2000

Systems of difference equations with asymptotically constant solutions

William F. Trench, *Trinity University*

SYSTEMS OF DIFFERENCE EQUATIONS WITH ASYMPTOTICALLY CONSTANT SOLUTIONS

William F. Trench
Trinity University, San Antonio, Texas, USA

Nonlinear Analysis, Theory, Methods, and Applications 40 (2000) 611–615

Dedicated to Professor V. Lakshmikantham on his 75th birthday

Key words: Difference equations, Asymptotically constant, Conditional convergence, Banach space, Schauder-Tychonoff theorem

We consider the system

$$\Delta x_n = A_n x_n + f(n, x_n), \tag{1}$$

where x_n and f are k-vectors (real or complex) and A_n is a $k \times k$ matrix. We give conditions implying that (1) has a solution $\{\hat{x}_n\}$ such that $\lim_{n\to\infty} \hat{x}_n = c$, a given constant vector.

If u is a k-vector and B is a $k \times k$ matrix, then $|u|$ and $|A|$ are the ∞-norms of u and A.

THEOREM 1 *Let c be a given k-vector, and suppose there is a constant $M > 0$ and an integer N such that $f(n, x)$ is continuous with respect to x and*

$$|f(n, x) - f(n, c)| \le R(n, |x - c|) \tag{2}$$

on the set

$$S = \{(n, x) \mid n \ge N,\ |x - c| \le M\},$$

where $R = R(n, \lambda)$ is defined on the set

$$\{(n, x) \mid n \ge N,\ 0 \le \lambda \le M\}$$

and nondecreasing in λ for each n, and

$$\sum_{n=N}^{\infty} |R(n, M)| < \infty. \tag{3}$$

Suppose that either

$$\sum_{n=N}^{\infty} |A_n| < \infty \tag{4}$$

or there is a positive integer q such that the sequences

$$A_n^{(r)} = \sum_{m=n}^{\infty} A_{m+1}^{(r-1)} A_m, \quad r = 1, 2, \dots, q \quad (\textit{with } A_m^{(0)} = I) \tag{5}$$

are all defined for $n \geq N$, and

$$\sum_{n=N}^{\infty} |A_{n+1}^{(q)} A_n| < \infty. \tag{6}$$

(*If* (4) *holds we let* $q = 0$ *and* (5) *is vacuous; note that* (4) *and* (6) *are equivalent in this case, since* $A_{m+1}^{(0)} = I$.)

Define

$$\Gamma_n = \sum_{r=0}^{q} A_n^{(r)}, \tag{7}$$

and suppose that $\sum_{n=N}^{\infty} \Gamma_{n+1} f(n, c)$ *converges* (*perhaps conditionally*).

Then, if n_0 *is sufficiently large, there is a solution* $\hat{X} = \{\hat{x}_n\}_{n=n_0}^{\infty}$ *of* (1) *such that*

$$|\hat{x}_n - c| \leq M, \quad n \geq n_0, \tag{8}$$

and

$$\lim_{n\to\infty} \hat{x}_n = c. \tag{9}$$

PROOF. Since $A_n^{(0)} = I$ and $\lim_{n\to\infty} A_n^{(r)} = 0$ if $r > 0$, $\lim_{n\to\infty} \Gamma_n = I$. Therefore Γ_n is invertible for large n. For now, choose $n_0 \geq N$ so that Γ_n is invertible if $n \geq n_0$; we will impose another condition on n_0 later. Define

$$h_n = (\Gamma_n^{-1} - I)c - \Gamma_n^{-1} \left(\sum_{m=n}^{\infty} A_{m+1}^{(q)} A_m c + \Gamma_{m+1} f(m, c) \right). \tag{10}$$

Let B be the Banach space of bounded sequences $U = \{u_n\}_{n_0}^{\infty}$ of k-vectors, with norm $\|U\| = \sup_{n\geq n_0} |u_n|$. Let B_M be the closed convex subset

$$B_M = \{U \in B \mid \|U\| \leq M\}$$

of B. From (2) and our assumption that $R(n, \lambda)$ is nondecreasing with respect to λ, if $U \in B_M$ then

$$|f(m, u_m + c) - f(m, c)| \leq R(m, |u_m|) \leq R(m, M). \tag{11}$$

Therefore (3) and (6) imply that if $U \in B_M$ then the sequence TU, with

$$(TU)_n = h_n - \Gamma_n^{-1} \sum_{m=n}^{\infty} \left[A_{m+1}^{(q)} A_m u_m + \Gamma_{m+1}[f(m, u_m + c) - f(m, c)] \right] \tag{12}$$

is well defined. We will show that if n_0 is sufficiently large then T is a continuous mapping of B_M into itself and $T(B_M)$ has compact closure. Given this, the Schauder-Tychonoff theorem [1, p. 405] implies that $T\hat{U} = \hat{U}$ for some $\hat{U} \in B_M$. We will then show that $\hat{X} = C + \hat{U}$ (with $C = \{c, c, c, \dots,\}_{n_0}^{\infty}$) satisfies (1), (8), and (9).

