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Asymptotically constant solutions of functional difference systems

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Dedicated to Allan Peterson on the Occasion of His 60th Birthday.

Abstract

We consider the functional difference system $(A)$ $\Delta x_i(n) = f_i(n;X)$, $1 \leq i \leq k$, where $X = (x_1, \ldots, x_k)$ and $f_1(:, X), \ldots, f_k(:, X)$ are real-valued functionals of $X$, which may depend quite arbitrarily on values of $X(\ell)$ for multiple values of $\ell \in \mathbb{Z}$. We give sufficient conditions for $(A)$ to have solutions that approach specified constant vectors as $n \to \infty$. Some of the results guarantee only that the solutions are defined for $n$ sufficiently large, while others are global. The proof of the main theorem is based on the Schauder-Tychonoff theorem. Applications to specific quasi-linear systems are included.

Keywords: Functional difference system; Nonsingular; Quasi-linear; Schauder-Tychonoff theorem; Singular

Mathematics Subject Classification (2000): 39A11

1 Introduction

Throughout this paper $\mathbb{Z}$ is the set of all integers. If $m$ is an integer, then $\mathbb{Z}_m = \{ n \in \mathbb{Z} \mid n \geq m \}$.

We consider the functional difference system

$$\Delta x_i(n) = f_i(n; X), \quad 1 \leq i \leq k,$$

where $X = (x_1, \ldots, x_k) : \mathbb{Z} \to \mathbb{R}^k$ and $f_1(\cdot; X), \ldots, f_k(\cdot; X)$ are real-valued functionals of $X$. We view $X = \{X(\ell)\}_{\ell \in \mathbb{Z}}$ as a two-way infinite sequence; for a given $n$, $f_i(n; X)$ may depend quite arbitrarily on values of $X(\ell)$ for multiple values of $\ell \in \mathbb{Z}$. We also write the system as

$$\Delta X(n) = F(n; X). \quad (1)$$
Definition 1 If \( n_0 \) is an integer, then \( C_{n_0} \) is the space of bounded sequences \( X = \{X(n)\}_{n \in \mathbb{Z}} \) that are constant for \( n \leq n_0 \), with norm \( \|X\| = \sup_{n \in \mathbb{Z}} |X(n)| \), where \( |U| = \max\{|u_1|, \ldots, |u_k|\} \) if \( U = (u_1, \ldots, u_k) \). If \( \{X_\nu\} \) is an infinite sequence of elements in \( C_{n_0} \), we say that \( X_\nu \to X \) if \( \lim_{\nu \to \infty} \|X_\nu - X\| = 0 \). We say that \( X \in C_{n_0} \) is a solution of (1) if \( \Delta X(n) = F(n; X) \) for \( n \geq n_0 \).

Note that \( C_{n_0} \) is a Banach space. We make the following standing assumption.

Assumption 1 Let \( m \) and \( r \) be integers, with \( 0 \leq r \leq k \), and let \( \rho_1, \ldots, \rho_k \) be arbitrary positive numbers. If \( X \in C_m \) and

\[
|x_i(n)| \leq \rho_i, \quad n \in \mathbb{Z}, \quad 1 \leq i \leq r, \tag{2}
\]

and

\[
|x_i(n)| \geq \rho_i, \quad n \in \mathbb{Z}, \quad r + 1 \leq i \leq k, \tag{3}
\]

then

\[
|f_i(n; X)| \leq w_i(n, \rho_1, \ldots, \rho_k), \quad n \in \mathbb{Z}_m, \quad 1 \leq i \leq k, \tag{4}
\]

where \( w_i : \mathbb{Z}_m \times (0, \infty)^k \to (0, \infty) \) and

\[
\sum_{n=m}^\infty w_i(n, \rho_1, \ldots, \rho_k) < \infty, \quad 1 \leq i \leq k, \tag{5}
\]

for all \( \rho_1, \ldots, \rho_k > 0 \). Finally, if \( \{X_\nu\} \subset C_m \) with

\[
|x_{i\nu}(n)| \leq \rho_i, \quad n \in \mathbb{Z}, \quad 1 \leq i \leq r, \tag{6}
\]

and

\[
|x_{i\nu}(n)| \geq \rho_i, \quad n \in \mathbb{Z}, \quad r + 1 \leq i \leq k, \tag{7}
\]

for all \( \nu \), and \( X_\nu \to X \), then

\[
\lim_{\nu \to \infty} F(n; X_\nu) = F(n; X), \quad n \in \mathbb{Z}_m. \tag{8}
\]

We say that the system (1) is nonsingular in \( x_1, \ldots, x_r \) and singular in \( x_{r+1}, \ldots, x_k \). We also say that (1) is purely singular if \( r = 0 \), purely nonsingular if \( r = k \), or mixed if \( 0 < r < k \).

