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Trench's canonical form for a disconjugate n th-order linear difference equation

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TRENCH'S CANONICAL FORM FOR A DISCONJUGATE n TH-ORDER LINEAR DIFFERENCE EQUATION

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ABSTRACT. We consider the disconjugate, n th order linear difference equation $l_n u(t) = u(t+n) + p_1(t)u(t+n-1) + \cdots + p_n(t)u(t) = 0$. We will prove the existence of a Trench factorization for $l_n u(t) = 0$. We will then use this factorization to find a set of solutions $\{u_0, u_1, \dots, u_{n-1}\}$ such that

$$\lim_{t \rightarrow \infty} \frac{u_i(t)}{u_{i+1}(t)} = 0 \text{ if } 0 \leq i \leq n-2.$$

A set of solutions with this property is called a *principal set* of solutions.

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1. INTRODUCTION

At the Midwest Differential Equations Conference in 1985, Bennette Harris presented a talk on the discrete Trench factorization of an n th order linear difference equation, but the material was never offered for publication. William Trench and later Robert Krueger worked independently on results related to the talk by Harris. This paper contains their combined results.

We consider the n th order linear difference equation

$$(1) \quad l_n u(t) = u(t+n) + p_1(t)u(t+n-1) + \cdots + p_n(t)u(t) = 0$$

with the condition

$$(2) \quad (-1)^n p_n(t) > 0$$

where t is defined on the discrete interval $[a, b] = \{a, a+1, \dots, b\}$. We will use the forward difference operator Δ , which is defined by $\Delta u(t) = u(t+1) - u(t)$.

Given that $u(t)$ is a real-valued function on $[a, b+n]$, we say $t_0 = a$ is a *generalized zero* for $u(t)$ if $u(a) = 0$ and that $t_0 > a$ is a *generalized zero* for $u(t)$ if either $u(t_0) = 0$ or there exists an integer k , $1 \leq k \leq t_0 - a$, such that $(-1)^k u(t_0 - k)u(t_0) > 0$ and if $k > 1$, $u(t_0 - k + 1) = \cdots = u(t_0 - 1) = 0$.

The difference equation $l_n u(t) = 0$ is said to be *disconjugate* on $[a, b+n]$ if no nontrivial solution has n or more generalized zeros in $[a, b+n]$. We will usually assume that our equation (1) is disconjugate on $[a, b+n]$.

In 1922, Polya [7] considered the n th order linear differential equation

$$(3) \quad L_n y = y^{(n)} + q_1(x)y^{(n-1)} + \cdots + q_n(x)y = 0$$

Date:

where the coefficient functions $q_i(x)$, $1 \leq i \leq n$, are assumed to be continuous on an interval I . If (3) is disconjugate on an open interval I , then certain Wronskians of solutions are positive on I . Polya [7] showed that if these conditions on the Wronskians are satisfied on an open interval I , then there are positive functions $s_i(x)$ on I of class $C^{n-i}(I)$ such that for any function u of class $C^n(I)$,

$$(4) \quad L_n u(x) = s_n(x) \frac{d}{dx} \left(s_{n-1}(x) \frac{d}{dx} \left(\dots \left(\frac{d}{dx} (s_0(x) u(x)) \right) \dots \right) \right)$$

for $x \in I$. Now (4) is called the Polya factorization of $L_n u(x)$. Coppel [2], Hartman [4], and Levin [6], showed this for an arbitrary interval. In 1974, Trench [8] proved that if $L_n u(x) = 0$ is disconjugate on $I = [a, b)$, $a < b \leq \infty$, then there is a Polya factorization of $L_n u(x)$ of the form (4) where

$$(5) \quad \int_a^b \frac{1}{s_i(x)} dx = \infty,$$

$1 \leq i \leq n-1$. When the conditions (5) hold, (4) is called a Trench factorization of (3). (See, for example, page 5 in [3].)

In 1978, Hartman [5] proved that if (1) is disconjugate on $[a, b+n]$ and there are solutions $u_1(t), \dots, u_n(t)$ such that the Wronskian (Casoratian) defined by

$$(6) \quad w_k(t) := \begin{vmatrix} u_1(t) & u_2(t) & \dots & u_k(t) \\ u_1(t+1) & u_2(t+1) & \dots & u_k(t+1) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(t+k-1) & u_2(t+k-1) & \dots & u_k(t+k-1) \end{vmatrix} > 0$$

on $[a, b+n-k+1]$ for $1 \leq k \leq n$, then for any $u(t)$ defined on $[a, b+n]$, we obtain the Polya factorization of $l_n u$,

$$(7) \quad l_n u(t) = \rho_n(t) \Delta(\rho_{n-1}(t) \Delta(\dots \Delta(\rho_0(t) u(t)) \dots))$$

for $t \in [a, b]$, where

$$\begin{aligned} \rho_0(t) &= \frac{1}{u_1(t)} > 0, \quad t \in [a, b+n] \\ \rho_i(t) &= \frac{w_i(t) w_i(t+1)}{w_{i-1}(t+1) w_{i+1}(t)} > 0, \\ &\quad t \in [a, b+n-i], \quad 1 \leq i \leq n-1 \\ \rho_n(t) &= \frac{w_n(t+1)}{w_{n-1}(t+1)} > 0, \quad t \in [a, b]. \end{aligned}$$

Our results are the discrete analogues of the work done by Trench in [8]. In particular, we will show that (7) can be written as

$$(8) \quad l_n u = \frac{1}{\beta_n(t)} \Delta \left[\dots \frac{1}{\beta_1(t)} \Delta \left(\frac{u(t)}{\beta_0(t)} \right) \right]$$

where

$$\sum_{t=a}^{\infty} \beta_i(t) = \infty$$

for $1 \leq i \leq n-1$.

