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SPECTRAL EVOLUTION OF A ONE–PARAMETER EXTENSION OF A REAL SYMMETRIC TOEPLITZ MATRIX*

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Abstract. Let $T_n = (t_{i-j})_{i,j=1}^n$ $(n \geq 3)$ be a real symmetric Toeplitz matrix such that T_{n-1} and T_{n-2} have no eigenvalues in common. We consider the evolution of the spectrum of T_n as the parameter $t = t_{n-1}$ varies over $(-\infty, \infty)$. It is shown that the eigenvalues of T_n associated with symmetric (reciprocal) eigenvectors are strictly increasing functions of t, while those associated with the skew–symmetric (anti–reciprocal) eigenvectors are strictly decreasing. Results are obtained on the asymptotic behavior of the eigenvalues and eigenvectors as $t \to \pm\infty$, and on the possible orderings of eigenvalues associated with symmetric and skew–symmetric eigenvectors.

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SPECTRAL EVOLUTION OF A ONE–PARAMETER EXTENSION OF A REAL SYMMETRIC TOEPLITZ MATRIX

William F. Trench

Following Andrew [1], we will say that an m–vector

$$X = [x_1, x_2, \dots, x_m]$$

is *symmetric* if

$$x_j = x_{m-j+1},\ 1 \le j \le m,$$

or *skew–symmetric* if

$$x_j = -x_{m-j+1},\ 1 \le j \le m.$$

(Some authors call such vectors *reciprocal* and *anti-reciprocal*.) Cantoni and Butler [2] have shown that if

$$T_m = (t_{i-j})_{i,j=1}^{m}$$

is a real symmetric Toeplitz matrix of order m, then R^m has an orthonormal basis consisting of $m - [m/2]$ symmetric and $[m/2]$ skew–symmetric eigenvectors of T_m, where $[x]$ is the integer part of x. A related result of Delsarte and Genin [4, Thm. 8] is that if λ is an eigenvalue of T_m with multiplicity greater than one, then the λ–eigenspace of T_m has an orthonormal basis which splits as evenly as possible between symmetric and skew–symmetric λ–eigenvectors of T_m. For convenience here, we will say that an eigenvalue λ of T_m is *even* (*odd*) if T_m has a symmetric (skew–symmetric) λ–eigenvector. The collection $S^+(T_m)(S^-(T_m))$ of even (odd) eigenvalues will be called the *even* (*odd*) *spectrum of* T_m. From the result of Delsarte and Genin, a multiple eigenvalue is in both the even and odd spectra of T_m.

This paper is motivated by considerations that arose in connection with the inverse eigenvalue problem for real symmetric Toeplitz matrices. Although we do not claim that our results provide much insight into this problem, they may nevertheless be of some interest in their own right.

The inverse eigenvalue problem for real symmetric Toeplitz matrices is usually stated as follows: Find a real symmetric Toeplitz matrix T_m with given spectrum

$$S(T_m) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m\}.$$

For our purposes it is convenient to impose an additional condition; namely, that T_m have even and odd spectra $S^+(T_m)$ and $S^-(T_m)$, containing, respectively, $m - [m/2]$ and $[m/2]$ given elements (counting repeated eigenvalues according to their multiplicities) of S. We will say that $S^+(T_m)$ and $S^-(T_m)$ are *interlaced* if whenever λ_k and λ_l are in $S^+(T_m)$ $(S^-(T_m))$ and $k < l$, there is an element λ_i in $S^-(T_m)$ $(S^+(T_m))$ such that $\lambda_k \leq \lambda_i \leq \lambda_l$. Delsarte and Genin [4] showed that if $m \leq 4$ then the inverse eigenvalue problem always has a solution (regardless of the numerical values of λ_1, λ_2, λ_3, and λ_4) if $S^+(T_m)$ and $S^-(T_m)$ are interlaced; however, if they are not, then the existence or nonexistence of a solution depends on the specific numerical values of the λ_i's. They also argue that this negative consequence of non–interlacement of $S^+(T_m)$ and $S^-(T_m)$ holds for all $m > 4$; that is, if the two desired spectra are not interlaced, then the inverse eigenvalue problem fails to have a solution for some choices of desired eigenvalues.