2 The Main Theorem

The following theorem is our main result.

Theorem 1 Suppose that \( n_0 \geq m \) and \( \rho_1, \ldots, \rho_k \), and \( \alpha_1, \ldots, \alpha_k \) are positive numbers such that \( \alpha_i < 1 \) if \( 1 \leq i \leq r \), and

\[
\sum_{n=n_0}^\infty w_i(n, \rho_1, \ldots, \rho_k) \leq \alpha_i \rho_i, \quad 1 \leq i \leq k. \tag{9}
\]
Let \( c_1, \ldots, c_k \) be constants such that
\[
|c_i| \leq (1 - \alpha_i)\rho_i, \quad 1 \leq i \leq r,
\]
and
\[
|c_i| \geq (1 + \alpha_i)\rho_i, \quad r + 1 \leq i \leq k.
\]
Then there is an \( \hat{X} \) in \( C_{n_0} \) such that
\[
\Delta \hat{X}(n) = F(n; \hat{X}), \quad n \in \mathbb{Z}_{n_0},
\]
\[
|\hat{x}_i(n) - c_i| \leq \alpha_i\rho_i, \quad n \in \mathbb{Z}, \quad 1 \leq i \leq k,
\]
and
\[
\lim_{n \to \infty} \hat{x}_i(n) = c_i, \quad 1 \leq i \leq k.
\]

**Proof.** We obtain \( \hat{X} \) as a fixed point of the transformation \( Y = TX \) defined by
\[
y_i(n) = \begin{cases} 
  c_i - \sum_{\ell=n_0}^{\infty} f_i(\ell; X), & n \geq n_0, \\
  c_i - \sum_{\ell=n_0}^{n-1} f_i(\ell; X), & n < n_0,
\end{cases} \quad 1 \leq i \leq k,
\]
acting on the subset \( S_{n_0} \) of \( C_{n_0} \) such that
\[
|x_i(n) - c_i| \leq \alpha_i\rho_i, \quad n \in \mathbb{Z}, \quad 1 \leq i \leq k.
\]

If \( \hat{X} = TX \) for some \( \hat{X} \in S_{n_0} \), then \( \hat{X} \) satisfies (9), (10), and (11).

Since \( S_{n_0} \) is a closed convex subset of a Banach space, the Schauder-Tychonoff theorem \([1, p. 405]\) asserts that \( \hat{X} = TX \) for some \( \hat{X} \) in \( S_{n_0} \) if
(a) \( T \) is defined on \( S_{n_0} \);
(b) \( T(S_{n_0}) \subset S_{n_0} \);
(c) \( TX_\nu \to TX \) if \( \{X_\nu\} \subset S_{n_0} \) and \( X_\nu \to X \);
(d) \( T(S_{n_0}) \) has compact closure.

For the rest of the proof we assume that \( X \in S_{n_0} \). Then (7) and (13) imply (2), while (8) and (13) imply (3). By Assumption 1, (2) and (3) imply (4); hence, (6) and (12) imply that \( Y = TX \) is defined, and that
\[
|y_i(n) - c_i| \leq \alpha_i\rho_i, \quad n \in \mathbb{Z}, \quad 1 \leq i \leq k.
\]

This establishes hypotheses (a) and (b) of the Schauder-Tychonoff theorem.

Now suppose \( \{X_\nu\} \subset S_{n_0} \) and \( X_\nu \to X \). Let \( Y_\nu = TX_\nu = (y_{1\nu}, \ldots, y_{k\nu}) \) and \( Y = TX = (y_1, \ldots, y_k) \). From (12),
\[
|y_{i\nu}(n) - y_i(n)| \leq \sum_{\ell=n_0}^{\infty} |f_i(\ell; X_\nu) - f_i(\ell; X)|, \quad n \in \mathbb{Z}, \quad 1 \leq i \leq k.
\]
From (6), if \( \epsilon > 0 \), there is an \( N > n_0 \) such that
\[
\sum_{\ell = N + 1}^{\infty} w_i(\ell, \rho_1, \ldots, \rho_k) < \epsilon, \quad 1 \leq i \leq k.
\]

Then (4) and (14) imply that
\[
|y_{i\nu}(n) - y_i(n)| \leq \sum_{\ell = n_0}^{N} |f_i(\ell; X) - f_i(\ell; X)| + 2\epsilon, \quad n \in \mathbb{Z}, \quad 1 \leq i \leq k. \quad (15)
\]

Since
\[
\lim_{\nu \to \infty} |f_i(\ell; X) - f_i(\ell; X)| = 0, \quad \ell \geq n_0,
\]
from (5), (15) implies that \( \lim_{n \to \infty} \|Y_\nu - Y\| = 0 \). This establishes hypothesis (c) of the Schauder-Tychonoff theorem.

