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ON THE INVERSION OF A HILBERT TYPE MATRIX*

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1. Introduction. In this paper we shall be interested in inverting the matrix

$$A(x) = \left(\left(\frac{1}{m-n+x};\, m, n = 1, 2, \cdots\right)\right), \qquad x \text{ nonintegral.} \tag{1.1}$$

The solution to this can be considered as a generalization of the problem in [7], where the case $x = \frac{1}{2}$ was emphasized. Here we shall consider both the finite and the infinite case via different methods, and we shall show that a particular inverse to the infinite matrix can be obtained as the limit as $n \to \infty$ of the inverse of the finite case.

We note that $A(x)$ resembles a Hilbert type matrix, the classic Hilbert matrix being

$$H(x) = \left(\left(\frac{1}{m+n+x};\, m, n = 1, 2, \cdots\right)\right). \tag{1.2}$$

However, one of the principal differences between (1.1) and (1.2) is that $H(x)$ is symmetric while $A(x)$ is not. One of the important similarities between the two forms is that it was shown in [4] that for fixed nonintegral x both (1.1) and (1.2) are bounded operators in l_2, the linear vector space of square summable sequences. In both cases the bound is $\pi|\csc \pi x|$.

2. Finite considerations. Consider the $k \times k$ submatrix of (1.1) formed from the elements in the upper left hand corner and denoted by

$$A_k(x) = \left(\left(\frac{1}{m-n+x};\, m, n = 1, \cdots, k\right)\right), \qquad x \text{ nonintegral.}$$

$A_k(x)$ can be easily inverted using the following lemma.

LEMMA 2.1. *Let C be a $k \times k$ matrix of the form*

$$C = \left(\left(\frac{1}{a_m+b_n};\, m, n = 1, \cdots, k\right)\right).$$

Then the elements of the inverse $D = C^{-1}$ are given by

$$d_{mn} = \frac{\prod_{q=1}^{k}(a_q+b_m)(a_n+b_q)}{(a_n+b_m)\prod_{\substack{r=1\\ r\neq m}}^{k}(b_r-b_m)\prod_{\substack{s=1\\ s\neq n}}^{k}(a_s-a_n)}, \tag{2.1}$$

provided $a_1, \cdots, a_k, -b_1, \cdots, -b_k$ are pairwise distinct.

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Proof. From a result of Pólya and Szegö [3, vol. 2, pp. 98, 299] it is known that

$$\det C = \frac{\prod_{1 \leq r < s \leq k} (a_r - a_s)(b_r - b_s)}{\prod_{r,s=1}^{k} (a_r + b_s)}. \tag{2.2}$$

Since all the minors of C are of the same form as $\det C$, but of order $k - 1$, (2.1) follows readily from (2.2).

To apply the result of Lemma 1 to $A_k(x)$ we let $a_m = m + x$, $b_n = -n$. Then, if $B_{mn}^{(k)}(x)$ is the element in the mth row and nth column of $A_k^{-1}(x)$,

$$B_{mn}^{(k)}(x) = \frac{\prod_{q=1}^{k} (q - m + x)(n - q + x)}{(n - m + x) \prod_{\substack{r=1 \\ r \neq m}}^{k} (m - r) \prod_{\substack{s=1 \\ s \neq m}}^{k} (s - n)}.$$

This can be rewritten as

$$B_{mn}^{(k)}(x) = \frac{x^2}{x + n - m} \prod_{q=1}^{m-1} \left(1 - \frac{x}{q}\right) \prod_{r=1}^{k-m} \left(1 + \frac{x}{r}\right) \prod_{s=1}^{n-1} \left(1 + \frac{x}{s}\right) \prod_{t=1}^{k-n} \left(1 - \frac{x}{t}\right).$$

Now, if we let $k \to \infty$, we can formally obtain the limit function

$$B_{mn}(x) = \frac{1}{\pi} \frac{x \sin \pi x}{x + n - m} \prod_{q=1}^{m-1} \left(1 - \frac{x}{q}\right) \prod_{s=1}^{n-1} \left(1 + \frac{x}{s}\right), \qquad x \text{ nonintegral}, \tag{2.3}$$

as the (m, n)th term of $B(x)$, the inverse of (1.1), where we used the product representation

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right).$$

Although a number of results are known [1, p. 24] which deal with the conditions under which passage to the limit is valid, they do not seem to apply in this case, and another approach will be considered in the next section.

3. The infinite case. In this section it is our intention to prove the following theorem for the inverse of the infinite case.