Delsarte and Genin [4] formulated the (still open) conjecture that the inverse eigenvalue problem always has a solution (for arbitrary m) provided that the desired even and odd spectra are interlaced. (This was apparently misinterpreted by Laurie [6], who cited a real symmetric Toeplitz matrix for which $S^+(T_m)$ and $S^-(T_m)$ are not interlaced as "a counterexample ... to the conjecture of Delsarte and Genin that the eigenvectors of a symmetric Toeplitz matrix, corresponding to eigenval-

ues arranged in decreasing order, alternate between reciprocal and anti-reciprocal vectors.")

In numerical experiments reported in [7] we computed the eigenvalues of hundreds of randomly generated real symmetric Toeplitz matrices with orders up to 1000. (Since then we have considered matrices of order 2000). The even and odd spectra of these matrices are certainly not necessarily interlaced, but they seem to be "almost interlaced," in that we seldom saw more than two or three successive even (or odd) eigenvalues. In unsuccessfully trying to formulate a definition of a measure of interlacement that would be useful in connection with the inverse eigenvalue problem, we were led to study the problem considered here; namely, if

$$T_{n-1} = \begin{bmatrix} t_0 & t_1 & \dots & t_{n-2} \\ t_1 & t_0 & \dots & t_{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n-2} & t_{n-3} & \dots & t_0 \end{bmatrix},$$

is a given real symmetric Toeplitz matrix of order $n-1$, then how does the spectrum of the n-th order matrix

$$T_n(t) = \begin{bmatrix} t_0 & t_1 & \dots & t_{n-2} & t \\ t_1 & t_0 & \dots & t_{n-3} & t_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n-2} & t_{n-3} & \dots & t_0 & t_1 \\ t & t_{n-2} & \dots & t_1 & t_0 \end{bmatrix} \tag{1}$$

evolve as t varies over $(-\infty, \infty)$?

We impose the following assumption throughout.

Assumption A: $n \geq 3$ and T_{n-2} and T_{n-1} have no eigenvalues in common.

Assumption A and Cauchy's interlace theorem imply that T_{n-2} and T_{n-1} have no repeated eigenvalues. Let

$$\alpha_1 < \alpha_2 < \cdots < \alpha_{n-1}$$

be the eigenvalues of T_{n-1} and let

$$\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_n(t)$$

be the eigenvalues of $T_n(t)$. The Cauchy interlace theorem implies that

$$\lambda_i(t) \leq \alpha_i \leq \lambda_{i+1}(t), \ 1 \leq i \leq n-1, \ -\infty < t < \infty.$$

It is also convenient to introduce distinct names for the even and odd eigenvalues of T_{n-2} and $T_n(t)$. Define

$$r = n - [n/2] \text{ and } s = [n/2];$$

thus $r = s$ if n is even and $r = s+1$ if n is odd. Denote the even and odd eigenvalues of T_{n-2} by

$$\beta_1 < \beta_2 < \cdots < \beta_{r-1}$$

and

$$\gamma_1 < \gamma_2 < \cdots < \gamma_{s-1},$$

respectively, and let

$$\mu_1(t) \leq \mu_2(t) \leq \cdots \leq \mu_r(t) \tag{2}$$

and

$$\nu_1(t) \leq \nu_2(t) \leq \cdots \leq \nu_s(t) \tag{3}$$

be the even and odd eigenvalues, respectively, of $T_n(t)$.

Now define

$$p_j(\lambda) = \det(T_j - \lambda I_j),\ 1 \le j \le n-1,$$

and

$$p_n(\lambda, t) = \det(T_n(t) - \lambda I_n).$$

As observed by Delsarte and Genin [4], a result of Cantoni and Butler [2] implies that $p_n(\lambda, t)$ can be factored in the form

$$p_n(\lambda, t) = p_n^+(\lambda, t) p_n^-(\lambda, t),$$

where p_n^+ and p_n^- are of degrees r and s respectively in λ,

$$p_n^+(\mu_i(t), t) = 0,\ 1 \le i \le r, -\infty < t < \infty, \tag{4}$$

and

$$p_n^-(\nu_j(t), t) = 0,\ 1 \le j \le s, -\infty < t < \infty. \tag{5}$$

Moreover, an argument of Delsarte and Genin [4, p. 203, 208] implies that the even (odd) eigenvalues of T_{n-2} separate the even (odd) eigenvalues of $T_n(t)$; i. e.,

