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GLOBAL SOLUTIONS OF NONLINEAR PERTURBATIONS OF LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

Conditions are given for a nonlinear perturbation of a linear differential equation to have a solution on a given semiinfinite interval which behaves asymptotically like a specified solution of the unperturbed equation. Unlike most previous results on this question, our integrability conditions allow conditional convergence of some of the improper integrals that arise. Our asymptotic estimates are more precise than those previously obtained, and our results apply to singular equations.

1. Introduction

We consider the asymptotic behavior of solutions of the differential equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = f(t, y, y', \ldots, y^{(n-1)}), \tag{1}$$

where $p_1, \ldots, p_n \in C[a, \infty)$. For convenience we write

$$f(t; y) = f(t, y, y', \ldots, y^{(n-1)}).$$

Let $\{x_1, x_2, \ldots, x_n\}$ be a fundamental system for the unperturbed equation

$$x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0 \tag{2}$$

and define

$$w_i = (-1)^{n-i} \frac{W(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)}{W(x_1, \ldots, x_n)},$$

where $W(u_1,\dots,u_m)$ is the Wronskian of $u_1,\dots,u_m$. The following well known identities will be useful below:

$$\sum_{i=1}^{n} x_i^{(r)} w_i = \delta_{r,n-1},\ 0 \le r \le n-1. \tag{3}$$

Our results are global, in that we give sufficient conditions for (1) to have solutions on the *given interval* $[a,\infty)$ which behave like solutions of (2) as $t \to \infty$. (We use "o" and "O" in the usual way to indicate asymptotic behavior as $t \to \infty$.) In this sense, this paper continues a theme developed by Kusano and the author.[3,4]

The following theorem is the main result of [4].

THEOREM 1. *Suppose that $f : [a,\infty) \times \mathcal{R}^n \to \mathcal{R}$ is continuous and*

$$|f(t,u_0,\dots,u_{n-1})| \le F(t,|u_0|,\dots,|u_{n-1}|), \tag{4}$$

where $F : [a,\infty) \times \mathcal{R}_+^n \to \mathcal{R}_+$ is continuous and $F(t,\mu_0,\dots,\mu_{n-1})$ is nondecreasing in each μ_r $(0 \le r \le n-1)$, and either (i) for fixed $(t,v_0,\dots,v_{n-1}) \in [a,\infty) \times \mathcal{R}_+^n$, $\lambda^{-1}F(t,\lambda v_0,\dots,\lambda v_{n-1})$ is nondecreasing in λ for $\lambda > 0$, and

$$\lim_{\lambda \to 0+} \lambda^{-1}F(t,\lambda v_0,\dots,\lambda v_{n-1}) = 0;$$

or (ii) for fixed $(t,v_0,\dots,v_{n-1}) \in [a,\infty) \times \mathcal{R}_+^n$, $\lambda^{-1}F(t,\lambda v_0,\dots,\lambda v_{n-1})$ is nonincreasing in λ for $\lambda > 0$, and

$$\lim_{\lambda \to \infty} \lambda^{-1}F(t,\lambda v_0,\dots,\lambda v_{n-1}) = 0.$$

Let $\bar{x}$ be a given nontrivial solution of (2) and suppose that there are positive continuous functions $\sigma_0,\dots,\sigma_{n-1}$ on $[a,\infty)$ and an integer k $(1 \le k \le n)$ such that

$$|\bar{x}^{(r)}(t)| \le \sigma_r(t),\ 0 \le r \le n-1, \tag{5}$$

$$|x_i^{(r)}(t)| \int_a^t |w_i(s)| F(s,\lambda\sigma_0(s),\dots,\lambda\sigma_{n-1}(s))\, ds = o\left(\sigma_r(t)\right),$$

$$1 \le i \le k-1,\ 0 \le r \le n-1,$$

and

$$|x_i^{(r)}(t)| \int_t^\infty |w_i(s)| F(s, \lambda\sigma_0(s), \dots, \lambda\sigma_{n-1}(s))\, ds = o(\sigma_r(t)),$$

$$k \le i \le n, \ 0 \le r \le n-1,$$

for $\lambda > 0$. *Let* θ *be an arbitrary positive number, and suppose that* c *is a given positive constant. Then (1) has a solution* $\overline{y}$ *on* $[a, \infty)$ *such that*

$$|\overline{y}^{(r)}(t) - c\overline{x}^{(r)}(t)| \le \theta c \sigma_r(t), \ 0 \le r \le n-1, \tag{6}$$

provided that c *is sufficiently small if (i) holds, or sufficiently large if (ii) holds. Moreover,*

$$\overline{y}^{(r)}(t) = c\overline{x}^{(r)}(t) + o(\sigma_r(t)), \ 0 \le r \le n-1. \tag{7}$$