THEOREM. *The matrix* $B(x) = ((B_{mn}(x),\ m,\ n \geqq 1))$, *where*

$$B_{mn}(x) = \frac{x \sin \pi x}{\pi} \frac{1}{x + n - m} \binom{m - x - 1}{m - 1} \binom{n + x - 1}{n - 1}$$

is (a) *a left inverse of* $A(x)$ *for nonintegral* $x < 1$, *and* (b) *a right inverse of* $A(x)$ *for nonintegral* $x > -1$. *It is thus an inverse of* $A(x)$ *in* $0 < |x| < 1$.

To prove this theorem, we shall need the following lemmas. We first have the following modification of an identity from Whittaker and Watson [4, p. 260].

LEMMA 3.1. *If* $a > 0$, *then*

$$\frac{\Gamma(z + 1)\Gamma(a)}{\Gamma(z + a + 1)} = \sum_{r=0}^{\infty} \binom{r - a - 1}{r - 1} \frac{1}{z + r}. \tag{3.1}$$

Note the restriction on a. The only restriction on z is that z cannot be a negative integer.

Writing (3.1) for two distinct values of z, say z_1 and z_2, and subtracting one from the other leads to

$$(3.2)\qquad \sum_{r=1}^{\infty}\binom{r-a-1}{r-1}\frac{1}{(z_1+r)(z_2+r)}=\frac{\dfrac{\Gamma(z_1+1)\Gamma(a)}{\Gamma(z_1+a+1)}-\dfrac{\Gamma(z_2+1)\Gamma(a)}{\Gamma(z_2+a+1)}}{z_1-z_2},$$

which is, of course, valid for $a > 0$.

LEMMA 3.2. *The equation* (3.2) *is valid for* $a > -1$.

Proof. By inspection, the right side of (3.2) is an analytic function of a (with removable singularity at $a = 0$) for Re $a > -1$. If one uses Raabe's test for convergence [2, p. 285] and Stirling's approximation to $\Gamma(x)$, one notes that on the left hand side of (3.2),

$$\binom{r-a}{r}=\frac{\Gamma(r-a+1)}{\Gamma(r+1)\Gamma(1-a)}\sim\frac{r^{-a}}{\Gamma(1-a)}.$$

Thus, the left side of (3.2) converges for Re $a > -1$ and is analytic for Re $a > -1$, provided z_1, z_2 are not negative integers. Thus, by analytic continuation the equality must hold for Re $a > -1$, even though the original relationship (3.1) holds only in Re $a > 0$.

If in (3.2) one sets $z_1 = z$ and lets $z_2 \to z_1$, then one gets

$$(3.3)\qquad \sum_{r=1}^{\infty}\binom{r-a-1}{r-1}\frac{1}{(z+r)^2}=-\Gamma(a)\frac{d}{dz}\left(\frac{\Gamma(z+1)}{\Gamma(z+a+1)}\right),\qquad a>-1.$$

The connection between the problem at hand and (3.2) and (3.3) is as follows. If in (3.2) we set $a = -x$, $z_1 = -m + x$, and $z_2 = -n + x$, where $m \neq n$ and $m, n \geqq 1$, we obtain, for x nonintegral,

$$(3.4)\qquad \sum_{r=1}^{\infty}\binom{r+x-1}{r-1}\frac{1}{(x+r-m)(x+r-n)}=0.$$

This is, of course, due to $\Gamma(z + a + 1)$ having a simple pole when $z + a + 1$ equals $1 - m$, or $1 - n$. By the same token, if in (3.3) we set $a = -x$ and $z = -n + x$, for $n \geqq 1$ and nonintegral $x < 1$, we get

$$(3.5)\qquad \sum_{r=1}^{\infty}\binom{r+x-1}{r-1}\frac{1}{(x+r-n)^2}=-\Gamma(-x)\lim_{z\to x-n}\frac{d}{dz}\left(\frac{\Gamma(z+1)}{\Gamma(z-x+1)}\right).$$

To evaluate the limit on the right hand side of (3.5) we write

$$(3.6)\qquad \frac{d}{dz}\left(\frac{\Gamma(z+1)}{\Gamma(z-x+1)}\right)=\frac{-\Gamma(z+1)\Gamma'(z-x+1)}{\Gamma^2(z-x+1)}+\frac{\Gamma'(z+1)}{\Gamma(z-x+1)}.$$

The second term in (3.6) can be ignored, as it goes to zero when z goes to $-n + x$. To evaluate the first term, we again use the fact that for every integer $m \geqq 0$, $\Gamma(z)$ has a simple pole at $z = -m$.

Denote the residue at such a pole by r_m. It can then be shown that $r_m = (-1)^m m!$.

