A numerical method for solving the one-dimensional heat and advection-diffusion equation using radial basis functions

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A numerical method for solving the one-dimensional heat and advection-diffusion equation using radial basis functions

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Abstract
In this paper, we introduce radial basis functions and the use of these basis functions is discussed for solving parabolic equations. We propose a numerical scheme and approximate the solution using Laguerre-Gaussians radial basis functions (LG-RBFs). This method will be used to reduce the problem to a set of algebraic equations. The results of numerical experiments are presented and compared with the other radial basis functions and the results of other methods to confirm the validity of this method.

Keywords: Radial basis functions, Laguerre-Gaussians functions, Parabolic partial differential equation

Mathematics Subject Classification [2010]: 65M99, 35K20

1 Introduction
Consider the one-dimensional advection-diffusion equation
\[ u_t(x,t) + \beta u_x(x,t) = \alpha u_{xx}(x,t), \quad 0 < x < L, 0 < t \leq T, \] (1)
with the initial and the boundary conditions
\[ u(x,0) = f(x), \quad 0 < x < L, \quad u(0,t) = g_0(t), \quad u(L,t) = g_1(t), \quad 0 < t \leq T, \] (2)
where \( \beta \) is an arbitrary constant which shows the speed of convection and the diffusion coefficient, i.e. \( \alpha \) is a positive constant. We assume that \( f(x), g_0(t) \) and \( g_1(t) \) are suitably given functions. Eq. (1) has been used to describe heat transfer in the dispersion of dissolved material in estuaries and coastal seas [1]. Mohebbi [4] proposed a class of new finite difference schemes for solving the heat and advection-diffusion equations. In [6], a high-order compact boundary value method was employed for solution of the heat equations.

Radial basis functions (RBFs), in irregular domains or higher dimensional geometry, has attracted the attention of researchers in science. Commonly used types of these functions include (\( r = \| x - x_i \|_2 \)):

- **Multiquadric (MQ):** \( \phi(r) = \sqrt{\varepsilon^2 + r^2} \)
- **Inverse Quadratic (IQ):** \( \phi(r) = \frac{1}{\varepsilon^2 + r^2} \)
- **Inverse Multiquadric (IMQ):** \( \phi(r) = \frac{1}{\sqrt{\varepsilon^2 + r^2}} \)
- **Gaussian (GA):** \( \phi(r) = e^{-\varepsilon^2 r^2} \)

where \( \varepsilon \) is a free positive parameter, often referred to as the shape parameter. The kind of RBFs, we will be mostly interested in, are the Gaussians \( \phi(r) = e^{-\varepsilon^2 r^2} \). Other families of radial basis functions are the Laguerre-Gaussians. The definition of Laguerre-Gaussians functions family comes from the generalized Laguerre polynomials of degree \( n \) and order \( s/2 \) [5]. Specific examples are listed in Table 1. A numerical method using Laguerre-Gaussians functions was proposed for solving the one-dimensional heat equation subject to initial-boundary conditions in [2]. However many researchers have attempted to develop algorithms for choosing optimal values of the shape parameter but the optimal choice of the shape parameter is still an open question and it is most often selected by brute force.

1 speaker

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2 Radial basis function approximation

Let \( \mathbb{R}^+ = \{ x \in \mathbb{R}, x \geq 0 \} \), \( \| \cdot \|_2 \) denotes the Euclidean norm and \( \phi : \mathbb{R}^+ \to \mathbb{R} \) be a continuous function with \( \phi(0) \geq 0 \). A radial basis function on \( \mathbb{R}^d \) is a function of the form:

\[
\phi(\|\mathbf{x} - \mathbf{x}_i\|),
\]

which depends only on the distance between \( \mathbf{x} \in \mathbb{R}^d \) and a fixed point \( \mathbf{x}_i \in \mathbb{R}^d \). So that the radial basis function \( \phi_i \) is radially symmetric about the center \( \mathbf{x}_i \). Let \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N \in \Omega \subset \mathbb{R}^d \) be a given set of scattered data. Let \( r \) be the Euclidean distance between a fixed point \( \mathbf{x}_i \in \mathbb{R}^d \) and \( \mathbf{x} \in \mathbb{R}^d \), i.e. \( \|\mathbf{x} - \mathbf{x}_i\|_2 \).

Let \( X = L^2(\Omega) \) where \( \Omega = [0, L] \times [0, T] \) and

\[
\{ \phi_{00}(x,t), ..., \phi_{0M}(x,t), \phi_{10}(x,t), ..., \phi_{1M}(x,t), ..., \phi_{N0}(x,t), ..., \phi_{NM}(x,t) \} \subset X
\]

be the set of RBFs and

\[
H = \text{span}\{ \phi_{00}(x,t), ..., \phi_{0M}(x,t), \phi_{10}(x,t), ..., \phi_{1M}(x,t), ..., \phi_{N0}(x,t), ..., \phi_{NM}(x,t) \},
\]

suppose that \( h \) be an arbitrary element in \( X \). Since \( H \) is a finite dimensional vector space, \( h \) has the unique best approximation out of \( H \) as \( h_{NM} \in H \), that is [3]:

\[
\forall g \in H, \| h - h_{NM} \|_2 \leq \| h - g \|_2.
\]