2. The Main Result

The main result of this paper is similar to Theorem 1 in that it guarantees the existence of the desired solution $\overline{y}$ on the given interval $[a, \infty)$, but different in that it provides more precise estimates of the differences $\overline{y}^{(r)}(t) - \overline{x}^{(r)}(t)$ as $t \to \infty$ in the case where $f(\cdot; c\overline{x})$ is oscillatory. It is also applicable in some situations where Theorem 1 is not.

DEFINITION 1. Let ω denote either $0+$ or ∞. If $r > 0$ then $I(r) = (0, r)$ if $\omega = 0+$, or $I(r) = (r, \infty)$ if $\omega = \infty$. We will say that an assertion is true for "c sufficiently close to ω" if it holds for all c in some interval $I(r)$, where r is sufficiently small if $\omega = 0+$ or sufficiently large if $\omega = \infty$.

THEOREM 2. *Let* $\overline{x}$ *be a given solution of (2) and let* $\rho_0, \dots, \rho_{n-1}$ *be continuous and positive on* $[a, \infty)$. *Suppose that there is a* $c_0 > 0$ *such that the following assumptions hold for all* c *in* $I(c_0)$:

(i) $f(\cdot; c\overline{x})$ *is continuous on* $[a, \infty)$;

(ii) For some $\theta > 0$, $f(t, u_0, \dots, u_{n-1})$ *is continuous and*

$$|f(t, u_0, \dots, u_{n-1}) - f(t; c\overline{x})| \le F(t, c) \tag{8}$$

on the set

$$\{(t, u_0, \dots, u_{n-1}) : |u_r - c\overline{x}^{(r)}(t)| \le \theta c \rho_r(t), t \ge a, \ 0 \le r \le n-1\},$$

where $F : [a,\infty) \times I(c_0) \to \mathcal{R}_+$ *is continuous and* $c^{-1}F(t,c)$ *is nondecreasing in* c *for each* $t \geq a$ *if* $\omega = 0+$, *or nonincreasing in* c *for each* $t \geq a$ *if* $\omega = \infty$. *In either case,*

$$\lim_{c \to \omega} c^{-1}F(t,c) = 0,\ t \geq a. \tag{9}$$

(iii) for some integer k, $1 \leq k \leq n+1$, *and* $t \geq a$,

$$\left| x_i^{(r)}(t) \int_a^t w_i(s) f(s; c\overline{x})\, ds \right| \leq \alpha(c)\rho_r(t),\ 1 \leq i \leq k-1,\ 0 \leq r \leq n-1, \tag{10}$$

$$\left| x_i^{(r)}(t) \int_t^\infty w_i(s) f(s; c\overline{x})\, ds \right| \leq \alpha(c)\rho_r(t),\ k \leq i \leq n,\ 0 \leq r \leq n-1, \tag{11}$$

where $\lim_{c \to \omega} c^{-1}\alpha(c) = 0$,

$$|x_i^{(r)}(t)| \int_a^t |w_i(s)| F(s,c)\, ds = o(\rho_r(t)),\ 1 \leq i \leq k-1, \tag{12}$$

and

$$|x_i^{(r)}(t)| \int_t^\infty |w_i(s)| F(s,c)\, ds = o(\rho_r(t)),\ k \leq i \leq n. \tag{13}$$