We will use this Trench factorization to find a set of solutions $\{u_0, u_1, \dots, u_{n-1}\}$ such that

$$\lim_{t \rightarrow \infty} \frac{u_i(t)}{u_{i+1}(t)} = 0 \text{ if } 0 \leq i \leq n-2.$$

A set of solutions with this property is called a *principal set* of solutions.

2. PRELIMINARIES

The following Lemmas will be the $n = 2$ and $n = 3$ cases for our induction in the proof of Theorem 3.

Lemma 1. *If*

$$M = \frac{1}{\alpha_2} \Delta \left(\frac{1}{\alpha_1} \Delta \left(\frac{\cdot}{\alpha_0} \right) \right)$$

with $\sum_{t=a}^{\infty} \alpha_1(t) < \infty$, then M can be rewritten as

$$M = \frac{1}{\beta_2} \Delta \left(\frac{1}{\beta_1} \Delta \left(\frac{\cdot}{\beta_0} \right) \right)$$

such that $\sum_{t=a}^{\infty} \beta_1(t) = \infty$.

Proof. Let

$$(9) \quad \beta_0(t) = \alpha_0(t) \sum_{s=t}^{\infty} \alpha_1(s),$$

$$(10) \quad \beta_1(t) = \frac{\alpha_1(t)}{\sum_{s=t}^{\infty} \alpha_1(s) \sum_{s=t+1}^{\infty} \alpha_1(s)},$$

$$(11) \quad \beta_2(t) = \alpha_2(t) \sum_{s=t+1}^{\infty} \alpha_1(s),$$

and let

$$(12) \quad \xi(t) = \sum_{s=t}^{\infty} \alpha_1(s).$$

Then

$$(13) \quad \Delta \xi(t) = -\alpha_1(t),$$

$$(14) \quad \Delta \left(\frac{1}{\xi(t)} \right) = \frac{-\Delta \xi(t)}{\xi(t)\xi(t+1)}.$$

Using (10), consider

$$\sum_{t=a}^{\infty} \beta_1(t) = \sum_{t=a}^{\infty} \left(\frac{\alpha_1(t)}{\sum_{s=t}^{\infty} \alpha_1(s) \sum_{s=t+1}^{\infty} \alpha_1(s)} \right).$$

By (12) and (13),

$$\sum_{t=a}^{\infty} \beta_1(t) = \sum_{t=a}^{\infty} \frac{-\Delta \xi(t)}{\xi(t)\xi(t+1)}.$$

By (14),

$$\begin{aligned}
\sum_{t=a}^{\infty} \beta_1(t) &= \sum_{t=a}^{\infty} \Delta \left(\frac{1}{\xi(t)} \right) \\
&= \lim_{b \rightarrow \infty} \sum_{t=a}^b \Delta \left(\frac{1}{\xi(t)} \right) \\
&= \lim_{b \rightarrow \infty} \left(\frac{1}{\xi(b+1)} - \frac{1}{\xi(a)} \right) \\
&= \lim_{b \rightarrow \infty} \left(\frac{1}{\sum_{s=b+1}^{\infty} \alpha_1(s)} - \frac{1}{\sum_{s=a}^{\infty} \alpha_1(s)} \right) \\
&= \infty.
\end{aligned}$$

Now we must show that the two factorizations define the same operator. If $u(t)$ is an arbitrary function defined on $[a, \infty)$ then

$$\begin{aligned}
\Delta \left(\frac{u(t)}{\beta_0(t)} \right) &= \Delta \left(\frac{\alpha_0^{-1}(t)u(t)}{\sum_{s=t}^{\infty} \alpha_1(s)} \right) \\
&= \frac{\sum_{s=t}^{\infty} \alpha_1(s) \Delta(\alpha_0^{-1}(t)u(t)) - \alpha_0^{-1}(t)u(t)(-\alpha_1(t))}{\sum_{s=t}^{\infty} \alpha_1(s) \sum_{s=t+1}^{\infty} \alpha_1(s)}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{\beta_1(t)} \Delta \left(\frac{u(t)}{\beta_0(t)} \right) &= \frac{\sum_{s=t}^{\infty} \alpha_1(s) \sum_{s=t+1}^{\infty} \alpha_1(s)}{\alpha_1(t)} \left(\frac{\sum_{s=t}^{\infty} \alpha_1(s) \Delta(\alpha_0^{-1}(t)u(t)) - \alpha_0^{-1}(t)u(t)(-\alpha_1(t))}{\sum_{s=t}^{\infty} \alpha_1(s) \sum_{s=t+1}^{\infty} \alpha_1(s)} \right) \\
&= \left[\sum_{s=t}^{\infty} \alpha_1(s) \right] [\alpha_1^{-1}(t) \Delta(\alpha_0^{-1}(t)u(t))] + \alpha_0^{-1}(t)u(t).
\end{aligned}$$

It follows that

$$\Delta \left(\frac{1}{\beta_1(t)} \Delta \left(\frac{u(t)}{\beta_0(t)} \right) \right) = \left[\sum_{s=t+1}^{\infty} \alpha_1(s) \right] \Delta [\alpha_1^{-1}(t) \Delta(\alpha_0^{-1}(t)u(t))].$$

