Systems of difference equations with asymptotically constant solutions

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SYSTEMS OF DIFFERENCE EQUATIONS WITH ASYMPTOTICALLY CONSTANT SOLUTIONS

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We consider the system

\[ \Delta x_n = A_n x_n + f(n, x_n), \]  

where \( x_n \) and \( f \) are \( k \)-vectors (real or complex) and \( A_n \) is a \( k \times k \) matrix. We give conditions implying that (1) has a solution \( \{ \hat{x}_n \} \) such that \( \lim_{n \to \infty} \hat{x}_n = c \), a given constant vector.

If \( u \) is a \( k \)-vector and \( B \) is a \( k \times k \) matrix, then \( |u| \) and \( |A| \) are the \( \infty \)-norms of \( u \) and \( A \).

**Theorem 1** Let \( c \) be a given \( k \)-vector, and suppose there is a constant \( M > 0 \) and an integer \( N \) such that \( f(n, x) \) is continuous with respect to \( x \) and

\[ |f(n, x) - f(n, c)| \leq R(n, |x - c|) \]

on the set

\[ S = \{(n, x) | n \geq N, |x - c| \leq M\}, \]

where \( R = R(n, \lambda) \) is defined on the set

\[ \{(n, x) | n \geq N, 0 \leq \lambda \leq M\} \]

and nondecreasing in \( \lambda \) for each \( n \), and

\[ \sum_{n=N}^{\infty} |R(n, M)| < \infty. \]

Suppose that either

\[ \sum_{n=N}^{\infty} |A_n| < \infty \]

or there is a positive integer \( q \) such that the sequences

\[ A_n^{(r)} = \sum_{m=n}^{\infty} A_{m+1}^{(r-1)} A_m, \quad r = 1, 2, \ldots, q \quad (\text{with } A_n^{(0)} = I) \]
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are all defined for \( n \geq N \), and

\[
\sum_{n=N}^{\infty} |A_{n+1}^{(q)} A_n| < \infty. \tag{6}
\]

(If (4) holds we let \( q = 0 \) and (5) is vacuous; note that (4) and (6) are equivalent in this case, since \( A_{m+1}^{(0)} = I \).

Define

\[
\Gamma_n = \sum_{r=0}^{m} A_{n+1}^{(r)}, \tag{7}
\]

and suppose that \( \sum_{n=N}^{\infty} \Gamma_n f(n, c) \) converges (perhaps conditionally).

Then, if \( n_0 \) is sufficiently large, there is a solution \( \hat{X} = \{\hat{x}_n\}_{n=n_0}^{\infty} \) of (1) such that

\[
|\hat{x}_n - c| \leq M, \quad n \geq n_0, \tag{8}
\]

and

\[
\lim_{n \to \infty} \hat{x}_n = c. \tag{9}
\]

**Proof.** Since \( A_n^{(0)} = I \) and \( \lim_{n \to \infty} A_{n+1}^{(r)} = 0 \) if \( r > 0 \), \( \lim_{n \to \infty} \Gamma_n = I \). Therefore \( \Gamma_n \) is invertible for large \( n \). For now, choose \( n_0 \geq N \) so that \( \Gamma_n \) is invertible if \( n \geq n_0 \); we will impose another condition on \( n_0 \) later. Define

\[
h_n = (\Gamma_n^{-1} - I)c - \Gamma_n^{-1} \left( \sum_{m=n}^{\infty} A_{m+1}^{(q)} A_m c + \Gamma_{m+1} f(m, c) \right). \tag{10}
\]

Let \( B \) be the Banach space of bounded sequences \( U = \{u_n\}_{n=n_0}^{\infty} \) of \( k \)-vectors, with norm \( \|U\| = \sup_{n \geq n_0} |u_n| \). Let \( B_M \) be the closed convex subset

\[
B_M = \{ U \in B \mid \|U\| \leq M \}
\]

of \( B \). From (2) and our assumption that \( R(n, \lambda) \) is nondecreasing with respect to \( \lambda \), if \( U \in B_M \) then

\[
|f(m, u_m + c) - f(m, c)| \leq R(m, |u_m|) \leq R(m, M). \tag{11}
\]

Therefore (3) and (6) imply that if \( U \in B_M \) then the sequence \( T U \), with

\[
(TU)_n = h_n - \Gamma_n^{-1} \sum_{m=n}^{\infty} \left[ A_{m+1}^{(q)} A_m u_m + \Gamma_{m+1} [f(m, u_m + c) - f(m, c)] \right] \tag{12}
\]

is well defined. We will show that if \( n_0 \) is sufficiently large then \( T \) is a continuous mapping of \( B_M \) into itself and \( T(B_M) \) has compact closure. Given this, the Schauder-Tychonoff theorem [1, p. 405] implies that \( TU = \hat{U} \) for some \( \hat{U} \in B_M \). We will then show that \( \hat{X} = C + \hat{U} \) (with \( C = \{c, c, c, \ldots\}_{n_0}^{\infty} \)) satisfies (1), (8), and (9).