Then (1) has a solution $\overline{y}$ *on* $[a,\infty)$ *such that*

$$|\overline{y}^{(r)}(t) - c\overline{x}^{(r)}(t)| \leq \theta c \rho_r(t),\ t \geq a,\ 0 \leq r \leq n-1, \tag{14}$$

provided that c *is sufficiently close to* ω. *Moreover,*

$$\overline{y}^{(r)}(t) = c\overline{x}^{(r)}(t) + g(t; c\overline{x}) + o\left(\rho_r(t)\right),\ 0 \leq r \leq n-1, \tag{15}$$

where

$$g(t; c\overline{x}) = \sum_{i=1}^{k-1} x_i(t) \int_a^t w_i(s) f(s; c\overline{x})\, ds - \sum_{i=k}^{n} x_i(t) \int_t^\infty w_i(s) f(s; c\overline{x})\, ds. \tag{16}$$

The main differences between Theorems 1 and 2 are as follows:

• The integrals in (10) and (11) do not involve the *absolute value* of $f(\cdot; c\overline{x})$. Therefore, Theorem 2 takes advantage of oscillatory behavior of $f(\cdot; c\overline{x})$; in particular, the integrals in (11) may converge conditionally.

• Assumption (5) automatically restricts the precision of the estimates in (6) and (7). There is no such restriction in Theorem 2; indeed, Theorem 2 may apply even if $\overline{\lim}_{t\to\infty}\overline{x}^{(r)}(t)|/\rho_r(t) = \infty$, as the examples in Section 4 will show.

• Theorem 2 does not require that f be majorized as in (4), with F nondecreasing in all arguments except t. This requirement makes Theorem 1 inapplicable, for example, to singular equations, as in Example 2 of Section 4.

We prove Theorem 2 by means of the following special case of the Schauder–Tychonoff theorem. The proof of this lemma is like that of a similar lemma of Coppel.[1]

LEMMA 1. *Let $C^{(n-1)}[a,\infty)$ be given the topology of uniform convergence on compact subintervals of $[a,\infty)$; i.e., if $\{y_j\}$ is a sequence in $C^{(n-1)}[a,\infty)$, then "$y_j \to y$" means that $\lim_{j\to\infty} y_j^{(r)}(t) = y^{(r)}(t)$ ($t \geq a$, $0 \leq r \leq n-1$) uniformly on $[a,T]$ for all $T > a$. Let $\mathcal{S}$ be a closed convex subset of $C^{(n-1)}[a,\infty)$ and suppose that $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ is continuous and the family of functions $\mathcal{T}(\mathcal{S})$ is uniformly bounded and equicontinuous on $[a,T]$ for all $T > a$. Then $\mathcal{T}\overline{y} = \overline{y}$ for some $\overline{y} \in \mathcal{S}$.*

PROOF OF THEOREM 2. For a given $c \in I(c_0)$ let $\mathcal{S}$ be the closed convex subset of $C^{(n-1)}[a,\infty)$ defined by

$$\mathcal{S} = \{y \in C^{(n-1)}[a,\infty) : |y^{(r)}(t) - c\overline{x}^{(r)}(t)| \leq \theta c \rho_r(t),\ t \geq a,\ 0 \leq r \leq n-1\}. \quad (17)$$

We will show that if c is sufficiently close to ω, then the transformation $\mathcal{T}$ defined by

$$(\mathcal{T}y)(t) = c\overline{x}(t) + \sum_{i=1}^{k-1} x_i(t) \int_a^t w_i(s) f(s;y)\,ds - \sum_{i=k}^{n} x_i(t) \int_t^\infty w_i(s) f(s;y)\,ds \quad (18)$$

satisfies the hypotheses of Lemma 1 on $\mathcal{S}$, and therefore that $\mathcal{T}\overline{y} = \overline{y}$ for some $\overline{y} \in \mathcal{S}$. It will then be easy to verify that $\overline{y}$ is a solution of (1) with the stated properties.