Finally,

$$\begin{aligned}
\frac{1}{\beta_2(t)} \Delta \left(\frac{1}{\beta_1(t)} \Delta \left(\frac{u(t)}{\beta_0(t)} \right) \right) &= \frac{1}{\alpha_2(t) \sum_{s=t+1}^{\infty} \alpha_1(s)} \left[\sum_{s=t+1}^{\infty} \alpha_1(s) \right] \Delta [\alpha_1^{-1}(t) \Delta(\alpha_0^{-1}(t)u(t))] \\
&= \frac{1}{\alpha_2(t)} \Delta \left(\frac{1}{\alpha_1(t)} \Delta \left(\frac{u(t)}{\alpha_0(t)} \right) \right).
\end{aligned}$$

□

Lemma 2. *If*

$$N = \frac{1}{\mu_3} \Delta \left(\frac{1}{\mu_2} \Delta \left(\frac{1}{\mu_1} \Delta \left(\frac{\cdot}{\mu_0} \right) \right) \right)$$

with $\sum_{t=a}^{\infty} \mu_1(t) = \infty$ and $\sum_{t=a}^{\infty} \mu_2(t) < \infty$, then N can be rewritten as

$$N = \frac{1}{\nu_3} \Delta \left(\frac{1}{\nu_2} \Delta \left(\frac{1}{\nu_1} \Delta \left(\frac{\cdot}{\nu_0} \right) \right) \right)$$

where $\sum_{t=a}^{\infty} \nu_1(t) = \infty$ and $\sum_{t=a}^{\infty} \nu_2(t) = \infty$.

Proof. Apply Lemma 1 to obtain

$$N = \frac{1}{\tilde{\nu}_3} \Delta \left(\frac{1}{\tilde{\nu}_2} \Delta \left(\frac{1}{\tilde{\nu}_1} \Delta \left(\frac{\cdot}{\tilde{\nu}_0} \right) \right) \right)$$

where

$$(15) \quad \tilde{\nu}_0(t) = \mu_0(t),$$

$$(16) \quad \tilde{\nu}_1(t) = \mu_1(t) \sum_{s=t}^{\infty} \mu_2(s),$$

$$(17) \quad \tilde{\nu}_2(t) = \frac{\mu_2(t)}{\sum_{s=t}^{\infty} \mu_2(s) \sum_{s=t+1}^{\infty} \mu_2(s)},$$

$$(18) \quad \tilde{\nu}_3(t) = \mu_3(t) \sum_{s=t+1}^{\infty} \mu_2(s),$$

and $\sum_{t=a}^{\infty} \tilde{\nu}_2(t) = \infty$.

If $\sum_{t=a}^{\infty} \tilde{\nu}_1(t) = \infty$, then there is nothing to show, so assume $\sum_{t=a}^{\infty} \tilde{\nu}_1(t) < \infty$. Now apply Lemma 1 again to obtain

$$N = \frac{1}{\nu_3} \Delta \left(\frac{1}{\nu_2} \Delta \left(\frac{1}{\nu_1} \Delta \left(\frac{\cdot}{\nu_0} \right) \right) \right)$$

where

$$(19) \quad \nu_0(t) = \tilde{\nu}_0(t) \sum_{s=t}^{\infty} \tilde{\nu}_1(s),$$

$$(20) \quad \nu_1(t) = \frac{\tilde{\nu}_1(t)}{\sum_{s=t}^{\infty} \tilde{\nu}_1(s) \sum_{s=t+1}^{\infty} \tilde{\nu}_1(s)},$$

$$(21) \quad \nu_2(t) = \tilde{\nu}_2(t) \sum_{s=t+1}^{\infty} \tilde{\nu}_1(s),$$

$$(22) \quad \nu_3(t) = \tilde{\nu}_3(t),$$

and $\sum_{t=a}^{\infty} \nu_1(t) = \infty$. We must show $\sum_{t=a}^{\infty} \nu_2(t) = \infty$. Consider

$$\sum_{t=a}^{b-1} \nu_2(t) = \sum_{t=a}^{b-1} \left[\tilde{\nu}_2(t) \sum_{s=t+1}^{\infty} \tilde{\nu}_1(s) \right]$$

by (21). Now by summation by parts, we obtain

$$\sum_{t=a}^{b-1} \nu_2(t) = \left(\sum_{s=t+1}^{\infty} \tilde{\nu}_1(s) \right) \left(\sum_{s=a}^{t-1} \tilde{\nu}_2(s) \right) \Big|_{t=a}^{t=b} + \sum_{t=a}^{b-1} \left(\tilde{\nu}_1(t+1) \sum_{s=a}^t \tilde{\nu}_2(s) \right).$$

Note: In the first term when $t = a$, the sum (by convention) equals zero. Now from (17) we obtain,

$$\begin{aligned} \sum_{t=a}^{b-1} \nu_2(t) &= \left(\sum_{t=b+1}^{\infty} \tilde{\nu}_1(t) \right) \left(\sum_{t=a}^{b-1} \frac{\mu_2(t)}{\sum_{s=t}^{\infty} \mu_2(s) \sum_{s=t+1}^{\infty} \mu_2(s)} \right) \\ &+ \sum_{t=a}^{b-1} \left(\tilde{\nu}_1(t+1) \sum_{s=a}^t \frac{\mu_2(s)}{\sum_{j=s}^{\infty} \mu_2(j) \sum_{j=s+1}^{\infty} \mu_2(j)} \right). \end{aligned}$$