Let

\[
\mu(n_0) = \sup_{m \geq n_0} |\Gamma_m^{-1}| \quad \text{and} \quad v(n_0) = \sup_{m \geq n_0} |\Gamma_{m+1}|.
\]
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From (11) and (12), if \( U \in B_M \) then

\[
|\langle T U \rangle_n| \leq |h_n| + \mu(n_0) \sum_{m=n}^{\infty} \left| A_{m+1} A_m^{(q)} |M + v(n_0) R(m, M)| \right|. \tag{13}
\]

Since \( \lim_{n_0 \to \infty} \mu(n_0) = \lim_{n_0 \to \infty} v(n_0) = 1 \), (3) and (6) enable us to choose \( n_0 \) so that the quantity on the right side of (13) is less than \( M \) if \( n \geq n_0 \). Then \( T(B_M) \subset B_M \).

We will now show that \( T \) is continuous on \( B_M \). Suppose that \( U = \lim_{r \to \infty} U^{(r)} \) where \( \{U^{(r)}\} \subset B_M \). Let \( V = TU \) and \( V^{(r)} = TU^{(r)} \). Then

\[
v^{(r)}_m - v_m = \Gamma_m^{-1} \sum_{m=n}^{\infty} \left[ A_{m+1} A_m^{(q)} (u_m - u^{(r)}_m) + \Gamma_m^{(r)} \left( f(m, u_m + c) - f(m, u^{(r)}_m + c) \right) \right].
\]

Therefore

\[
\|v^{(r)} - v\| \leq \mu(n_0) \sum_{m=n_0}^{\infty} \sigma^{(r)}_m, \tag{14}
\]

where

\[
\sigma^{(r)}_m = \left| A_{m+1} A_m^{(q)} |u_m - u^{(r)}_m| + v(n_0) \left| f(m, u_m + c) - f(m, u^{(r)}_m + c) \right| \right|.
\]

Note that

\[
\lim_{r \to \infty} \sigma^{(r)}_m = 0, \quad m \geq n_0,
\]

because of the continuity assumption on \( f \), and

\[
\sigma^{(r)}_m \leq \sigma_m = 2 \left( M |A_{m+1} A_m^{(q)}| + |v(n_0)| R(m, M) \right), \tag{15}
\]

(see (11), applied to \( U \) and \( U^{(r)} \)) because \( U \) and \( U^{(r)} \) are in \( B_M \). Because of (3) and (6), \( \sum_{m=n_0}^{\infty} \sigma_m < \infty \). Given \( \epsilon > 0 \), choose \( n_1 \geq n_0 \) so that \( \sum_{m=n_1+1}^{\infty} \sigma_m < \epsilon \). Then (14) and (15) imply that

\[
\|v^{(r)} - v\| \leq \mu(n_0) \left( \sum_{m=n_0}^{n_1} \sigma^{(r)}_m + \epsilon \right). \tag{16}
\]

Now choose \( r_0 \) so that

\[
\sigma^{(r)}_m < \frac{\epsilon}{(n_1 - n_0 + 1)} \text{ for } m = n_0, \ldots, n_1 \text{ if } r \geq r_0.
\]

Then (16) implies that

\[
\|v^{(r)} - v\| < 2\mu(n_0)\epsilon \text{ if } r \geq r_0,
\]

which shows that \( T \) is continuous on \( B_M \).
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We will now show that \( \overline{T(B_M)} \) (the closure of \( T(B_M) \)) is compact. From (11) and (12), \( \overline{T(B_M)} \) is a subset of \[ A = \{ v \in B \mid |v_n| \leq \rho(n), n \geq n_0 \}, \]
where \[ \rho(n) = |h_n| + \mu(n_0) \left( M \sum_{m=n}^{\infty} |A_m^{(q)} A_m| + \sum_{m=n}^{\infty} v(n_0) R(m, M) \right). \]
Therefore, it suffices to show that \( A \) is compact. From [2, pp. 51-53], this is true if \( A \) is totally bounded; that is, for every \( \varepsilon > 0 \) there is a finite subset \( A_\varepsilon \) of \( B \) such that for each \( v \in A \) there is a \( \tilde{v} \in A_\varepsilon \) that satisfies the inequality \( \| v - \tilde{v} \| < \varepsilon \). To establish the existence of \( A_\varepsilon \), choose an integer \( n_1 \geq n_0 \) such that \( \rho(n) < \varepsilon \) if \( n > n_1 \), and let \( p \) be an integer such that \( \rho(n) < M \). Then, since \( |v_n| \leq M \) for all \( n \geq n_0 \), the finite set \( A_\varepsilon \) consisting of sequences of the form \[ a = (a_{n_0}, \ldots, a_{n_1}, 0, 0, \ldots) \]
where the components of the \( k \)-vectors \( \{a_{n_0}, \ldots, a_{n_1}\} \) are all in the set \[ \{-p\varepsilon, -(p-1)\varepsilon, \ldots, 0, (p-1)\varepsilon, p\varepsilon\}, \quad n = n_0, \ldots, n_1, \]
has the desired property.