If the improper integrals on the right of (18) converge, then (3) implies that

$$(\mathcal{T}y)^{(r)}(t) = c\overline{x}^{(r)}(t) + \sum_{i=1}^{k-1} x_i^{(r)}(t) \int_a^t w_i(s) f(s;y)\, ds$$

$$- \sum_{i=k}^{n} x_i^{(r)}(t) \int_t^{\infty} w_i(s) f(s;y)\, ds, \ 0 \le r \le n-1, \tag{19}$$

and

$$(\mathcal{T}y)^{(n)}(t) = -p_1(t)(\mathcal{T}y)^{(n-1)}(t) - \cdots - p_n(t)(\mathcal{T}y)(t) + f(t;y). \tag{20}$$

It is convenient to rewrite (19) as

$$(\mathcal{T}y)^{(r)}(t) = c\overline{x}^{(r)}(t) + g^{(r)}(t;c\overline{x}) + (\mathcal{H}y)^{(r)}(t) \tag{21}$$

(see (16)), where

$$(\mathcal{H}y)^{(r)}(t) = \sum_{i=1}^{k-1} x_i^{(r)}(t) \int_a^t w_i(s)[f(s;y) - f(s;c\overline{x})]\, ds$$

$$- \sum_{i=k}^{n} x_i^{(r)}(t) \int_t^{\infty} w_i(s)[f(s;y) - f(s;c\overline{x})]\, ds. \tag{22}$$

From (3) and (16),

$$g^{(r)}(t;c\overline{x}) = \sum_{i=1}^{k-1} x_i^{(r)}(t) \int_a^t w_i(s) f(s;c\overline{x})\, ds$$

$$- \sum_{i=k}^{n} x_i^{(r)}(t) \int_t^{\infty} w_i(s) f(s;c\overline{x})\, ds, \ 0 \le r \le n-1;$$

therefore, (10) and (11) imply that

$$|g^{(r)}(t;c\overline{x})| \le n\alpha(c)\rho_r(t), \ t \ge a, \ 0 \le r \le n-1. \tag{23}$$

In order to estimate $(\mathcal{H}y)^{(r)}$ we argue as in the proof of Theorem 1.[4] For c in $I(c_0)$ define

$$\phi_i(t,c) = \begin{cases} \displaystyle\int_a^t |w_i(s)|F(s,c)ds, & 1 \le i \le k-1, \\ \displaystyle\int_t^\infty |w_i(s)|F(s,c)\,ds, & k \le i \le n, \end{cases} \tag{24}$$

and

$$\Phi_r(t,c) = \sum_{i=1}^{n} |x_i^{(r)}(t)|\phi_i(t,c), \; 0 \le r \le n-1. \tag{25}$$

Then (12) and (13) imply that

$$\Phi_r(t,c) = o\left(\rho_r(t)\right), \; 0 \le r \le n-1. \tag{26}$$

Now let γ be an arbitrary positive number. Then

$$\Phi_r(t,c) \le \gamma c \rho_r(t), \; 0 \le r \le n-1, \tag{27}$$

for $t \ge a$ if c is sufficiently close to ω. To see this, choose $T \ge a$ such that

$$\Phi_r(t,c_0) \le \gamma c_0 \rho_r(t), \; t \ge T, \; 0 \le r \le n-1.$$

(This is possible, because of (26).) Then the monotonicity of $c^{-1}F(t,c)$ implies (27) for $t \ge T$ and $c \in I(c_0)$. If $a \le t \le T$, then (24) and (25) imply that

$$\Phi_r(t,c) \le \sum_{i=1}^{k-1} |x_i^{(r)}(t)|\phi_i(T,c) + \sum_{i=k}^{n} |x_i^{(r)}(t)|\phi_i(a,c), \; 0 \le r \le n-1. \tag{28}$$

From (9), (24), and Lebesgue's bounded convergence theorem,

$$\lim_{c\to\omega} c^{-1}\phi_i(\tau,c) = 0, \; 1 \le i \le n, \tag{29}$$

for any fixed $\tau \geq a$. Since the functions $\rho_r^{-1}|x_i^{(r)}|$ $(0 \leq r \leq n-1,\ 1 \leq i \leq k)$ are all bounded on $[a, T]$, (28) and (29) now imply (27) on $[a, T]$ if c is sufficiently close to ω.