Using the same idea as in (12)-(14) yields

$$\begin{aligned} \sum_{t=a}^{b-1} \nu_2(t) &= \left(\sum_{t=b+1}^{\infty} \tilde{\nu}_1(t) \right) \sum_{t=a}^{b-1} \Delta_t \left[\frac{1}{\sum_{s=t}^{\infty} \mu_2(s)} \right] + \sum_{t=a}^{b-1} \left(\tilde{\nu}_1(t+1) \sum_{s=a}^t \Delta_s \left[\frac{1}{\sum_{j=s}^{\infty} \mu_2(j)} \right] \right) \\ &= \left(\sum_{t=b+1}^{\infty} \tilde{\nu}_1(t) \right) \left(\frac{1}{\sum_{t=b}^{\infty} \mu_2(t)} - \frac{1}{\sum_{t=a}^{\infty} \mu_2(t)} \right) \\ &+ \sum_{t=a}^{b-1} \left(\tilde{\nu}_1(t+1) \left(\frac{1}{\sum_{s=t+1}^{\infty} \mu_2(s)} - \frac{1}{\sum_{s=a}^{\infty} \mu_2(s)} \right) \right). \end{aligned}$$

Using (16), we obtain

$$\begin{aligned} \sum_{t=a}^{b-1} \nu_2(t) &= \left(\sum_{t=b+1}^{\infty} \tilde{\nu}_1(t) \right) \left(\frac{1}{\sum_{t=b}^{\infty} \mu_2(t)} - \frac{1}{\sum_{t=a}^{\infty} \mu_2(t)} \right) \\ &+ \sum_{t=a}^{b-1} \left[\mu_1(t+1) \left(\sum_{s=t+1}^{\infty} \mu_2(s) \right) \left(\frac{1}{\sum_{s=t+1}^{\infty} \mu_2(s)} - \frac{1}{\sum_{s=a}^{\infty} \mu_2(s)} \right) \right] \\ &= \left(\sum_{t=b+1}^{\infty} \tilde{\nu}_1(t) \right) \left(\frac{1}{\sum_{t=b}^{\infty} \mu_2(t)} - \frac{1}{\sum_{t=a}^{\infty} \mu_2(t)} \right) \\ &+ \sum_{t=a}^{b-1} \mu_1(t+1) - \sum_{t=a}^{b-1} \frac{\mu_1(t+1) \sum_{s=t+1}^{\infty} \mu_2(s)}{\sum_{s=a}^{\infty} \mu_2(s)}. \end{aligned}$$

Changing the index of summation in the last two terms,

$$\begin{aligned} \sum_{t=a}^{b-1} \nu_2(t) &= \left(\sum_{t=b+1}^{\infty} \tilde{\nu}_1(t) \right) \left(\frac{1}{\sum_{t=b}^{\infty} \mu_2(t)} - \frac{1}{\sum_{t=a}^{\infty} \mu_2(t)} \right) \\ &+ \sum_{t=a+1}^b \mu_1(t) - \sum_{t=a+1}^b \frac{\mu_1(t) \sum_{s=t}^{\infty} \mu_2(s)}{\sum_{s=a}^{\infty} \mu_2(s)}. \end{aligned}$$

So by (16),

$$\sum_{t=a}^{b-1} \nu_2(t) = \frac{\sum_{t=b+1}^{\infty} \tilde{\nu}_1(t)}{\sum_{t=b}^{\infty} \mu_2(t)} - \frac{\sum_{t=b+1}^{\infty} \tilde{\nu}_1(t)}{\sum_{t=a}^{\infty} \mu_2(t)} + \sum_{t=a+1}^b \mu_1(t) - \frac{\sum_{t=a+1}^b \tilde{\nu}_1(t)}{\sum_{s=a}^{\infty} \mu_2(s)}.$$

Combining the second and last terms, we obtain

$$\sum_{t=a}^{b-1} \nu_2(t) = \frac{\sum_{t=b+1}^{\infty} \tilde{\nu}_1(t)}{\sum_{t=b}^{\infty} \mu_2(t)} - \frac{\sum_{t=a+1}^{\infty} \tilde{\nu}_1(t)}{\sum_{t=a}^{\infty} \mu_2(t)} + \sum_{t=a+1}^b \mu_1(t).$$

Letting $b \rightarrow \infty$, we obtain the desired result. Using a method similar to the proof of Lemma 1, we can show the two factorizations define the same operator. \square

3. MAIN RESULTS

Note that the operator l_n in Theorem 3 below is in a Polya factorization form where $\rho_i(t) = \frac{1}{\alpha_i(t)}$ for $0 \leq i \leq n$.

Theorem 3. *Any operator*

$$(23) \quad l_n = \frac{1}{\alpha_n} \Delta \left(\frac{1}{\alpha_{n-1}} \Delta \dots \Delta \left(\frac{1}{\alpha_1} \Delta \left(\frac{\cdot}{\alpha_0} \right) \right) \right)$$

with $\alpha_i(t) > 0$ on $[a, \infty)$, $0 \leq i \leq n$, can be written as

$$(24) \quad l_n = \frac{1}{\beta_n} \Delta \left(\frac{1}{\beta_{n-1}} \Delta \dots \Delta \left(\frac{1}{\beta_1} \Delta \left(\frac{\cdot}{\beta_0} \right) \right) \right)$$

with $\sum_{t=a}^{\infty} \beta_i(t) = \infty$ for $1 \leq i \leq n-1$ and $\beta_i(t) > 0$ on $[a, \infty)$, $0 \leq i \leq n$.