We will now show that \( \overline{T(S_{n_0})} \) is compact. Let \( C = (c_1, \ldots, c_k) \) and \( \Gamma = (\gamma_1, \ldots, \gamma_k) \), with
\[
\gamma_i(n) = \sum_{\ell = n}^{\infty} w_i(\ell, \rho_1, \ldots, \rho_k), \quad 1 \leq i \leq k, \quad n \geq n_0.
\]

From (4) and (12),
\[
\overline{T(S_{n_0})} \subset A = \{ V \in C_{n_0} \mid |V(n) - C| \leq |\Gamma(n)| \},
\]
so it suffices to show that \( A \) is compact. From [2, pp. 51-53], this is true if \( A \) is totally bounded; that is, for every \( \epsilon > 0 \) there is a finite subset \( A_\epsilon \) of \( C_{n_0} \) such that for each \( V \in A \) there is a \( \tilde{V} \in A_\epsilon \) that satisfies the inequality \( \|V - \tilde{V}\| < \epsilon \). To establish the existence of \( A_\epsilon \), choose an integer \( n_1 \geq n_0 \) such that \( |\Gamma(n_1)| < \epsilon \). Now let
\[
M = \max \{|\Gamma(n)| \mid n_0 \leq n \leq n_1 - 1 \},
\]
let \( p \) be an integer such that \( p\epsilon > M \), and let \( Q = \{ r\epsilon \mid r = \text{ integer }, -p \leq r \leq p \} \).

Let \( A_\epsilon \) be the finite set of \( k \)-vector functions \( A \) on \( Z \) defined as follows:

(i) If \( n \geq n_1 \), then \( A(n) = C \).

(ii) If \( n \leq n_0 \), then \( A(n) = A(n_0) \).

(iii) If \( n_0 \leq n \leq n_1 - 1 \), then \( A(n) = (c_1 + q_1(n), \ldots, c_k + q_k(n)) \), where \( q_1(n), \ldots, q_k(n) \) are in \( Q \).

Then, since \( |V(n) - C| \leq M \) for \( n_0 \leq n \leq n_1 - 1 \) if \( V \in A \), the set \( A_\epsilon \) has the desired property. Therefore the Schauder-Tychonoff theorem implies that \( T\hat{X} = \hat{X} \) for some \( \hat{X} \) in \( S_{n_0} \).
3 Applications of Theorem 1

Since all our results follow from Theorem 1, we will simply verify (6), (7), and (8) in each case, without specifically citing Theorem 1. We say that the problem $P_{r}(n_{0}; c_{1}, \ldots, c_{k})$ has a solution if there is a sequence $\hat{X}$ in $C_{n_{0}}$ such that $\Delta \hat{X}(n) = F(n; \hat{X})$, $n \geq n_{0}$, and $\lim_{n \to \infty} \hat{x}_{i}(n) = c_{i}$, $1 \leq i \leq k$. Some of our results are local at $\infty$, in that a solution is shown to exist only if $n_{0}$ is sufficiently large. Others are global, in that a solution is shown to exist for all $n \geq m$.

**Theorem 2** If $c_{i} \neq 0$ for $r + 1 \leq i \leq k$, then $P_{r}(n_{0}; c_{1}, \ldots, c_{k})$ has a solution if $n_{0}$ is sufficiently large.

**Proof.** Let $\alpha_{1}, \ldots, \alpha_{k}$ be positive, with $\alpha_{i} < 1$ for $1 \leq i \leq r$. Choose $\rho_{1}, \ldots, \rho_{k}$ to satisfy (7) and (8). Then choose $n_{0}$ to satisfy (6). □

**Theorem 3** If $\sum_{n=n_{0}}^{\infty} w_{i}(n, \rho_{1}, \ldots, \rho_{k}) < \rho_{i}$, $1 \leq i \leq r$, then $P_{r}(n_{0}, c_{1}, \ldots, c_{k})$ has a solution if $|c_{1}|, \ldots, |c_{r}|$ are sufficiently small and $|c_{r+1}|, \ldots, |c_{k}|$ are sufficiently large.