From assumption (ii), (17), (22), (24), and (25),

$$|(\mathcal{H}y)^{(r)}(t)| \leq \Phi_r(t, c),\ t \geq a,\ 0 \leq r \leq n-1. \tag{30}$$

Now choose $c_1 \in I(c_0)$ so that

$$n\alpha(c) \leq \frac{\theta c}{2} \text{ and } \Phi_r(t, c) \leq \frac{\theta c}{2}\rho_r(t),\ t \geq a,\ 0 \leq r \leq n-1,$$

if $c \in I(c_1)$. (We let $\gamma = \theta/2$ in (27).) Then (21), (23), and (30) imply that

$$|(\mathcal{T}y)^{(r)}(t) - c\overline{x}^{(r)}(t)| \leq \theta c \rho_r(t),\ t \geq a,\ 0 \leq r \leq n-1; \tag{31}$$

i.e., that $\mathcal{T}(\mathcal{S}) \subset \mathcal{S}$. From (31), the families

$$\{(\mathcal{T}y)^{(r)} : y \in \mathcal{S}\},\ 0 \leq r \leq n-1, \tag{32}$$

are equibounded on finite intervals. Moreover, from assumption (ii) and (17),

$$|f(t; y)| \leq |f(t; c\overline{x})| + |f(t; y) - f(t; c\overline{x})| \leq |f(t; c\overline{x})| + F(t, c),\ y \in \mathcal{S}.$$

Therefore, (20) now implies that the family $\{(\mathcal{T}y)^{(n)} : y \in \mathcal{S}\}$ is also equibounded on finite intervals. We can therefore conclude that the families (32) are equicontinuous on finite intervals.

We will now show that $\mathcal{T}$ is continuous. Let $\{y_n\}$ be a sequence in $\mathcal{S}$ such that $y_n \to y$. From (19), if $T \geq a$,

$$|(\mathcal{T}y_n)^{(r)}(t) - (\mathcal{T}y)^{(r)}(t)| \leq \sum_{i=1}^{k-1} |x_i^{(r)}(t)| \int_a^T |w_i(s)||f(s; y_n) - f(s; y)|\, ds$$

$$+ \sum_{i=k}^{n} |x_i^{(r)}(t)| \int_a^\infty |w_i(s)||f(s; y_n) - f(s; y)|\, ds,\ a \leq t \leq T,\ 0 \leq r \leq n-1. \tag{33}$$

The integrands on the right converge to zero as $t \to \infty$, and they are respectively dominated by $2|w_i(s)|F(s,c)$, $1 \le i \le n$. Therefore, our integrability conditions on F and Lebesgue's dominated convergence theorem imply that the integrals in (33) converge to zero as $t \to \infty$. Since $x_1^{(r)}, \ldots, x_n^{(r)}$ are bounded on $[a,T]$, this implies that $\mathcal{T} y_n \to \mathcal{T} y$; hence, $\mathcal{T}$ is continuous.

We have now verified that $\mathcal{T}$ satisfies the hypotheses of Lemma 1 on $\mathcal{S}$. Therefore, $\mathcal{T}\overline{y} = \overline{y}$ for some $\overline{y}$ in $\mathcal{S}$. Setting $y = \overline{y}$ in (20) and (31) shows that $\overline{y}$ is a solution of (1) and that $\overline{y}$ satisfies (14). Setting $y = \overline{y}$ in (30) and recalling (26) shows that $(\mathcal{H}y)^{(r)}(t) = o(\rho_r(t))$, $0 \le r \le n-1$; therefore, setting $y = \overline{y}$ in (21) verifies (15). This completes the proof of Theorem 2.

3. Perturbations of a Disconjugate Equation

If no nontrivial solution of (2) has more that $n-1$ zeros (counting multiplicities) on $[a,\infty)$ then we say that (2) is *disconjugate on* $[a,\infty)$. In this case (2) has a fundamental system $\{x_1, \ldots, x_n\}$ such that[2,5,6]

$$x_i > 0,\ |w_i| > 0,\ t \ge a,\ 1 \le i \le n, \tag{34}$$

$$\left|\frac{w_i}{w_j}\right|' > 0, \text{ and } \lim_{t\to\infty} \frac{w_j(t)}{w_i(t)} = \lim_{t\to\infty} \frac{x_i(t)}{x_j(t)} = 0,\ 1 \le i < j \le n. \tag{35}$$

In order to state our next theorem we need the following elementary lemma, which can be proved by integration by parts.