Proof. Proof is by induction. Lemma 1 and Lemma 2 imply the desired results for $n = 2$ and $n = 3$, respectively. Suppose $n \geq 4$ and assume the theorem is satisfied for any $(n-1)$ st order operator l_n . This implies

$$(25) \quad \sum_{t=a}^{\infty} \alpha_j(t) = \infty$$

for $1 \leq j \leq n-2$. If $\sum_{t=a}^{\infty} \alpha_{n-1}(t) = \infty$, then we are done, so assume $\sum_{t=a}^{\infty} \alpha_{n-1}(t) < \infty$.

Construct a sequence of operators

$$(26) \quad l_n = \frac{1}{\alpha_{n,i}} \Delta \left(\frac{1}{\alpha_{n-1,i}} \Delta \dots \Delta \left(\frac{1}{\alpha_{1,i}} \Delta \left(\frac{\cdot}{\alpha_{0,i}} \right) \right) \right)$$

with $\alpha_{j,0} = \alpha_j$ for $0 \leq j \leq n$. So for $i \geq 1$,

$$(27) \quad \alpha_{j,i}(t) = \alpha_{j,i-1}(t), \quad j \neq n-i+1, n-i, n-i-1,$$

$$(28) \quad \alpha_{n-i+1,i}(t) = \alpha_{n-i+1,i-1}(t) \sum_{\tau=t+1}^{\infty} \alpha_{n-i,i-1}(\tau),$$

$$(29) \quad \alpha_{n-i,i}(t) = \frac{\alpha_{n-i,i-1}(t)}{\sum_{\tau=t}^{\infty} \alpha_{n-i,i-1}(\tau) \sum_{\tau=t+1}^{\infty} \alpha_{n-i,i-1}(\tau)},$$

$$(30) \quad \alpha_{n-i-1,i}(t) = \alpha_{n-i-1,i-1}(t) \sum_{\tau=t}^{\infty} \alpha_{n-i,i-1}(\tau).$$

This process stops at the i th step if

$$(31) \quad \sum_{t=a}^{\infty} \alpha_{j,i}(t) = \infty$$

for all $j \in \{1, \dots, n-1\}$.

Now (31) holds by the induction hypothesis and (27)-(30) except possibly for $j = n-i-1$. Hence, this process terminates when

$$(32) \quad \sum_{t=a}^{\infty} \alpha_{n-i-1,i}(t) = \infty$$

or when $i = n-1$, whichever is first. So if the process terminates at $i = r$, then $\beta_j = \alpha_{j,r}$, $0 \leq j \leq n$.

Finally, we can show that the two factorizations define the same operator using the same method as in the proof of Lemma 1. \square

Corollary 4. *If $l_n u = 0$ is disconjugate on $[a, \infty)$, then the operator l_n has a Trench factorization.*

We say $\{u_0(t), u_1(t), \dots, u_{n-1}(t)\}$ is a *principal set of solutions* of $l_n u(t) = 0$ on $[a, \infty)$ provided, for each i , $0 \leq i \leq n-1$, $u_i(t) > 0$ in a neighborhood of infinity and

$$\lim_{t \rightarrow \infty} \frac{u_i(t)}{u_{i+1}(t)} = 0 \text{ if } 0 \leq i \leq n-2.$$

To show the Trench factorization leads to a principal set of solutions, first we need a theorem which is Theorem 1.7.9 in Agarwal [1].

Theorem 5. (Discrete L'Hospital's Rule) *Let $u(k)$ and $v(k)$ be defined on $[a, \infty)$ and assume $v(k) > 0$ and $\Delta v(k) > 0$ for all large $k \in [a, \infty)$, then, for $0 \leq c \leq \infty$, if*

$$\lim_{k \rightarrow \infty} v(k) = \infty$$

and

$$\lim_{k \rightarrow \infty} \frac{\Delta u(k)}{\Delta v(k)} = c,$$

then

$$\lim_{k \rightarrow \infty} \frac{u(k)}{v(k)} = c.$$

We now prove the existence of a principal set of solutions of (1).

Theorem 6. *If $l_n u(t) = 0$ is disconjugate on $[a, \infty)$, then there exists a principal set of solutions on $[a, \infty)$.*

Proof. By Corollary 4, we have a Trench factorization (8). So assume

$$(33) \quad \frac{1}{\beta_n(t)} \Delta \left(\frac{1}{\beta_{n-1}(t)} \Delta \dots \Delta \left(\frac{1}{\beta_1(t)} \Delta \left(\frac{1}{\beta_0(t)} u(t) \right) \right) \dots \right) = 0$$

is a Trench factorization of $l_n u = 0$.

First, if we set $u_0(t) = \beta_0(t)$, then $u_0(t)$ is a solution of (33).

Second, we take $u_1(t)$ to be the solution of the IVP

$$\frac{1}{\beta_1} \Delta \left(\frac{1}{\beta_0} u \right) = 1, \quad u(a) = 0.$$

That would imply

$$u_1(t) = \beta_0(t) \sum_{s=a}^{t-1} \beta_1(s).$$

Finally, we take $u_k(t)$, $1 \leq k \leq n-1$, to be the solution of the IVP

$$\frac{1}{\beta_k} \Delta \left(\frac{1}{\beta_{k-1}} \Delta \dots \Delta \left(\frac{1}{\beta_0} u \right) \dots \right) = 1$$

$$u(a) = \Delta u(a) = \dots = \Delta^{k-1} u(a) = 0.$$

Solving this IVP yields

$$(34) \quad u_k(t) = \beta_0(t) \sum_{s_1=a}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a}^{s_1-1} \dots \sum_{s_{k-1}=a}^{s_{k-2}-1} \left(\beta_{k-1}(s_{k-1}) \sum_{s_k=a}^{s_{k-1}-1} \beta_k(s_k) \right) \dots \right).$$

To show that $\{u_0(t), u_1(t), \dots, u_{n-1}(t)\}$ is a principal set of solutions for (1), first we must show $u_k(t) > 0$ in a neighborhood of infinity for $0 \leq k \leq n-1$.