**Proof.** Choose $\alpha_{1}, \ldots, \alpha_{k}$ sufficiently large to satisfy (6). (Because of (16), this can be achieved with $\alpha_{i} < 1$, $1 \leq i \leq r$.) Then $P(n_{0}, c_{1}, \ldots, c_{k})$ has a solution if (7) and (8) hold. □

**Theorem 4** If $|c_{1}|, \ldots, |c_{k}|$ are sufficiently large, then $P_{0}(m; c_{1}, \ldots, c_{k})$ has a solution.

**Proof.** Let $\rho_{1}, \ldots, \rho_{k}$ be positive. Choose $\alpha_{1}, \ldots, \alpha_{k}$ to satisfy (6) with $n_{0} = m$. Then choose $c_{1}, \ldots, c_{k}$ to satisfy (8) with $r = 0$. □

**Theorem 5** Suppose that

$$\lim_{\rho \to 0+} \rho^{-1} \sum_{n=m}^{\infty} w_{i}(n, \rho, \ldots, \rho) = \psi_{i} < 1, \quad 1 \leq i \leq k. \quad (17)$$

Then $P_{k}(m; c_{1}, \ldots, c_{k})$ has a solution if $|c_{1}|, \ldots, |c_{k}|$ are sufficiently small.

**Proof.** Let $\psi_{i} < \alpha_{i} < 1$, $1 \leq i \leq k$. From (17), we can choose $\rho_{0}$ so small that

$$\sum_{n=m}^{\infty} w_{i}(n, \rho_{0}, \ldots, \rho_{0}) \leq \alpha_{i} \rho_{0}, \quad 1 \leq i \leq k.$$

Now choose $c_{1}, \ldots, c_{k}$ so that $|c_{i}| \leq (1 - \alpha_{i}) \rho_{0}$, $1 \leq i \leq k$. □
THEOREM 6 Suppose that
\[ \limsup_{\rho \to \infty} \rho^{-1} \sum_{n=m}^{\infty} w_i(n, \rho, \ldots, \rho) = \eta_i < 1, \quad 1 \leq i \leq k. \] (18)

Let \( c_1, \ldots, c_k \) be arbitrary. Then \( P_k(m; c_1, \ldots, c_k) \) has a solution.

PROOF. Let \( \eta_i < \alpha_i < 1, \ 1 \leq i \leq k. \) From (18), we can choose \( \rho_0 \) so large that
\[ \sum_{n=m}^{\infty} w_i(n, \rho_0, \ldots, \rho_0) \leq \alpha_i \rho_0, \quad 1 \leq i \leq k. \]

\( \square \)

ASSUMPTION 2 In addition to Assumption 1, assume that (1) is mixed (that is, \( 0 < r < k \)), and
\[ w_i(n, \rho_1, \rho_2, \ldots, \rho_k) = u_i(n, \rho_1, \ldots, \rho_r) + v_i(n, \rho_{r+1}, \ldots, \rho_k), \quad 1 \leq i \leq r, \]
where \( u_i \) and \( v_i \) are positive, and
\[ \lim_{\rho \to \infty} \sum_{n=m}^{\infty} v_i(n, \rho, \ldots, \rho) = 0, \quad 1 \leq i \leq r. \] (19)

THEOREM 7 If Assumption 2 holds, then \( P_r(n_0, c_1, \ldots, c_k) \) has a solution if \( n_0 \) and \( |c_{r+1}|, \ldots, |c_k| \) are sufficiently large and \( |c_1|, \ldots, |c_r| \) are sufficiently small.

PROOF. Let \( \rho_1 > 0 \) and \( 0 < \alpha_1 < 1, \) and choose \( n_0 \geq m \) so that
\[ \sum_{n=n_0}^{\infty} u_i(n, \rho_1, \ldots, \rho_1) < \alpha_1 \rho_1, \quad 1 \leq i \leq r. \]

From (19), we can choose \( \rho_2 \) so large that
\[ \sum_{n=n_0}^{\infty} (u_i(n, \rho_1, \ldots, \rho_1) + v_i(n, \rho_2, \ldots, \rho_2)) \leq \alpha_1 \rho_1, \quad 1 \leq i \leq r. \]