From (3.6) and the value of r_m it follows that for nonintegral $x < 1$ and $n \geqq 1$,

$$\sum_{r=1}^{\infty} \binom{r+x-1}{r-1} \frac{1}{(x+r-n)^2} = (-1)^n \Gamma(-x)\Gamma(-n+x+1)\Gamma(n). \tag{3.7}$$

By using the fundamental property $\Gamma(x) = (x-1)\Gamma(x-1)$ again and again, as well as the identity $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$, one reduces (3.7) to

$$\sum_{r=1}^{\infty} \binom{r+x-1}{r-1} \frac{1}{(x+r-n)^2} = \frac{\pi}{x \sin \pi x \binom{n-x-1}{n-1}}. \tag{3.8}$$

Equations (3.4) and (3.8) can be combined to yield

$$\binom{m-x-1}{m-1} \frac{x \sin \pi x}{\pi} \sum_{r=1}^{\infty} \binom{r+x-1}{r-1} \frac{1}{(x+r-m)(x+r-n)} = \delta_{m,n}, \tag{3.9}$$

for nonintegral $x < 1$ and $m, n \geqq 1$, where $\delta_{m,n}$ denotes the Kronecker delta.

In re-examining (2.3) we note that $B_{mn}(x)$ can also be written as

$$B_{mn}(x) = \frac{x \sin \pi x}{\pi} \frac{1}{x+n-m} \binom{m-x-1}{m-1}\binom{n+x-1}{n-1}. \tag{3.10}$$

Equation (3.10) thus shows that $B(x)$, with terms $B_{mn}(x)$, is a left inverse of $A(x)$ for nonintegral $x < 1$, i.e., $B(x)A(x) = I$. If in (3.9) we replace x by $-x$, and interchange m and n, we obtain

$$\binom{n+x-1}{n-1} \frac{x \sin \pi x}{\pi} \sum_{r=1}^{\infty} \binom{r-x-1}{r-1} \frac{1}{(x+n-r)(x+m-r)} = \delta_{m,n}, \tag{3.11}$$

for $m, n \geqq 1$ and nonintegral $x > -1$. But (3.11) then says that $B(x)$ is a right inverse of $A(x)$, i.e., $A(x)B(x) = I$ for nonintegral $x > -1$. Thus for $0 < |x| < 1$, $B(x)$ is both a right and a left inverse of $A(x)$. This proves our theorem.

Unfortunately, the inverse obtained in our theorem is not unique, and a whole host of different left and right inverses can be generated from the expression (3.10). We see this in the following lemma.

LEMMA 3.3. *Let $B_{m,n}^{(p)}(x) = B_{m+p,n}(x+p)$, where $B_{m,n}(x)$ is defined by* (3.10). *Then the matrix $B^{(p)}(x)$ with elements $B_{m,n}^{(p)}(x)$ is a left inverse of $A(x)$ in $x < 1 - p$, x nonintegral.*

Proof. From (3.9) we have that

$$\sum_{r=1}^{\infty} \frac{B_{m,r}(x)}{x+r-n} = \delta_{m,n}, \qquad m, n \geqq 1 \quad \text{and} \quad x <' 1,$$

where by $<'$ we mean less than, and nonintegral. Clearly

$$\sum_{r=1}^{\infty} \frac{B_{m,r}(x+p)}{x+r-(n-p)} = \delta_{m,n}, \qquad m, n \geqq 1 \quad \text{and} \quad x <' 1 - p.$$

If we replace n by $n + p$, and m by $m + p$, we get

$$\sum_{r=1}^{\infty} \frac{B_{m+p,r}(x+p)}{x+r-n} = \delta_{m+p,n+p} = \delta_{m,n}, \qquad m, n \geqq 1-p \text{ and } x <' 1-p. \tag{3.12}$$

In particular (3.12) holds for $m, n \geqq 1$, proving the lemma. A quick check would show that $B_{m+p,r}(x+p)$ is different from $B_{m,r}(x)$.

It is, of course, clear that a suitably modified lemma could have been proved for right inverses. It is interesting to note that a completely different set of right inverses can be obtained from (3.12). For, replace x by $-x$ and interchange m and n. Then

$$\sum_{r=1}^{\infty} \frac{B_{n+p,r}(-x+p)}{-x+r-m} = \delta_{m,n}, \qquad m, n \geqq 1 \quad \text{and} \quad x >' p-1,$$

or

$$\sum_{r=1}^{\infty} \frac{B_{n+p,r}(-x+p)}{x+m-r} = -\delta_{m,n}, \qquad m, n \geqq 1 \quad \text{and} \quad x >' p-1.$$

Hence $((-B_{n+p,m}(-x+p), m, n \geqq 1))$ is a right inverse for $x >' p-1$.

Note that by a theorem in [5, p. 224] the multiplicity of right and left inverses tells us that these inverses are all unbounded in l_2.

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