$$\mu_i(t) \le \beta_i \le \mu_{i+1}(t),\ 1 \le i \le r-1, \tag{6}$$

and

$$\nu_i(t) \le \gamma_i \le \nu_{i+1}(t),\ 1 \le i \le s-1. \tag{7}$$

It now follows that (2) and (3) can be replaced by the stronger inequalities

$$\mu_1(t) < \mu_2(t) < \cdots < \mu_r(t)$$

and

$$\nu_1(t) < \nu_2(t) < \cdots < \nu_s(t).$$

To see this, suppose for example that $\mu_i(\hat{t}) = \mu_{i+1}(\hat{t})$ for some i and $\hat{t}$. Then (6) implies that β_i, an eigenvalue of T_{n-2}, is a repeated eigenvalue of $T_n(\hat{t})$. Cauchy's theorem then implies that β_i is also an eigenvalue of T_{n-1}, which violates Assumption A.

Since $p_n^+(\lambda, t)$ and $p_n^-(\lambda, t)$ have distinct roots for all t, (4), (5), (6), and (7) define $\mu_1(t), \ldots, \mu_r(t)$ and $\nu_1(t), \ldots \nu_s(t)$ as continuously differentiable functions on $(-\infty, \infty)$. However, (4) and (5) do not provide convenient representations for the derivatives of these functions. The next two lemmas will enable us to find such representations.

LEMMA 1. *Suppose that Assumption A holds, and let α_i $(1 \le i \le n-1)$ be an eigenvalue of T_{n-1}. Then there is exactly one value τ_i of t such that α_i is an eigenvalue of $T_n(\tau_i)$. Moreover, α_i is in fact an eigenvalue of $T_n(\tau_i)$ with multiplicity two, and $\tau_1, \ldots, \tau_{n-1}$ are the only values of t for which $T_n(t)$ has repeated eigenvalues.*

PROOF. By an argument of Iohvidov [5, p. 98], based on Sylvester's identity, it can be shown that

$$(8) \qquad p_n(\lambda, t)p_{n-2}(\lambda) = p_{n-1}^2(\lambda) - \begin{vmatrix} t_1 & t_2 & t_3 & \ldots & t_{n-2} & t \\ t_0 - \lambda & t_1 & t_2 & \ldots & t_{n-3} & t_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n-3} & t_{n-4} & t_{n-5} & \ldots & t_0 - \lambda & t_1 \end{vmatrix}^2$$

for all λ and t. Expanding the determinant on the right in cofactors of its first row shows that (8) can be rewritten as

$$(9) \qquad p_n(\lambda, t)p_{n-2}(\lambda) = p_{n-1}^2(\lambda) - [(-1)^{n+1}p_{n-2}(\lambda)t + k_{n-2}(\lambda)]^2,$$

where $k_{n-2}(\lambda)$ is independent of t. Therefore, $p_n(\alpha_i, \tau_i) = 0$ if and only if

$$\tau_i = \frac{(-1)^n k_{n-2}(\alpha_i)}{p_{n-2}(\alpha_i)}.$$

Obviously, α_i is a repeated zero of the polynomial obtained by setting $t = \tau_i$ on the right of (9), and therefore α_i is an eigenvalue of $T_n(\tau_i)$ with multiplicity $m > 1$. To see that $m = 2$, suppose to the contrary that $m > 2$. Then either $\mu_l(\tau_i) = \mu_{l+1}(\tau_i) = \alpha_i$ or $\nu_l(\tau_i) = \nu_{l+1}(\tau_i) = \alpha_i$ for some l. But then (6) and (7) imply that $\alpha_i = \beta_l$ or $\alpha_i = \gamma_l$ for some l, which contradicts Assumption A; hence, $m = 2$. To conclude the proof, we simply observe that a repeated eigenvalue of $T_n(t)$ must be an eigenvalue of T_{n-1}.