Since \( h_{NM} \in H \), there exist unique coefficients \( r_{00}, ..., r_{0M}, r_{10}, ..., r_{1M}, ..., r_{N0}, ..., r_{NM} \) such that:

\[
h \simeq h_{NM} = \sum_{i=0}^{N} \sum_{j=0}^{M} r_{ij} \phi_{ij}(x,t) = R^T \Phi_{NM}(x,t) = \Phi_{NM}^T(x,t)R,
\]

where \( R \) and \( \Phi_{NM}(x,t) \) are vectors with the form:

\[
R = [r_{00}, ..., r_{0M}, r_{10}, ..., r_{1M}, ..., r_{N0}, ..., r_{NM}]^T,
\]

\[
\Phi_{NM}(x,t) = [\phi_{00}(x,t), ..., \phi_{0M}(x,t), \phi_{10}(x,t), ..., \phi_{NM}(x,t)]^T.
\]

In the rest of this section we discuss the application of the radial basis functions for solving parabolic partial differential equation. Let

\[
u_t(x,t) + \beta u_x(x,t) = \alpha u_{xx}(x,t), \quad (x,t) \in (0, L) \times (0, T),
\]

with the following initial and boundary conditions:

\[
u(x,t) = f(x), \quad (x,t) \in (0, L) \times \{0\},
\]

\[
u(x,t) = g_0(t), \quad (x,t) \in \{0\} \times (0, T],
\]

\[
u(x,t) = g_1(t), \quad (x,t) \in \{L\} \times (0, T].
\]

Let \( \Xi = \{(x_i, t_j) | x_i = L \frac{i}{N}, t_j = T \frac{j}{M}, i = 0, 1, \ldots, N, j = 0, 1, \ldots, M \} \). Using a RBFs method, the solution of the problem is considered as

\[
\hat{u}(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{M} r_{ij} \phi_{ij}(x,t),
\]
where \( r_{ij} \) are unknown which remain to be determined and \( \phi_{ij}(x, t) \) is the Laguerre-Gaussians, i.e. 
\[
\phi_{ij}(x, t) = (2 - \varepsilon^2((x - x_i)^2 + (t - t_j)^2))e^{-\varepsilon^2((x-x_i)^2+(t-t_j)^2)}.
\]
Now by the collocation approach we impose the approximate solution \( \tilde{u} \) to satisfy the differential equation and the initial and boundary conditions at \((x_i, t_j), i = 0, 1, ..., N, j = 0, 1, ..., M\). So, we have
\[
\tilde{u}_t(x_i, t_j) + \beta \tilde{u}_x(x_i, t_j) = \alpha \tilde{u}_{xx}(x_i, t_j), \quad (x_i, t_j) \in (0, L) \times (0, T],
\]
\[
\tilde{u}(x_i, t_j) = f(x_i), \quad (x_i, t_j) \in (0, L) \times \{0\},
\]
\[
\tilde{u}(x_i, t_j) = g_0(t_j), \quad (x_i, t_j) \in \{0\} \times (0, T],
\]
\[
\tilde{u}(x_i, t_j) = g_1(t_j), \quad (x_i, t_j) \in \{L\} \times (0, T],
\]
which results a linear system of equations. Solving the resulted system, the unknown values \( r_{ij}, i = 0, 1, ..., N, j = 0, 1, ..., M \) can be found. Similarly, we approximate the solution for MQ and IMQ basis functions.

![Figure 1. Analytical (line) and estimated (point) solutions with \( dx = dt = 0.0714 \) and \( \varepsilon = 0.5 \) for \( T = 1 \) from Example 1.](image)

### 3 Numerical examples

**Example 3.1.** Consider Eqs. (1)-(2) with \( L = 2, T = 1 \) and
\[
f(x) = \sin(x), \quad g_0(t) = e^{-\alpha t} \sin(-\beta t), \quad g_1(t) = e^{-\alpha t} \sin(1 - \beta t),
\]
which has the exact solution
\[
u(x, t) = e^{-\alpha t} \sin(x - \beta t).
\]
For this problem we put \( \beta = 1 \). In Table 2 we give the absolute errors with \( dx = dt = 0.0714 \) for LG-RBFs with \( \varepsilon = 0.5 \), and for MQ and IMQ basis functions with \( \varepsilon = 5.3 \) at final time \( T = 1 \). To compare our result we give the absolute errors for the Compact finite difference scheme [1]. Analytical and numerical solutions for \( T = 1 \) are given in Fig. 1.

**Example 3.2.** Consider the heat equation
\[
u_t(x, t) = \frac{1}{\pi^2} u_{xx}(x, t),
\]
with \( L = 1, T = 1 \) and
\[
f(x) = \sin(\pi x), \quad g_0(t) = 0, \quad g_1(t) = 0,
\]
which has the exact solution
\[
u(x, t) = e^{-t} \sin(\pi x).
\]
Table 2: Computational results for Example 1.