LEMMA 2. *Suppose that* $u \in C[a,\infty)$, *and* $\int^\infty u(s)\,ds$ *converges (perhaps conditionally), and* $\sup_{\tau \ge t} \left|\int_\tau^\infty u(s)\,ds\right| \le \psi(t)$, *where* ψ *is continuous and nonincreasing. Let* $p' \in C[a,\infty)$ *and* $p \ge 0$.

(a) If $p' < 0$ *then* $\left|\int_t^\infty u(s)p(s)\,ds\right| \le 2\psi(t)p(t)$, $t \ge a$.

(b) If $p' > 0$ *then* $\left|\int_a^t u(s)p(s)\,ds\right| \le \psi(a)p(a) + \psi(t)p(t) + \int_a^t \psi(s)p'(s)\,ds$,

and the quantity on the right is $o(p(t))$ *if* $\lim_{t\to\infty} p(t) = \infty$.

THEOREM 3. *Let (2) have a fundamental system* $\{x_1,\dots,x_n\}$ *which satisfies 34) and (35), and let* ϕ *be continuous and nonincreasing on* $[a,\infty)$. *Let* k *be an nteger,* $1\le k\le n$, *and define*

$$\mu_{ik}=\begin{cases}\left|\dfrac{w_i}{w_k}\right| \quad, & 1\le i\le k-1,\\ \phi \quad, & i=k,\\ 2\phi\left|\dfrac{w_i}{w_k}\right|, & k+1\le i\le n,\end{cases} \tag{36}$$

nd

$$\rho_{rk}=\max_{1\le i\le n}\{\mu_{ik}|x_i^{(r)}|\}. \tag{37}$$

et $\overline{x}$ *be a given solution of (2) and suppose that there is a* $c_0>0$ *such that for all* $\in I(c_0)$ *assumptions (i) and (ii) of Theorem 2 hold with* $\rho_r=\rho_{rk}$ $(0\le r\le n-1)$. *uppose also that*

$$\int_t^\infty w_k(s)F(s,c)\,ds=o\,(\phi(t)) \tag{38}$$

nd

$$\sup_{\tau\ge t}\left|\int_\tau^\infty w_k(s)f(s;c\overline{x})\,ds\right|\le\beta(c)\phi(t), \tag{39}$$

here $\lim_{c\to\omega}c^{-1}\beta(c)=0$. *Then the conclusions of Theorem 2 hold with* $\rho_r=\rho_{rk}$.

PROOF. Because of (36), (39), and Lemma 2,

$$\left|\int_a^t w_i(s)f(s;c\overline{x})\,ds\right|\le J_i\beta(c)\mu_{ik}(t),\ 1\le i\le k-1$$

vhere $J_1,\dots,J_{k-1}$ are suitable constants), and

$$\left|\int_t^\infty w_i(s)f(s;c\overline{x})\,ds\right|\le\beta(c)\mu_{ik}(t),\ k\le i\le n,$$

for all $c \in I(c_0)$. Also, (36), (38), and Lemma 2 imply that

$$\int_a^t w_i(s)F(s,c)\,ds = o\left(\mu_{ik}(t)\right), \quad 1 \le i \le k-1.$$

and

$$\int_t^\infty w_i(s)F(s,c)\,ds = o\left(\mu_{ik}(t)\right), \quad k \le i \le n$$

for all $c \in I(c_0)$. The last four equations and (37) imply (10), (11), (12), and (13) with $\rho_r = \rho_{rk}$ and $\alpha = J\beta$, where $J = \max\{J_1, \dots, J_{k-1}, 1\}$.

4. Examples

Consider the equation

$$y''' + y' = \left(a_0 y^{\gamma_0} + a_1 (y')^{\gamma_1} + a_2 (y'')^{\gamma_3}\right) \phi(t) \sin \delta t, \tag{40}$$

where a_0, a_1, and a_2 are constants, γ_0, γ_1, and γ_2 are rational numbers greater than one with odd denominators, δ is not an integer, ϕ is positive, continuously differentiable, and nonincreasing on $[a, \infty)$, $\lim_{t\to\infty} \phi(t)=0$, and

$$\int_t^\infty \phi^2(s)\,ds = o\left(\phi(t)\right). \tag{41}$$

The functions $x_1(t) = 1$, $x_2(t) = \cos t$, and $x_3(t) = \sin t$ form a fundamental system for the unperturbed equation

$$x''' + x' = 0, \tag{42}$$

and, correspondingly, $w_1(t) = 1$, $w_2(t) = -\cos t$, and $w_3(t) = -\sin t$. Eq. (40) is of the form (1) with

$$f(t, u_0, u_1, u_2) = \left(a_0 u_0^{\gamma_0} + a_1 u_1^{\gamma_1} + a_2 u_2^{\gamma_2}\right) \phi(t) \sin \delta t.$$