This lemma is related to Theorem 3 of Cybenko [3], who also considered questions connected with the eigenstructure of $T_n(t)$ regarded as an extension of T_{n-1}.

Now define

$$q_n(\lambda, t) = \frac{p_n(\lambda, t)}{p_{n-1}(\lambda)}.$$

The next lemma can be proved by partitioning $T_n(t) - \lambda I_n$ in the form

$$T_n(t) - \lambda I_n = \begin{bmatrix} t_0 - \lambda & U_{n-1}^T(t) \\ U_{n-1}(t) & T_{n-1} - \lambda I_{n-1} \end{bmatrix},$$

where $U_{n-1}(t)$ is defined in (11), below. (For details, see the proof of Theorem 1 of [7].)

LEMMA 2. *If λ is not an eigenvalue of T_{n-1}, let*

$$X_{n-1}(\lambda, t) = \begin{bmatrix} x_{1,n-1}(\lambda, t) \\ x_{2,n-1}(\lambda, t) \\ \vdots \\ x_{n-1,n-1}(\lambda, t) \end{bmatrix}$$

be the solution of the system

$$(T_{n-1} - \lambda I_{n-1})X_{n-1}(\lambda, t) = U_{n-1}(t), \tag{10}$$

where

$$U_{n-1}(t) = \begin{bmatrix} t_1 \\ \vdots \\ t_{n-2} \\ t \end{bmatrix}. \tag{11}$$

Then

$$q_n(\lambda, t) = t_0 - \lambda - U_{n-1}^T(t)X_{n-1}(\lambda, t);$$

moreover, if $q_n(\lambda, t) = 0$, *then the vector*

$$Y_n(\lambda, t) = \begin{bmatrix} -1 \\ X_{n-1}(\lambda, t) \end{bmatrix}$$

is a λ*–eigenvector of* $T_n(t)$*; hence*

$$x_{n-1,n-1}(\lambda, t) = (-1)^{q+1}, \tag{12}$$

where

$$q = \begin{cases} 0 \text{ if } \lambda \text{ is an even eigenvalue of } T_{n-1}(t), \\ 1 \text{ if } \lambda \text{ is an odd eigenvalue of } T_{n-1}(t). \end{cases}$$

We will call q the *parity* of the eigenvalue λ.

Now suppose that $\lambda(t)$ is one of the functions $\mu_1(t), \dots, \mu_r(t)$ or $\nu_1(t), \dots, \nu_s(t)$. Lemmas 1 and 2 imply that

$$t_0 - \lambda(t) - U_{n-1}^T(t)X_{n-1}(\lambda(t), t) = 0, \ t \in J,$$

where J is any interval which does not contain any of the exceptional points $\tau_1, \ldots, \tau_{n-1}$ defined in Lemma 1. Differentiating this yields

$$
\begin{aligned}
&\left(1 + U_{n-1}^T(t)\frac{\partial}{\partial \lambda}X_{n-1}(\lambda(t), t)\right)\lambda'(t) + \frac{\partial U_{n-1}^T(t)}{\partial t}X_{n-1}(\lambda(t), t) \\
&\qquad + U_{n-1}^T(t)\frac{\partial}{\partial t}X_{n-1}(\lambda(t), t) = 0.
\end{aligned} \tag{13}
$$

However, if λ is any number which is not an eigenvalue of T_{n-1}, then

$$U_{n-1}^T(t) = X_{n-1}^T(\lambda, t)(T_{n-1} - \lambda I_{n-1}) \tag{14}$$

(see (10)), and differentiating (10) yields

$$(T_{n-1} - \lambda I_{n-1})\frac{\partial}{\partial \lambda}X_{n-1}(\lambda, t) = X_{n-1}(\lambda, t) \tag{15}$$

and

$$(T_{n-1} - \lambda I_{n-1})\frac{\partial}{\partial t}X_{n-1}(\lambda, t) = \frac{\partial}{\partial t}U_{n-1}(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{16}$$

(see (11)). Setting $\lambda = \lambda(t)$ in (14), (15), and (16) and substituting the results into (13) yields

$$\left(1 + \|X_{n-1}(\lambda(t), t)\|^2\right)\lambda'(t) + 2x_{n-1,n-1}(\lambda(t), t) = 0$$