<table>
<thead>
<tr>
<th>x</th>
<th>Method [1]</th>
<th>Present method</th>
<th>LG-RBF</th>
<th>MQ-RBF</th>
<th>IMQ-RBF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P_e=1000</td>
<td>P_e=10,000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>1.4e-06</td>
<td>1.0e-06</td>
<td>2.9e-09</td>
<td>1.2e-05</td>
<td>3.2e-06</td>
</tr>
<tr>
<td>0.50</td>
<td>1.7e-06</td>
<td>4.9e-07</td>
<td>5.8e-09</td>
<td>2.2e-05</td>
<td>6.3e-06</td>
</tr>
<tr>
<td>0.75</td>
<td>1.9e-06</td>
<td>6.9e-06</td>
<td>5.8e-09</td>
<td>4.0e-05</td>
<td>1.1e-05</td>
</tr>
<tr>
<td>1.00</td>
<td>9.7e-07</td>
<td>2.6e-05</td>
<td>1.8e-08</td>
<td>5.9e-05</td>
<td>1.6e-05</td>
</tr>
<tr>
<td>1.25</td>
<td>1.1e-07</td>
<td>7.0e-05</td>
<td>1.3e-08</td>
<td>8.3e-05</td>
<td>1.8e-05</td>
</tr>
<tr>
<td>1.50</td>
<td>2.0e-07</td>
<td>1.6e-04</td>
<td>3.7e-07</td>
<td>9.9e-05</td>
<td>1.5e-05</td>
</tr>
<tr>
<td>1.75</td>
<td>2.9e-07</td>
<td>2.7e-04</td>
<td>1.2e-07</td>
<td>8.9e-05</td>
<td>2.7e-06</td>
</tr>
</tbody>
</table>

Figure 2. Analytical (line) and estimated (point) solutions with \(dx=dt=0.1\) and \(\varepsilon=0.5\) for \(T=1\) from Example 2.

In Table 3 we give the absolute errors for LG-RBFs with \(dx=dt=0.1\) and \(\varepsilon=0.5\), and with \(dx=dt=0.0667\), \(dx=dt=0.05\) and \(\varepsilon=0.4\) at final time \(T=1\). In Table 4 maximum errors obtained for LG-RBF are presented. Also we give maximum errors for MQ and IMQ basis functions with \(dx=dt=0.1\) and \(\varepsilon=5.6\), and with \(dx=dt=0.05\) and \(\varepsilon=4.6\). We compared our method together with high-order compact boundary value method [6] and compact finite difference scheme [4]. Analytical and numerical solutions for \(T=1\) are given in Fig. 2.

Table 3: Computational results for Example 2.

<table>
<thead>
<tr>
<th>x</th>
<th>Present method with M=N=10,(\varepsilon=0.5)</th>
<th>M=N=15,(\varepsilon=0.4)</th>
<th>M=N=20,(\varepsilon=0.4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.4239e-07</td>
<td>3.8969e-09</td>
<td>1.5686e-10</td>
</tr>
<tr>
<td>0.2</td>
<td>2.7072e-07</td>
<td>1.0942e-08</td>
<td>8.6829e-11</td>
</tr>
<tr>
<td>0.3</td>
<td>3.7207e-07</td>
<td>1.3111e-08</td>
<td>8.8855e-11</td>
</tr>
<tr>
<td>0.4</td>
<td>4.3692e-07</td>
<td>1.0763e-08</td>
<td>1.6711e-10</td>
</tr>
<tr>
<td>0.5</td>
<td>4.5931e-07</td>
<td>1.0829e-08</td>
<td>1.0856e-10</td>
</tr>
<tr>
<td>0.6</td>
<td>4.3732e-07</td>
<td>2.6629e-09</td>
<td>2.8712e-10</td>
</tr>
<tr>
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<td>3.7285e-07</td>
<td>1.2411e-08</td>
<td>9.8855e-11</td>
</tr>
<tr>
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<td>6.1578e-09</td>
<td>7.8830e-11</td>
</tr>
<tr>
<td>0.9</td>
<td>1.4323e-07</td>
<td>1.4603e-08</td>
<td>2.1134e-11</td>
</tr>
</tbody>
</table>

4 Conclusion

A RBF-based numerical method was proposed for solving the one-dimensional heat and advection-diffusion equations. The Laguerre-Gaussians radial basis functions (LG-RBFs) on interval \(t\in[0,L]\) and \(x\in[0,T]\)
Table 4: Maximum errors obtained for Example 2.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>LG-RBF</td>
</tr>
<tr>
<td>10</td>
<td>1.5e-05</td>
<td>1.5e-05</td>
<td>4.6e-07</td>
</tr>
<tr>
<td>20</td>
<td>9.5e-07</td>
<td>9.4e-07</td>
<td>2.9e-10</td>
</tr>
</tbody>
</table>

were employed. The RBFs technique provided a closed form approximation of the solution. The method was based upon reducing the system into a set of algebraic equations. This algorithm proposed was tested for MQ and IMQ functions on several examples from the literature. The obtained results showed that this approach using LG-RBFs can solve the problem effectively.

References


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