We will use Theorem 2 with $k = 1$ to show that (40) has solutions on $[a, \infty)$ which behave like $c\overline{x}$, where $\overline{x}$ is an arbitrary solution of (42).

If $\overline{x}$ is a solution of (42), then our assumptions on γ_1, γ_2 and γ_3 imply that $\overline{x}^{\gamma_0}$, $(\overline{x}')^{\gamma_1}$ and $(\overline{x}'')^{\gamma_2}$ are continuously differentiable and have period 2π on $(-\infty, \infty)$; hence, their Fourier series converge uniformly on $(-\infty, \infty)$, and the coefficients in these series approach zero at least as fast as $1/n$ as $n \to \infty$. Since δ is not an integer, it follows that

$$\left| \int_a^t w_i(s) \left(\overline{x}^{(j)}(s) \right)^{\gamma_j} \sin \delta s \, ds \right| \le B, \; t \ge a, \;\; 1 \le i \le 3, \;\; 0 \le j \le 2,$$

for some constant B. This and integration by parts imply (11), with $k = 1$,

$$\alpha(c) = 2B(a_0 c^{\gamma_0} + a_1 c^{\gamma_1} + a_2 c^{\gamma_2}),$$

and $\rho_r(t) = \phi(t)$, $r = 0, 1, 2$.

If $\theta > 0$ and $|u_r - c\overline{x}^{(r)}(t)| \le \theta c \phi(t)$ $(r = 0, 1, 2)$, then the mean value theorem implies (8) with

$$F(t, c) = (A_0 c^{\gamma_0} + A_1 c^{\gamma_1} + A_2 c^{\gamma_2}) \, \phi^2(t),$$

where A_0, A_1, and A_2 are constants independent of c. Therefore, (41) implies (13). Since α and F satisfy the hypotheses of Theorem 2 with $\omega = 0+$, we can now infer from Theorem 2 that if $\theta > 0$ and c is a sufficiently small positive number, then (40) has a solution $\overline{y}$ on $[a, \infty)$ such that

$$|\overline{y}^{(r)}(t) - c\overline{x}^{(r)}(t)| \le \theta c \phi(t), \; t \ge a, \; r = 0, 1, 2.$$

Condition (41) holds, for example, if $\phi(t) = \epsilon(t)/t$, where $\epsilon(t) \to 0$ monotonically as $t \to \infty$. In this case, Theorem 2 implies the existence of solutions $\overline{y}$ of (40) on $[a, \infty)$ such that

$$\overline{y}^{(r)}(t) = c\overline{x}^{(r)}(t) + O\left(\epsilon(t)/t\right), \; r = 0, 1, 2.$$

Theorem 1 does not apply unless $\int^{\infty} \epsilon(t)/t \, dt < \infty$. Because of the restriction (5), the best estimate that could be obtained from Theorem 1 in this case is that

$$\overline{y}^{(r)}(t) = c\overline{x}^{(r)}(t) + o(1) \; (r = 0, 1, 2).$$

EXAMPLE 2. Consider the equation

$$y''' - \frac{6}{t}y'' + \frac{15}{t^2}y' - \frac{15}{t^3}y = \frac{\epsilon(t)\sin t}{t^\lambda}y^\gamma, t \geq a > 0, \tag{43}$$

which is of the form (1) with

$$f(t,u) = u^\gamma \epsilon(t) t^{-\lambda} \sin t. \tag{44}$$

We assume that $\gamma \neq 0, 1$, and ϵ is positive and nonincreasing. The functions $x_1 = t, x_2 = t^3$, and $x_3 = t^5$ form a fundamental system for the unperturbed equation

$$x''' - \frac{6}{t}x'' + \frac{15}{t^2}x' - \frac{15}{t^3}x = 0,$$

and, correspondingly, $w_1 = t/8, w_2 = -1/4t$, and $w_3 = 1/8t^3$.