(Euclidean norm); therefore, from (12),

$$\lambda'(t) = \frac{(-1)^q 2}{1 + \|X_{n-1}(\lambda(t), t)\|^2}. \tag{17}$$

Because of (12) we can write

$$X_{n-1}(\lambda(t),t) = \begin{bmatrix} \hat{X}_{n-2}(\lambda(t),t) \\ (-1)^{q+1} \end{bmatrix}, \tag{18}$$

where q is the parity of $\lambda(t)$ and $\hat{X}_{n-2}(\lambda(t),t)$ is symmetric if $q = 0$ or skew–symmetric if $q = 1$; then (17) becomes

$$\lambda'(t) = \frac{(-1)^q 2}{2 + \|\hat{X}_{n-2}(\lambda(t),t)\|^2}, \tag{19}$$

which is valid for $t \neq \tau_i,\ 1 \leq i \leq n-1$. This formula does not yet apply at these exceptional points, simply because the vectors $\hat{X}_{n-2}(\lambda(\tau_i),\tau_i) = \hat{X}_{n-2}(\alpha_i,\tau_i),\ 1 \leq i \leq n-1$ are as yet undefined. This is easily remedied; by Lemma 2,

$$\big(T_n(t) - \lambda(t) I_n\big) \begin{bmatrix} -1 \\ \hat{X}_{n-2}(\lambda(t),t) \\ (-1)^{q+1} \end{bmatrix} = 0 \tag{20}$$

for all $t \neq \tau_i,\ 1 \leq i \leq n-1$. This and (1) imply that

$$\big(T_{n-2} - \lambda(t) I_{n-2}\big)\hat{X}_{n-2}(\lambda(t),t) = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_{n-2} \end{bmatrix} + (-1)^q \begin{bmatrix} t_{n-2} \\ t_{n-3} \\ \vdots \\ t_1 \end{bmatrix} \tag{21}$$

for all $t \neq \tau_i,\ 1 \leq i \leq n-1$. However, this system has a unique solution when $t = \tau_i$, since the matrix $T_{n-2} - \lambda(\tau_i) I_{n-2} = T_{n-2} - \alpha_i I_{n-2}$ is nonsingular, by Assumption A. Defining this solution to be $\hat{X}_{n-2}(\lambda(\tau_i),\tau_i)$ extends $\hat{X}_{n-2}(\lambda(t),t)$ so as to make it continuous on $(-\infty,\infty)$. Since $\lambda'(t)$ is also continuous for all t, (19) must hold for all t.

For future reference, notice from (12) and the continuity of $x_{n-1,n-1}(\lambda_i(t), t)$ that the parity $q_i(t)$ of $\lambda_i(t)$ is constant on any interval J which does not contain any of the exceptional points $\tau_1, \dots, \tau_{n-1}$.

THEOREM 1. *The even eigenvalues $\mu_1(t), \dots, \mu_r(t)$ are strictly increasing on $(-\infty, \infty)$ and the inequalities (6) can be replaced by the strict inequalities*

$$\mu_i(t) < \beta_i < \mu_{i+1}(t), -\infty < t < \infty; \ 1 \le i \le r-1; \tag{22}$$

moreover,

$$\lim_{t\to\infty} \mu_i(t) = \begin{cases} \beta_i, & 1 \le i \le r-1, \\ \infty, & i = r, \end{cases} \tag{23}$$

and

$$\lim_{t\to-\infty} \mu_i(t) = \begin{cases} \beta_{i-1}, & 2 \le i \le r, \\ -\infty, & i = 1. \end{cases} \tag{24}$$

