamma(1+\alpha)} \frac{T^\alpha}{N^\alpha} \sum_{k=0}^{N-1} f_k E_a \left( -i^\alpha (2\pi)^\alpha n^\alpha k^\alpha / N^\alpha \right) \]
\[ = \frac{1}{\Gamma(1+\alpha)} \frac{T^\alpha}{N^\alpha} \sum_{k=0}^{N-1} \varphi(k) E_a \left( -i^\alpha (2\pi)^\alpha n^\alpha k^\alpha / N^\alpha \right) \]

In the same manner, if
\[ f(t) = \frac{1}{(2\pi)^\omega} \int_{-\omega}^{\omega} E_a(i^\alpha \omega^\alpha t^\alpha) f^{F,\alpha}_{\omega}(\omega)(d\omega)^\alpha, \]
then we can write
\[ f_k(\Delta t) \approx \frac{1}{(2\pi)^\omega} \sum_{n=0}^{N-1} f^{F,\alpha}_{\omega}(n\Delta \omega)(\Delta \omega)^\alpha E_a \left( i^\alpha n^\alpha (\Delta \omega)^\alpha \right) \]
with \( \omega = N\Delta \omega. \)

Taking \( t = k\Delta t \) and \( \frac{2\pi}{T} = \Delta \omega \) in (2.15) implies that
\[ \varphi(k) = f_k(\Delta t) \]
\[ \approx \frac{1}{(2\pi)^\omega} \sum_{n=0}^{N-1} f^{F,\alpha}_{\omega}(n\Delta \omega)(\Delta \omega)^\alpha E_a \left( i^\alpha n^\alpha (\Delta \omega)^\alpha \right) \]
\[ = \frac{1}{T^\alpha} \sum_{n=0}^{N-1} \varphi(n) E_a \left( i^\alpha n^\alpha k^\alpha (2\pi)^\alpha / N^\alpha \right). \]

Combing the formulas (2.14) and (2.16), we have the following results:
\[ \phi(n) = \frac{1}{\Gamma(1+\alpha)} \frac{T^\alpha}{N^\alpha} \sum_{k=0}^{N-1} \varphi(k) E_a \left( -i^\alpha (2\pi)^\alpha n^\alpha k^\alpha / N^\alpha \right) \]
and
\[ \varphi(k) = \frac{1}{T^\alpha} \sum_{n=0}^{N-1} \varphi(n) E_a \left( i^\alpha n^\alpha k^\alpha (2\pi)^\alpha / N^\alpha \right). \]

Setting \( F(n) = \frac{1}{T^\alpha} \phi(n) \) and interchanging \( k \) and \( n \), we get
\[ \varphi(n) = \sum_{k=0}^{N-1} F(k) E_a \left( i^\alpha n^\alpha k^\alpha (2\pi)^\alpha / N^\alpha \right) \]
and
\[ F(k) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^\alpha} \sum_{n=0}^{N-1} \varphi(n) E_a \left( -i^\alpha (2\pi)^\alpha n^\alpha k^\alpha / N^\alpha \right). \]
3 Discrete Yang-Fourier transforms of discrete-time fractal signal

**Definition 1**

Suppose that \( F(k) \) be a periodic discrete-time fractal signal with period \( N \). From (2.20) the sequence \( f(n) \) is defined by

\[
F(k) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^{\alpha}} \sum_{n=0}^{N-1} f(n) E_{\alpha} \left( -i^{\alpha} \left( 2\pi \right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha} \right),
\]

which is called \( N \)-point discrete Yang-Fourier transform of \( F(n) \), denoted by \( f(n) \leftrightarrow F(k) \).

**Definition 2**

Inverse discrete Yang-Fourier transform

From (2.19), the transform assigning the signal \( F(k) \) to \( f(n) \) is called the inverse discrete Yang-Fourier transform, which is rewritten as

\[
f(n) = \sum_{k=0}^{N-1} F(k) E_{\alpha} \left( i^{\alpha} n^{\alpha} k^{\alpha} \left( 2\pi \right)^{\alpha} / N^{\alpha} \right).
\]

Suppose that \( f(n) \leftrightarrow F(k) \), \( f_1(n) \leftrightarrow F_1(k) \) and \( f_2(n) \leftrightarrow F_2(k) \), the following relations are valid:

**Property 1**

\[
a f_1(n) + b f_2(n) \leftrightarrow a F_1(k) + b F_2(k).
\]

Proof. Taking into account the linear transform of discrete Yang-Fourier transform, we directly deduce the result.

**Property 2**

Let \( f(k) \) be a periodic discrete fractal signal with period \( N \). Then we have

\[
\sum_{n=j}^{j+N-1} f(n) = \sum_{n=0}^{N-1} f(n).
\]

Proof. We directly deduce the result when \( j = mN + l \) with \( 0 \leq l \leq N - 1 \).

**Theorem 3**

Suppose that

\[
F(n) = \frac{1}{\Gamma(1+\alpha)} \frac{1}{N^{\alpha}} \sum_{k=0}^{N-1} f(k) E_{\alpha} \left( -i^{\alpha} \left( 2\pi \right)^{\alpha} n^{\alpha} k^{\alpha} / N^{\alpha} \right),
\]

then we have

\[
f(k) = \sum_{n=0}^{N-1} F(n) E_{\alpha} \left( i^{\alpha} n^{\alpha} k^{\alpha} \left( 2\pi \right)^{\alpha} / N^{\alpha} \right).
\]

Proof. From the formulas (2.11)-(2.20) we deduce to the results.
4 Conclusions

In the present paper we discuss a model for the Yang-Fourier transforms of discrete approximation. As well, we give the discrete Yang-Fourier transforms of fractal signal as follows:

\[ F(k) = \frac{1}{\Gamma(1+\alpha)} \sum_{n=0}^{N-1} f(n)E_{\alpha}^{-\alpha} \left( -i^{\alpha} \frac{(2\pi)^{\alpha} n^\alpha k^\alpha}{N^\alpha} \right) \]

and

\[ f(n) = \sum_{k=0}^{N-1} F(k) E_{\alpha}^{-\alpha} \left( i^{\alpha} \frac{n^\alpha k^\alpha (2\pi)^{\alpha}}{N^\alpha} \right). \]

Furthermore, some results are discussed.

References