When $s_{k-1} = a$ in equation (34), then (by convention)

$$\sum_{s_k=a}^{a-1} \beta_k(s_k) = 0.$$

Therefore,

$$(35) \quad \sum_{s_{k-1}=a}^{s_{k-2}-1} (\dots) = \sum_{s_{k-1}=a+1}^{s_{k-2}-1} (\dots).$$

When $s_{k-2} = a+1$, we obtain a similar expression which is zero. Eventually, we can rewrite (34) as

$$\begin{aligned} u_k(t) &= \beta_0(t) \sum_{s_1=a+k-1}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a+k-2}^{s_1-1} \cdots \sum_{s_{k-1}=a+1}^{s_{k-2}-1} \left(\beta_{k-1}(s_{k-1}) \sum_{s_k=a}^{s_{k-1}-1} \beta_k(s_k) \right) \cdots \right) \\ &\geq \beta_0(t) \sum_{s_1=a+k-1}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a+k-2}^{s_1-1} \cdots \sum_{s_{k-1}=a+1}^{s_{k-2}-1} (\beta_{k-1}(s_{k-1}) \cdot \beta_k(a)) \cdots \right) \\ &\geq \beta_0(t) \sum_{s_1=a+k-1}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a+k-2}^{s_1-1} \cdots (\beta_{k-1}(a+1) \cdot \beta_k(a)) \cdots \right) \\ &\geq \beta_k(a) \cdot \beta_{k-1}(a+1) \cdots \beta_2(a+k-2) \cdot \beta_0(t) \sum_{s_1=a+k-1}^{t-1} (\beta_1(s_1)) \\ &\rightarrow \infty \text{ as } t \rightarrow \infty \end{aligned}$$

because (33) is a Trench factorization. Therefore, $u_k(t) > 0$ near infinity and we will be able to use Theorem 5 since $u_k(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Now consider

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{u_1(t)}{u_0(t)} &= \lim_{t \rightarrow \infty} \frac{\beta_0(t) \sum_{s=a}^{t-1} \beta_1(s)}{\beta_0(t)} \\ &= \lim_{t \rightarrow \infty} \sum_{s=a}^{t-1} \beta_1(s) \\ &= \sum_{s=a}^{\infty} \beta_1(s) = \infty \end{aligned}$$

since (33) is a Trench factorization. Therefore,

$$\lim_{t \rightarrow \infty} \frac{u_0(t)}{u_1(t)} = 0.$$

Similarly, consider

$$\lim_{t \rightarrow \infty} \frac{u_k(t)}{u_{k-1}(t)} = \lim_{t \rightarrow \infty} \frac{\beta_0(t) \sum_{s_1=a}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a}^{s_1-1} \cdots \sum_{s_{k-1}=a}^{s_{k-2}-1} \left(\beta_{k-1}(s_{k-1}) \sum_{s_k=a}^{s_{k-1}-1} \beta_k(s_k) \right) \cdots \right)}{\beta_0(t) \sum_{s_1=a}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a}^{s_1-1} \cdots \sum_{s_{k-1}=a}^{s_{k-2}-1} (\beta_{k-1}(s_{k-1})) \cdots \right)}.$$

By cancellation and using Theorem 5 $k - 1$ times, we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{u_k(t)}{u_{k-1}(t)} &= \lim_{t \rightarrow \infty} \frac{\beta_{k-1}(t) \sum_{s=a}^{t-1} \beta_k(s)}{\beta_{k-1}(t)} \\ &= \sum_{s=a}^{\infty} \beta_k(s) = \infty \end{aligned}$$

for $1 \leq k \leq n - 1$ because (33) is a Trench factorization. Therefore,

$$\lim_{t \rightarrow \infty} \frac{u_{k-1}(t)}{u_k(t)} = 0 \text{ for } 1 \leq k \leq n - 1.$$

□

Example 7. $l_3 u(t) = u(t + 3) - 9u(t + 2) + 26u(t + 1) - 24u(t) = 0$

The third order difference equation $l_3 u(t) = 0$ has solutions

$$u_1(t) = 3^t, \quad u_2(t) = -2^t, \quad u_3(t) = 4^t$$

such that

$$\begin{aligned} w_1(t) &= 3^t > 0, \\ w_2(t) &= 6^t > 0, \\ w_3(t) &= 2(24)^t > 0. \end{aligned}$$

Thus, we have the Polya factorization

$$(36) \quad l_3 u(t) = 8(4)^t \Delta \left(\left(\left(\frac{1}{2} \right)^t \Delta \left(3 \left(\frac{3}{2} \right)^t \Delta \left(\left(\frac{1}{3} \right)^t u(t) \right) \right) \right) \right)$$

where

$$\begin{aligned} \rho_0(t) &= \left(\frac{1}{3} \right)^t > 0, \\ \rho_1(t) &= 3 \left(\frac{3}{2} \right)^t > 0, \\ \rho_2(t) &= \left(\frac{1}{2} \right)^t > 0, \\ \rho_3(t) &= 8(4)^t > 0. \end{aligned}$$