PROOF. Setting $q = 0$ in (17) shows that $\mu_1(t), \dots \mu_r(t)$ are strictly increasing for all t; therefore, (6) implies (22). For convenience, define $\beta_r = \infty$ and suppose that

$$\lim_{t\to\infty} \mu_i(t) = \zeta_i < \beta_i \tag{25}$$

for some i in $\{1, \dots, r\}$. Since $\beta_{i-1} < \zeta_i < \beta_i$, the system

$$(T_{n-2} - \zeta_i I_{n-2})\tilde{X}_i = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_{n-2} \end{bmatrix} + (-1)^q \begin{bmatrix} t_{n-2} \\ t_{n-3} \\ \vdots \\ t_1 \end{bmatrix}$$

has a unique solution, and, from (21) with $\lambda(t) = \mu_i(t)$,

$$\lim_{t\to\infty} \hat{X}_{n-2}(\mu_i(t), t) = \tilde{X}_i.$$

Consequently, (19) implies that

$$\lim_{t\to\infty} \mu_i'(t) = \frac{2}{2 + \|\tilde{X}_i\|^2} > 0,$$

and therefore $\lim_{t\to\infty} \mu_i(t) = \infty$, which contradicts (25). This implies (23). A similar argument implies (24).

The proof of the next theorem is similar to this.

Theorem 2. *The odd eigenvalues $\nu_1(t), \dots, \nu_s(t)$ are strictly decreasing on $(-\infty, \infty)$ and the inequalities (7) can be replaced by the strict inequalities*

$$\nu_i(t) < \gamma_i < \nu_{i+1}(t), -\infty < t < \infty;\ 1 \le i \le s-1;$$

moreover,

$$\lim_{t\to\infty} \nu_i(t) = \begin{cases} \gamma_{i-1}, & 2 \le i \le s, \\ -\infty, & i = 1, \end{cases} \tag{26}$$

and

$$\lim_{t\to-\infty} \nu_i(t) = \begin{cases} \gamma_i, & 1 \le i \le s-1, \\ \infty, & i = s. \end{cases} \tag{27}$$

The remaining theorems deal with the asymptotic behavior of the vectors $\hat{X}_{n-2}(\lambda(t), t)$ (see (18)) and with the orders of convergence in (23), (24), (26), and (27).

Theorem 3. *Let*

$$A_i = \begin{bmatrix} a_1^{(i)} \\ a_2^{(i)} \\ \vdots \\ a_{n-2}^{(i)} \end{bmatrix}$$

be the β_i–eigenvector of T_{n-2} which is normalized so that

$$t_{n-2}a_1^{(i)} + t_{n-3}a_2^{(i)} + \cdots + t_1 a_{n-2}^{(i)} = 1.$$

Then

$$\lim_{t\to\infty} \frac{\hat{X}_{n-2}(\mu_i(t),t)}{t} = A_i,\ 1 \le i \le r-1, \tag{28}$$

and

$$\mu_i(t) = \beta_i - \frac{2(1+o(1))}{\|A_i\|^2 t},\ t \to \infty,\ 1 \le i \le r-1. \tag{29}$$

Also,

$$\lim_{t\to-\infty} \frac{\hat{X}_{n-2}(\mu_i(t),t)}{t} = A_{i-1},\ 2 \le i \le r, \tag{30}$$

and

$$\mu_i(t) = \beta_{i-1} - \frac{2(1+o(1))}{\|A_{i-1}\|^2 t},\ t \to -\infty,\ 2 \le i \le r. \tag{31}$$

PROOF. It is easy to verify that the vector

$$\begin{bmatrix} A_i \\ 0 \end{bmatrix}$$

is the last column of $(T_{n-1} - \beta_i I_{n-1})^{-1}$. Setting $\lambda = \mu_i(t)$ in (10) shows that

$$X_{n-1}(\mu_i(t),t) = (T_{n-1} - \mu_i(t) I_{n-1})^{-1} U_{n-1}(t)$$

for $|t|$ sufficiently large. Therefore, (11), (23), and (24) imply (28) and (30). From (19) with $\lambda(t) = \mu_i(t)$ and (28)

$$\mu_i'(t) = \frac{2(1+o(1))}{\|A_i\|^2 t^2},\ t \to \infty,\ 1 \le i \le r-1. \tag{32}$$

Similarly, (19) and (30) imply that

$$\mu_i'(t) = \frac{2(1+o(1))}{\|A_{i-1}\|^2 t^2},\ t \to -\infty,\ 2 \le i \le r. \tag{33}$$

Since (32) and (33) imply (29) and (31), the proof is complete.

A similar argument yields the following theorem.