Let $\overline{x} = t^{2m-1}$, where $m = 1, 2$ or 3 and suppose that $\lambda > 1 + (2m-1)\gamma$. Let

$$\alpha = \lambda - 1 - (2m-1)\gamma. \tag{45}$$

We will use Theorem 3 with $k = 1$ to show that if $c^{\gamma-1}$ is sufficiently large, then (43) has a solution $\overline{y}$ on $[a, \infty)$ such that

$$\overline{y}^{(r)}(t) = c\overline{x}^{(r)}(t) + O\left(\epsilon(t)t^{-\alpha-r+1}\right), r = 1, 2, 3. \tag{46}$$

If $m = 2$ or 3 this will require no additional assumptions on α or ϵ; if $m = 1$ we will need an additional assumption, stated below.

With f as in (44), integration by parts shows that

$$\left|\int_t^\infty w_1(s)f(s; c\overline{x})\,ds\right| = \frac{c^\gamma}{8}\left|\int_t^\infty s^{-\alpha}\epsilon(s)\sin s\,ds\right| \leq \frac{1}{4}c^\gamma t^{-\alpha}\epsilon(t),$$

which implies (39) with $k = 1$, $\beta(c) = c^\gamma/4$, and $\phi(t) = t^{-\alpha}\epsilon(t)$. With this ϕ, calculating from (36) and (37) yields

$$\rho_{01}(t) = 4t^{-\alpha+1}\epsilon(t), \ \rho_{11}(t) = 10t^{-\alpha}\epsilon(t), \text{ and } \rho_{21}(t) = 40t^{-\alpha-1}\epsilon(t).$$

If θ is sufficiently small and

$$|u - ct^{2m-1}| \leq \theta c \rho_{01}(t) = 4\theta c t^{-\alpha+1}\epsilon(t), \tag{47}$$

then

$$0 < \frac{1}{2}ct^{2m-1} \leq u \leq \frac{3}{2}ct^{2m-1},\ t \geq a,$$

which implies that

$$u^{\gamma-1} \leq 2^{|\gamma-1|}c^{\gamma-1}t^{(2m-1)(\gamma-1)},\ t \geq a, \tag{48}$$

for all γ. By the mean value theorem,

$$|f(t,u) - f(t, ct^{2m-1})| \leq |\gamma|\tilde{u}^{\gamma-1}|u - ct^{2m-1}|\epsilon(t)t^{-\lambda}, \tag{49}$$

where (47) (and therefore (48)) holds with $u = \tilde{u}$. Therefore, from (45), (47), and (49),

$$|f(t,u) - f(t, ct^{2m-1})| \leq F(t,c) =_{\text{Df.}} Kc^{\gamma}t^{-2\alpha-2m+1}\epsilon^2(t),$$

where K is independent of c. In the present situation, the integrability assumption (38) becomes

$$\int_t^\infty s^{-2(m+\alpha-1)}\epsilon^2(s)\,ds = o(\epsilon(t)t^{-\alpha}).$$

This automatically holds (for any nonincreasing ϵ) if $m = 2$ or 3; therefore, we need no further assumptions in this case. If $m = 1$, then we must assume that

$$\int_t^\infty s^{-2\alpha}\epsilon^2(s)\,ds = o\left(\epsilon(t)t^{-\alpha}\right),$$

which certainly holds if $\alpha > 1$, so that

$$\int_t^\infty |w_1(s)|(\overline{x}_m(s))^{\gamma}\,ds < \infty, \tag{50}$$

but also, for example, if $\alpha = 1$ and $\epsilon(t) = O\left((\log t)^{-1}\right)$, in which case (50) does not hold.

It should be noted that estimates more precise than (46) can be obtained by recalling (15). Also, notice that Theorem 1 cannot be applied to (43) unless $\gamma > 0$, and $\alpha > 1$; moreover, even in this case the asymptotic estimates obtained from Theorem 1 are not as precise as (46), due to the limitation (5).

5. References

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