Now the Schauder-Tychonoff theorem implies that \( T \) has a fixed point \( \hat{U} \). Since \( \hat{U} = T\hat{U} \), (10) and (12) imply that if \( \hat{X} = C + \hat{U} \) then
\[ \hat{x}_n = \Gamma_n^{-1} \left( c - \sum_{m=n}^{\infty} \left[ A_m^{(q)} A_m \hat{x}_m + \Gamma_m f(m, \hat{x}_m) \right] \right). \] (17)
Therefore, \( \lim_{n \to \infty} \hat{x}_n = c \). If \( q = 0 \) then (17) reduces to
\[ \hat{x}_n = c - \sum_{m=n}^{\infty} (A_m \hat{x}_m + f(m, x_m)), \]
so
\[ \Delta \hat{x}_n = A_n \hat{x}_n + f(n, \hat{x}_n). \] (18)
If \( q > 0 \) then (17) implies that
\[ \Delta \hat{x}_n = \Gamma_n^{-1} A_n^{(q)} A_n \hat{x}_n + f(n, \hat{x}_n) + (\Delta \Gamma_n^{-1}) \Gamma_n \hat{x}_n. \] (19)
Since \( \Delta \Gamma_n^{-1} = -\Gamma_n^{-1} (\Delta \Gamma_n) \Gamma_n^{-1} \), (19) implies that
\[ \Delta \hat{x}_n = \Gamma_n^{-1} A_n^{(q)} A_n - \Delta \Gamma_n \hat{x}_n + f(n, \hat{x}_n). \] (20)
However, (5) and (7) imply that
\[ \Delta \Gamma_n = -\sum_{r=1}^{q} A_{n+1}^{(r-1)} A_n. \]
so

\[ A_{n+1}^{(q)}A_n - \Delta \Gamma_n = \Gamma_{n+1}A_n, \]

and therefore (20) implies (18). ■

The hypotheses of Theorem 1 may hold for some constant vectors \( c \) and fail to hold for others. In the following corollary \( c \) may be chosen arbitrarily.

**Corollary 1** Let \( A_n \) satisfy the hypotheses of Theorem 1. Suppose there is an integer \( N \) such that \( f(n, x) \) is continuous with respect to \( x \) for all \( n \geq N \) and all \( x \), and

\[ |f(n, x_1) - f(n, x_2)| \leq R(n, |x_1 - x_2|) \]

where \( R = R(n, \lambda) \) is defined on

\[ \{(n, x) \mid n \geq N, 0 \leq \lambda \leq \infty\} \]

and nondecreasing in \( \lambda \) for each \( n \), and \( \sum_{n=N}^{\infty} |R(n, M)| < \infty \) for some constant \( M > 0 \). Suppose also that \( \sum_{n=N}^{\infty} \Gamma_{n+1}f(n, c) \) converges (perhaps conditionally) for every constant vector \( c \). Let \( c \) be a given constant vector. Then, if \( n_0 \) is sufficiently large, there is a solution \( \hat{X} = \{\hat{x}_n\}_{n=n_0}^{\infty} \) of (1) that satisfies (8) and (9).

The following corollary applies to the linear system

\[ \Delta x_n = (A_n + B_n)x_n + g_n, \tag{21} \]

where \( A_n \) and \( B_n \) are \( k \times k \) matrices and \( g_n \) is a \( k \)-vector.

**Corollary 2** Suppose that \( A_n \) satisfies the hypotheses of Theorem 1, while \( \sum_{n=0}^{\infty} |B_n| < \infty \) and \( \sum_{n=0}^{\infty} \Gamma_{n+1}g_n \) converges (perhaps conditionally). Let \( c \) be an arbitrary vector. Then (21) has a solution \( \hat{X} \) such that \( \lim_{n \to \infty} \hat{x}_n = c \).

**References**