THEOREM 4. *Let*

$$B_i = \begin{bmatrix} b_1^{(i)} \\ b_2^{(i)} \\ \vdots \\ b_{n-2}^{(i)} \end{bmatrix}$$

be the γ_i–eigenvector of T_{n-2} which is normalized so that

$$t_{n-2}b_1^{(i)} + t_{n-3}b_2^{(i)} + \cdots + t_1 b_{n-2}^{(i)} = 1.$$

Then

$$\lim_{t\to\infty} \frac{\hat{X}_{n-2}(\nu_i(t),t)}{t} = B_{i-1},\ 2 \le i \le s,$$

and

$$\nu_i(t) = \gamma_{i-1} + \frac{2(1+o(1))}{\|B_{i-1}\|^2 t},\ t \to \infty,\ 2 \le i \le s.$$

Also,

$$\lim_{t\to-\infty} \frac{\hat{X}_{n-2}(\nu_i(t),t)}{t} = B_i,\ 1 \le i \le s-1,$$

and

$$\nu_i(t) = \gamma_i + \frac{2(1+o(1))}{\|B_i\|^2 t},\ t \to -\infty,\ 1 \le i \le s-1.$$

Theorems 3 and 4 provide no information on the asymptotic behavior of the eigenvalues which tend to infinite limits as $t \to \pm\infty$. The next theorem fills this gap.

Theorem 5. *Let*

$$\Gamma_q = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_{n-2} \end{bmatrix} + (-1)^q \begin{bmatrix} t_{n-2} \\ t_{n-3} \\ \vdots \\ t_1 \end{bmatrix}. \tag{34}$$

Then

$$\mu_r(t) = t + t_0 + \frac{\|\Gamma_0\|^2}{2t} + o\Big(\frac{1}{t}\Big) \quad (t \to \infty); \tag{35}$$

$$\mu_1(t) = t + t_0 + \frac{\|\Gamma_0\|^2}{2t} + o\Big(\frac{1}{t}\Big) \quad (t \to -\infty); \tag{36}$$

$$\lim_{t\to\infty} t\hat{X}_{n-2}(\mu_r(t), t) = \lim_{t\to-\infty} t\hat{X}_{n-2}(\mu_1(t), t) = -\Gamma_0; \tag{37}$$

$$\nu_1(t) = -t + t_0 - \frac{\|\Gamma_1\|^2}{2t} + o\Big(\frac{1}{t}\Big) \quad (t \to \infty); \tag{38}$$

$$\nu_s(t) = -t + t_0 - \frac{\|\Gamma_1\|^2}{2t} + o\Big(\frac{1}{t}\Big) \quad (t \to -\infty); \tag{39}$$

and

$$\lim_{t\to\infty} t\hat{X}_{n-2}(\nu_1(t), t) = \lim_{t\to-\infty} t\hat{X}_{n-2}(\nu_s(t), t) = \Gamma_1. \tag{40}$$

Proof. We will prove (35) and (38) and verify the first limits in (37) and (40); the proof of (36) and (39) and the verification of the second limits in (37) and

(40) are similar. Let $\lambda(t) = \mu_r(t)$ and $q = 0$ or $\lambda(t) = \nu_1(t)$ and $q = 1$. We know from Theorems 1 and 2 that

$$\lim_{t\to\infty} |\lambda(t)| = \infty. \tag{41}$$

From (21) and (34),

$$\big(|\lambda(t)| - \|T_{n-2}\|\big)\|\hat{X}_{n-2}(\lambda(t),t)\| \le \|\Gamma_q\|;$$

therefore, (41) implies that

$$\lim_{t\to\infty} \|\hat{X}_{n-2}(\lambda(t),t)\| = 0.$$

From this and (19),

$$\lim_{t\to\infty} \lambda'(t) = (-1)^q,$$

and therefore

$$\lim_{t\to\infty} \frac{\lambda(t)}{t} = (-1)^q,$$

by L'Hôpital's rule. Now (21) implies that

$$\lim_{t\to\infty} t\hat{X}_{n-2}(\lambda(t),t) = (-1)^{q+1}\Gamma_q, \tag{42}$$

with Γ_q as in (34). This verifies the first limits in (37) and (40). Since the first component of the vector on the left of (20) is identically zero,

$$\lambda(t) - t_0 + [t_1, t_2, \ldots, t_{n-2}]\hat{X}_{n-2}(\lambda(t),t) + (-1)^{q+1}t = 0. \tag{43}$$