Let

$$\mu(n_0) = \sup_{m\geq n_0} |\Gamma_m^{-1}| \quad \text{and} \quad \nu(n_0) = \sup_{m\geq n_0} |\Gamma_{m+1}|.$$

From (11) and (12), if $U \in B_M$ then

$$|(TU)_n| \le |h_n| + \mu(n_0) \sum_{m=n}^{\infty} \left[|A_{m+1} A_m^{(q)}| M + \nu(n_0) R(m, M)| \right]. \qquad (13)$$

Since $\lim_{n_0 \to \infty} \mu(n_0) = \lim_{n_0 \to \infty} \nu(n_0) = 1$, (3) and (6) enable us to choose n_0 so that the quantity on the right side of (13) is less than M if $n \ge n_0$. Then $T(B_M) \subset B_M$.

We will now show that T is continuous on B_M. Suppose that $U = \lim_{r \to \infty} U^{(r)}$ where $\{U^{(r)}\} \subset B_M$. Let $V = TU$ and $V^{(r)} = TU^{(r)}$. Then

$$v_n^{(r)} - v_n = \Gamma_n^{-1} \sum_{m=n}^{\infty} \left[A_{m+1}^{(q)} A_m (u_m - u_m^{(r)}) + \Gamma_{m+1} \left(f(m, u_m + c) - f(m, u_m^{(r)} + c) \right) \right].$$

Therefore

$$\|v^{(r)} - v\| \le \mu(n_0) \sum_{m=n_0}^{\infty} \sigma_m^{(r)}, \qquad (14)$$

where

$$\sigma_m^{(r)} = |A_{m+1}^{(q)} A_m| |u_m^{(r)} - u_m| + \nu(n_0) \left| f(m, u_m^{(r)} + c) - f(m, u_m + c) \right|.$$

Note that

$$\lim_{r \to \infty} \sigma_m^{(r)} = 0, \quad m \ge n_0,$$

because of the continuity assumption on f, and

$$\sigma_m^{(r)} \le \sigma_m = 2 \left(M |A_{m+1}^{(q)} A_m| + |\nu(n_0)| R(m, M) \right) \qquad (15)$$

(see (11), applied to U and $U^{(r)}$) because U and $U^{(r)}$ are in B_M. Because of (3) and (6), $\sum_{m=n_0}^{\infty} \sigma_m < \infty$. Given $\epsilon > 0$, choose $n_1 \ge n_0$ so that $\sum_{m=n_1+1}^{\infty} \sigma_m < \epsilon$. Then (14) and (15) imply that

$$\|v^{(r)} - v\| \le \mu(n_0) \left(\sum_{m=n_0}^{n_1} \sigma_m^{(r)} + \epsilon \right). \qquad (16)$$

Now choose r_0 so that

$$\sigma_m^{(r)} < \frac{\epsilon}{(n_1 - n_0 + 1)} \text{ for } m = n_0, \ldots, n_1 \text{ if } r \ge r_0.$$

Then (16) implies that

$$\|v^{(r)} - v\| < 2\mu(n_0)\epsilon \text{ if } r \ge r_0,$$

which shows that T is continuous on B_M.

We will now show that $\overline{T(B_M)}$ (the closure of $T(B_M)$) is compact. From (11) and (12), $\overline{T(B_M)}$ is a subset of

$$A = \{v \in B \mid |v_n| \leq \rho(n),\ n \geq n_0\},$$

where

$$\rho(n) = |h_n| + \mu(n_0)\left(M \sum_{m=n}^{\infty} |A_{m+1}^{(q)} A_m| + \sum_{m=n}^{\infty} \nu(n_0) R(m, M)\right).$$

Therefore, it suffices to show that A is compact. From [2, pp. 51-53], this is true if A is totally bounded; that is, for every $\epsilon > 0$ there is a finite subset A_ϵ of B such that for each $v \in A$ there is a $\tilde{v} \in A_\epsilon$ that satisfies the inequality $\|v - \tilde{v}\| < \epsilon$. To establish the existence of A_ϵ, choose an integer $n_1 \geq n_0$ such that $\rho(n) < \epsilon$ if $n > n_1$, and let p be an integer such that $p\epsilon > M$. Then, since $|v_n| \leq M$ for all $n \geq n_0$, the finite set A_ϵ consisting of sequences of the form

$$a = (a_{n_0}, \ldots, a_{n_1}, 0, 0, \ldots)$$

where the components of the k-vectors $\{a_{n_0}, \ldots, a_{n_1}\}$ are all in the set

$$\{-p\epsilon, -(p-1)\epsilon, \ldots, 0, \ldots (p-1)\epsilon, p\epsilon\}, \quad n = n_0, \ldots, n_1,$$

has the desired property.