Now choose \( \alpha_2 \) so that
\[ \sum_{n=n_0}^{\infty} w_i(n, \rho_1, \ldots, \rho_k) \leq \alpha_2 \rho_2, \quad r + 1 \leq i \leq k, \]
if \( \rho_1 = \rho_1, \ 1 \leq i \leq r, \) and \( \rho_1 = \rho_2, \ r + 1 \leq i \leq k. \) Then choose \( |c_i| \leq (1 - \alpha_1) \rho_1, \ 1 \leq i \leq r, \) and \( |c_i| \geq (1 + \alpha_2) \rho_2, \ r + 1 \leq i \leq k. \) \( \square \)
Theorem 8 In addition to Assumption 2, suppose that
\[ \limsup_{\rho \to 0^+} \sum_{n=m}^{\infty} u_i(n, \rho, \ldots, \rho) = \psi_i < 1, \quad 1 \leq i \leq r. \]
Then \( P_r(m, c_1, \ldots, c_k) \) has a solution if \( |c_1|, \ldots, |c_r| \) are sufficiently small and \( |c_{r+1}|, \ldots, |c_k| \) are sufficiently large.

Proof. Let \( \psi_i < \alpha_i < 1, 1 \leq i \leq r \). Choose \( \rho_1 \) so small that
\[ \sum_{n=m}^{\infty} u_i(n, \rho_1, \ldots, \rho_1) < \alpha_i \rho_1, \quad 1 \leq i \leq r. \]
Now apply the argument used in the proof of Theorem 7, with \( n_0 = m \).

Theorem 9 In addition to Assumption 2, suppose that
\[ \limsup_{\rho \to \infty} \sum_{n=m}^{\infty} u_i(n, \rho, \ldots, \rho) = \eta_i < 1, \quad 1 \leq i \leq r. \]
Then \( P_r(m, c_1, \ldots, c_k) \) has a solution if \( |c_{r+1}|, \ldots, |c_k| \) are sufficiently large.

Proof. Let \( \eta_i < \alpha_i < 1, 1 \leq i \leq r \). Choose \( \rho_1 \) so large that \( \rho_1 \geq |c_i|/(1-\alpha_i) \), \( 1 \leq i \leq r \), and
\[ \sum_{n=m}^{\infty} u_i(n, \rho_1, \ldots, \rho_1) < \alpha_i \rho_1, \quad 1 \leq i \leq r. \]
Now apply the argument used in the proof of Theorem 7, with \( n_0 = m \).

4 Quasi-linear Systems: I

Consider the system
\[ \Delta x_i(n) = \sum_{j=1}^{k} a_{ij}(n) g_{ij}(x_j(\phi_{ij}(n))), \quad 1 \leq i \leq k, \quad (20) \]
assuming throughout that, for some integer \( m \) and \( 1 \leq i \leq j \leq k \), \( \phi_{ij} : \mathbb{Z}_m \to \mathbb{Z} \), \( g_{ij} : \mathbb{Z} \to \mathcal{R} \), \( a_{ij} : \mathbb{Z}_m \to \mathcal{R} \),
\[ |g_{ij}(u)| = |u|^\gamma_{ij} \quad \text{and} \quad \sum_{n=m}^{\infty} |a_{ij}(n)| < \infty. \]
We assume that for some \( r \) in \( \{0, 1, \ldots, k\} \), \( \gamma_{ij} > 0 \) if \( 1 \leq i \leq r \) and \( \gamma_{ij} < 0 \) if \( r + 1 \leq i \leq k \), for \( 1 \leq j \leq k \). Then Assumption 1 holds, with
\[ w_i(n, \rho_1, \rho_2, \ldots, \rho_k) = \sum_{j=1}^{k} |a_{ij}(n)| \rho_j^{\gamma_{ij}}. \]
It is to be understood throughout this section that this is the definition of $w_i$.

If $0 < r < k$, then (20) satisfies Assumption 2 with

$$u_i(n, \rho_1, \ldots, \rho_r) = \sum_{j=1}^{r} |a_{ij}(n)| \rho_j^{\gamma_{ij}}$$

and

$$v_i(n, \rho_{r+1}, \ldots, \rho_k) = \sum_{j=r+1}^{k} |a_{ij}(n)| \rho_j^{\gamma_{ij}}.$$  

**Theorem 10** If $\gamma_{ij} > 0$, $1 \leq i, j \leq k$, there is an $n_0 \geq m$, which depends upon $c_1, \ldots, c_k$, such that $P_k(n_0; c_1, \ldots, c_k)$ has a solution.