Consider

$$\begin{aligned} \alpha_0(t) &= \frac{1}{\rho_0(t)} = 3^t, \\ \alpha_1(t) &= \frac{1}{\rho_1(t)} = \frac{1}{3} \left(\frac{2}{3} \right)^t, \\ \alpha_2(t) &= \frac{1}{\rho_2(t)} = 2^t, \\ \alpha_3(t) &= \frac{1}{\rho_3(t)} = \frac{1}{8} \left(\frac{1}{4} \right)^t. \end{aligned}$$

Notice that

$$\sum_{t=a}^{\infty} \alpha_1(t) = \left(\frac{2}{3} \right)^a < +\infty.$$

Therefore, (36) is not a Trench factorization. So by the proof of Lemma 2, define

$$\beta_0(t) := \alpha_0(t) \sum_{s=t}^{\infty} \alpha_1(s) = 2^t,$$

$$\beta_1(t) := \frac{\alpha_1(t)}{\sum_{s=t}^{\infty} \alpha_1(s) \sum_{s=t+1}^{\infty} \alpha_1(s)} = \frac{1}{2} \left(\frac{3}{2} \right)^t,$$

$$\beta_2(t) := \alpha_2(t) \sum_{s=t+1}^{\infty} \alpha_1(s) = \frac{2}{3} \left(\frac{4}{3} \right)^t,$$

$$\beta_3(t) := \alpha_3(t) = \frac{1}{8} \left(\frac{1}{4} \right)^t.$$

Since

$$\sum_{s=t}^{\infty} \beta_1(s) = \infty \text{ and } \sum_{s=t}^{\infty} \beta_2(s) = \infty,$$

$$l_3 u(t) = 8(4)^t \Delta \left(\frac{3}{2} \left(\frac{3}{4} \right)^t \Delta \left(2 \left(\frac{2}{3} \right)^t \Delta \left(\left(\frac{1}{2} \right)^t u(t) \right) \right) \right)$$

is a Trench factorization of l_3 .

We know from the proof in Theorem 6 that $l_3 u = 0$ has solutions of the form

$$u_0(t) = \beta_0(t) = \boxed{2^t}$$

$$u_1(t) = \beta_0(t) \sum_{s=a}^{t-1} \beta_1(s) = \boxed{3^t + C_0 2^t}$$

$$\begin{aligned} u_2(t) &= \beta_0(t) \sum_{s_1=a}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a}^{s_1-1} \beta_2(s_2) \right) \\ &= \boxed{4^t + C_1 3^t + C_2 2^t} \end{aligned}$$

where the C_i are known quantities. These solutions form a principal set of solutions.

Now we will explore the essential uniqueness of the Trench factorization. First, a theorem about principal sets of solutions for the operator l_n .

Theorem 8. *If $\{x_0, \dots, x_{n-1}\}$ and $\{y_0, \dots, y_{n-1}\}$ are principal sets of solutions for $l_n u(t) = 0$ on $[a, \infty)$, then*

$$y_i = \sum_{j=0}^i a_{ij} x_j$$

where a_{ij} is a constant and $a_{ii} > 0$ for $0 \leq j \leq i$ and $0 \leq i \leq n-1$.

Proof. Since $\{x_0, \dots, x_{n-1}\}$ is a linearly independent set of solutions to the n th order difference equation $l_n u(t) = 0$, We can write the solutions

$$y_i = \sum_{j=0}^{n-1} a_{ij} x_j$$

for $0 \leq i \leq n-1$ where a_{ij} is a constant for $0 \leq j \leq i$ and $0 \leq i \leq n-1$. Consider

$$\frac{y_i(t)}{y_{n-1}(t)} = \frac{a_{i,0}x_0(t) + \cdots + a_{i,n-2}x_{n-2}(t) + a_{i,n-1}x_{n-1}(t)}{a_{n-1,0}x_0(t) + \cdots + a_{n-1,n-2}x_{n-2}(t) + a_{n-1,n-1}x_{n-1}(t)}$$

for $0 \leq i \leq n-2$. Dividing through the numerator and the denominator by $x_{n-1}(t)$ and letting $t \rightarrow \infty$, we obtain

$$0 = \lim_{t \rightarrow \infty} \frac{y_i(t)}{y_{n-1}(t)} = \frac{a_{i,n-1}}{a_{n-1,n-1}}.$$

Therefore $a_{i,n-1} = 0$ for $0 \leq i \leq n-2$ and $a_{n-1,n-1} \neq 0$. Next consider

$$\frac{y_i(t)}{y_{n-2}(t)} = \frac{a_{i,0}x_0(t) + \cdots + a_{i,n-3}x_{n-3}(t) + a_{i,n-2}x_{n-2}(t)}{a_{n-2,0}x_0(t) + \cdots + a_{n-2,n-3}x_{n-3}(t) + a_{n-2,n-2}x_{n-2}(t)}$$

for $0 \leq i \leq n-3$. Dividing through the numerator and the denominator by $x_{n-2}(t)$ and letting $t \rightarrow \infty$, we obtain

$$0 = \lim_{t \rightarrow \infty} \frac{y_i(t)}{y_{n-2}(t)} = \frac{a_{i,n-2}}{a_{n-2,n-2}}.$$

Therefore $a_{i,n-2} = 0$ for $0 \leq i \leq n-3$ and $a_{n-2,n-2} \neq 0$. Continuing this process yields

$$(37) \quad y_i = \sum_{j=0}^i a_{ij}x_j$$

for $0 \leq i \leq n-1$ where $a_{ii} \neq 0$.