Now the Schauder-Tychonoff theorem implies that T has a fixed point $\hat{U}$. Since $\hat{U} = T\hat{U}$, (10) and (12) imply that if $\hat{X} = C + \hat{U}$ then

$$\hat{x}_n = \Gamma_n^{-1}\left(c - \sum_{m=n}^{\infty}\left[A_{m+1}^{(q)} A_m \hat{x}_m + \Gamma_{m+1} f(m, \hat{x}_m)\right]\right). \tag{17}$$

Therefore, $\lim_{n\to\infty} \hat{x}_n = c$. If $q = 0$ then (17) reduces to

$$\hat{x}_n = c - \sum_{m=n}^{\infty}(A_m \hat{x}_m + f(m, x_m)),$$

so

$$\Delta\hat{x}_n = A_n \hat{x}_n + f(n, \hat{x}_n). \tag{18}$$

If $q > 0$ then (17) implies that

$$\Delta\hat{x}_n = \Gamma_{n+1}^{-1} A_{n+1}^{(q)} A_n \hat{x}_n + f(n, \hat{x}_n) + (\Delta\Gamma_n^{-1})\Gamma_n \hat{x}_n. \tag{19}$$

Since $\Delta\Gamma_n^{-1} = -\Gamma_{n+1}^{-1}(\Delta\Gamma_n)\Gamma_n^{-1}$, (19) implies that

$$\Delta\hat{x}_n = \Gamma_{n+1}^{-1}\left[A_{n+1}^{(q)} A_n - \Delta\Gamma_n\right]\hat{x}_n + f(n, \hat{x}_n). \tag{20}$$

However, (5) and (7) imply that

$$\Delta\Gamma_n = -\sum_{r=1}^{q} A_{n+1}^{(r-1)} A_n,$$

so
$$A_{n+1}^{(q)} A_n - \Delta \Gamma_n = \Gamma_{n+1} A_n,$$
and therefore (20) implies (18). ■

The hypotheses of Theorem 1 may hold for some constant vectors c and fail to hold for others. In the following corollary c may be chosen arbitrarily.

COROLLARY 1 *Let A_n satisfy the hypotheses of Theorem* 1. *Suppose there is an integer N such that $f(n, x)$ is continuous with respect to x for all $n \geq N$ and all x, and*
$$|f(n, x_1) - f(n, x_2)| \leq R(n, |x_1 - x_2|)$$
where $R = R(n, \lambda)$ is defined on
$$\{(n, x) \mid n \geq N,\ 0 \leq \lambda \leq \infty\}$$
and nondecreasing in λ for each n, and $\sum_{n=N}^{\infty} |R(n, M)| < \infty$ for some constant $M > 0$. Suppose also that $\sum_{n=N}^{\infty} \Gamma_{n+1} f(n, c)$ converges (*perhaps conditionally*) *for every constant vector c. Let c be a given constant vector. Then, if n_0 is sufficiently large, there is a solution $\hat{X} = \{\hat{x}_n\}_{n=n_0}^{\infty}$ of* (1) *that satisfies* (8) *and* (9).

The following corollary applies to the linear system
$$\Delta x_n = (A_n + B_n) x_n + g_n, \tag{21}$$
where A_n and B_n are $k \times k$ matrices and g_n is a k-vector.

COROLLARY 2 *Suppose that A_n satisfies the hypotheses of Theorem* 1, *while $\sum^{\infty} |B_n| < \infty$ and $\sum^{\infty} \Gamma_{n+1} g_n$ converges* (*perhaps conditionally*). *Let c be an arbitrary vector. Then* (21) *has a solution $\hat{X}$ such that* $\lim_{n \to \infty} \hat{x}_n = c$.

References

[1] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, Inc., New York, London, Sydney, 1964.

[2] A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis*, v. 1 (translated from the 1954 Russian edition by L. F . Boron), Graylock Press, Rochester, N.Y., 1957.