**Proof.** If $0 < \alpha < 1$, choose $\rho_1, \rho_2, \ldots, \rho_k$ so that $|c_i| \leq (1 - \alpha)\rho_i$, $1 \leq i \leq k$. Then choose $n_0$ so that

$$\sum_{n=n_0}^{\infty} w_i(n, \rho_1, \rho_2, \ldots, \rho_k) \leq \alpha \rho_i, \quad 1 \leq i \leq k.$$  

$\Box$

**Theorem 11** If $\gamma_{ij} = 1$, $1 \leq i, j \leq k$, there is an $n_0 \geq m$, independent of $c_1, \ldots, c_k$, such that $P_k(n_0; c_1, \ldots, c_k)$ has a solution.

**Proof.** If $0 < \alpha < 1$, choose $n_0$ so that

$$\sum_{n=n_0}^{\infty} w_i(n, 1, \ldots, 1) < \alpha, \quad 1 \leq i \leq k.$$  

Then

$$\sum_{n=n_0}^{\infty} w_i(n, \rho, \ldots, \rho) < \alpha \rho, \quad 1 \leq i \leq k,$$

for any $\rho > 0$. For arbitrary $c_1, \ldots, c_k$ choose $\rho$ so that $|c_i| \leq (1 - \alpha)\rho$, $1 \leq i \leq k$. $\Box$

Theorems 5-9 imply the following theorems.

**Theorem 12** If $\gamma_{ij} > 1$, $1 \leq i, j \leq k$, then $P_k(m; c_1, \ldots, c_k)$ has a solution if $|c_1|, \ldots, |c_k|$ are sufficiently small.

**Theorem 13** If $0 < \gamma_{ij} < 1$, $1 \leq i, j \leq k$, and $c_1, \ldots, c_k$ are arbitrary, then $P_k(m; c_1, \ldots, c_k)$ has a solution.

**Theorem 14** If $0 < r < k$, then $P_r(n_0; c_1, \ldots, c_k)$ has a solution if $n_0$ and $|c_r|$, $|c_{r+1}|$, $\ldots, |c_k|$ are sufficiently large.

**Theorem 15** If $0 < r < k$ and $\gamma_{ij} > 1$, $1 \leq i \leq r, 1 \leq j \leq k$, then $P_r(m; c_1, \ldots, c_k)$ has a solution if $|c_1|, \ldots, |c_r|$ are sufficiently small and $|c_{r+1}|$, $\ldots, |c_k|$ are sufficiently large.

**Theorem 16** If $0 < r < k$ and $0 < \gamma_{ij} < 1$, $1 \leq i \leq r, 1 \leq j \leq k$, then $P_r(m; c_1, \ldots, c_k)$ has a solution if $|c_{r+1}|, \ldots, |c_k|$ are sufficiently large.

8
5 Quasi-linear Systems: II

In this section we consider

\[ \Delta x_i(n) = \sum_{j=1}^{k} \beta_{ij}^{n} \sum_{\ell=0}^{n} p_{ij}(n-\ell) g_{ij}(x_{j}(\phi_{ij}(n))), \quad 1 \leq i \leq k, \quad n \geq 0, \quad (21) \]

where \( g_{ij} \) and \( \phi_{ij} \) are as in the previous section, \( |\beta_{ij}| < 1 \), and

\[ \sum_{n=0}^{\infty} |\beta_{ij}^{n} p_{ij}(n)| < \infty, \quad 1 \leq i, j \leq n. \]

Here we can take

\[ w_i(n, \rho_1, \ldots, \rho_k) = \sum_{j=1}^{k} \rho_j^{\gamma_{ij}} \sum_{\ell=0}^{n} |\beta_{ij}^{n} p_{ij}(n-\ell)|. \]

Therefore Assumption 1 holds with

\[ \sum_{n=0}^{\infty} w_i(n, \rho_1, \ldots, \rho_k) = \sum_{j=1}^{k} \sigma_{ij} \rho_j^{\gamma_{ij}}, \]

where

\[ \sigma_{ij} = \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} |\beta_{ij}^{n} p_{ij}(n-\ell)| = \sum_{\ell=0}^{\infty} \sum_{n=\ell}^{\infty} |\beta_{ij}^{n} p_{ij}(n-\ell)| \]

\[ = \frac{1}{1-|\beta_{ij}|} \sum_{n=0}^{\infty} |\beta_{ij}^{n} p_{ij}(n)|. \]

All the arguments used in the previous section can now be used with \( |a_{ij}| \) replaced by \( \sigma_{ij} \); therefore, Theorems 10-16 all hold (with \( m = 0 \)) for (21).

References