By assumption, $x_0(t) > 0$ and $y_0(t) > 0$ near infinity. Thus, by (37), $a_{00} > 0$. Similarly, by assumption and (37), $x_i(t) > 0$ and

$$y_i(t) = a_{i,0}x_0(t) + \cdots + a_{i,i-1}x_{i-1}(t) + a_{i,i}x_i(t) > 0$$

near infinity. Thus,

$$\frac{a_{i,0}x_0(t)}{x_i(t)} + \cdots + \frac{a_{i,i-1}x_{i-1}(t)}{x_i(t)} + \frac{a_{i,i}x_i(t)}{x_i(t)} > 0$$

near infinity. Letting $t \rightarrow \infty$, we obtain $a_{ii} > 0$ for $0 \leq i \leq n-1$. □

In the following theorem, we will obtain the essential uniqueness of the Trench factorization.

Theorem 9. *If*

$$(38) \quad l_n u(t) = \frac{1}{\alpha_n(t)} \Delta \left(\frac{1}{\alpha_{n-1}(t)} \Delta \cdots \Delta \left(\frac{1}{\alpha_1(t)} \Delta \left(\frac{u(t)}{\alpha_0(t)} \right) \right) \cdots \right)$$

and

$$(39) \quad l_n u(t) = \frac{1}{\beta_n(t)} \Delta \left(\frac{1}{\beta_{n-1}(t)} \Delta \cdots \Delta \left(\frac{1}{\beta_1(t)} \Delta \left(\frac{u(t)}{\beta_0(t)} \right) \right) \cdots \right)$$

are Trench factorizations for l_n , then

$$\alpha_i(t) = d_i \beta_i(t)$$

where the d_i are positive constants for $0 \leq i \leq n$.

Proof. Let $\{y_0, \dots, y_{n-1}\}$ be the principal set of solutions for $l_n u(t) = 0$ using the operator (38) as in the proof of Theorem 6 and let $\{x_0, \dots, x_{n-1}\}$ be the principal set of solutions for $l_n u(t) = 0$ using the operator (39) as in the proof of Theorem 6. Hence,

$$y_0(t) = \alpha_0(t) \text{ and } x_0(t) = \beta_0(t).$$

By Theorem 8,

$$y_0(t) = a_{00}x_0(t).$$

Thus we can conclude that

$$\alpha_0(t) = a_{00}\beta_0(t).$$

Furthermore,

$$y_1(t) = \alpha_0(t) \sum_{s_1=a}^{t-1} \alpha_1(s_1) \text{ and } x_1(t) = \beta_0(t) \sum_{s_1=a}^{t-1} \beta_1(s_1).$$

By Theorem 8,

$$y_1(t) = a_{10}x_0(t) + a_{11}x_1(t).$$

Thus we obtain

$$\begin{aligned} \alpha_0(t) \sum_{s_1=a}^{t-1} \alpha_1(s_1) &= a_{10}\beta_0(t) + a_{11}\beta_0(t) \sum_{s_1=a}^{t-1} \beta_1(s_1) \\ a_{00}\beta_0(t) \sum_{s_1=a}^{t-1} \alpha_1(s_1) &= a_{10}\beta_0(t) + a_{11}\beta_0(t) \sum_{s_1=a}^{t-1} \beta_1(s_1) \\ a_{00} \sum_{s_1=a}^{t-1} \alpha_1(s_1) &= a_{10} + a_{11} \sum_{s_1=a}^{t-1} \beta_1(s_1). \end{aligned}$$

Taking the difference of both sides yields

$$a_{00}\alpha_1(t) = a_{11}\beta_1(t).$$

Since $a_{00} > 0$,

$$\alpha_1(t) = \frac{a_{11}}{a_{00}}\beta_1(t).$$

Continuing this process, we get

$$\alpha_i(t) = \frac{a_{i,i}}{a_{i-1,i-1}}\beta_i(t)$$

for $0 \leq i \leq n-1$.

Thus, we could rewrite (38) as

$$(40) \quad l_n u(t) = \frac{1}{a_{n-1,n-1}} \frac{1}{\alpha_n(t)} \Delta \left(\frac{1}{\beta_{n-1}(t)} \Delta \dots \Delta \left(\frac{1}{\beta_1(t)} \Delta \left(\frac{u(t)}{\beta_0(t)} \right) \right) \dots \right).$$

Define

$$x_n(t) = \beta_0(t) \sum_{s_1=a}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a}^{s_1-1} \dots \sum_{s_{n-1}=a}^{s_{n-2}-1} \left(\beta_{n-1}(s_{n-1}) \sum_{s_n=a}^{s_{n-1}-1} \beta_n(s_n) \right) \dots \right).$$

Using (39), $l_n x_n(t) = 1$, while from (40),

$$l_n x_n(t) = \frac{1}{a_{n-1,n-1}} \frac{\beta_n(t)}{\alpha_n(t)}.$$

Hence, $\alpha_n(t) = \frac{1}{a_{n-1,n-1}} \beta_n(t)$.

Therefore, letting

$$\begin{aligned} d_0 &= a_{00}, \\ d_i &= \frac{a_{i,i}}{a_{i-1,i-1}}, \text{ for } 1 \leq i \leq n-1, \\ d_n &= \frac{1}{a_{n-1,n-1}}, \end{aligned}$$

we obtain the desired result. □

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