From (35) and (42),

$$[t_1, t_2, \ldots, t_{n-2}]\hat{X}_{n-2}(\lambda(t),t) = (-1)^{q+1}\frac{[t_1, t_2, \ldots, t_{n-2}]\Gamma_q}{t} + o\big(\frac{1}{t}\big)$$

$$= (-1)^{q+1}\frac{\|\Gamma_q\|^2}{2t} + o\big(\frac{1}{t}\big).$$

Substituting this into (43) and solving for $\lambda(t)$ yields

$$\lambda(t) = t_0 + (-1)^q \big[t + \frac{\|\Gamma_q\|^2}{2t} + o\big(\frac{1}{t}\big)\big],$$

which proves (35) and (38).

We conclude with a comment on the possible orderings of even and odd eigenvalues of $T_n(t)$. Let the eigenvalues of T_{n-2} be

$$\omega_1 < \omega_2 < \cdots < \omega_{n-2};$$

i. e.,

$$\{\omega_1, \ldots, \omega_{n-2}\} = \{\beta_1, \ldots, \beta_{r-1}\} \cup \{\gamma_1, \ldots, \gamma_{s-1}\}.$$

Define

$$Q_0 = [q(\omega_1), q(\omega_2), \ldots, q(\omega_{n-2})],$$

where $q(\omega_i)$ is the parity of ω_i. Suppose that the elements $\tau_1, \ldots, \tau_{n-1}$ of the exceptional set discussed in Lemma 1 are distinct, and ordered so that

$$\tau_{i_1} < \tau_{i_2} < \cdots < \tau_{i_{n-1}}. \tag{44}$$

Let

$$J_l = \begin{cases} (-\infty, \tau_{i_1}), & l = 1, \\ (\tau_{i_{l-1}}, \tau_{i_l}), & 2 \le l \le n-1, \\ (\tau_{i_l}, \infty), & l = n. \end{cases}$$

The eigenvalues of $T_n(t)$ satisfy the strict inequalities

$$\lambda_1(t) < \lambda_2(t) < \cdots < \lambda_n(t)$$

for all t in each interval $J_1, \dots, J_{n-1}$. Recalling that the parities of $\lambda_1(t), \dots, \lambda_n(t)$ are constant on each J_l, we can define the n-vectors

$$Q_l = [q_{l1}, q_{l2}, \dots, q_{ln}], \quad 1 \le l \le n,$$

where q_{lj} is the parity of $\lambda_j(t)$ on J_l. Since $\lambda_i(t) \le \alpha_i \le \lambda_{i+1}(t)$ for all t and α_i is an eigenvalue with multiplicity two of $T_n(\tau_i)$, we must have $\lambda_i(\tau_i) = \lambda_{i+1}(\tau_i) = \alpha_i$. Therefore, α_i is in both the even and odd spectrum of $T_n(\tau_i)$. From the monotonicity properties of the even and odd eigenvalues of $T_n(t)$, it follows that $\lambda_i(t)$ changes from even to odd and $\lambda_{i+1}(t)$ changes from odd to even as t increases through τ_i. This and (23), (24), (26), and (27) imply the following theorem.

THEOREM 6. *If $\tau_1, \dots, \tau_{n-1}$ satisfy (44), then*

$$Q_1 = [0, Q_0, 1], \qquad Q_n = [1, Q_0, 0],$$

and, for $1 \le l \le n-1$,

$$q_{i,l+1} = q_{i,l} \text{ if } i \ne i_l \text{ and } i \ne i_{l+1}, \tag{45}$$

and

$$q_{i_l,l} = 0, \; q_{i_l+1,l} = 1, \; q_{i_l,l+1} = 1, \; \text{and } q_{i_l+1,l+1} = 0. \tag{46}$$

From (45) and (46), Q_{l+1} is obtained by interchanging the zero and one which must be in columns i_l and i_{l+1}, respectively, of Q_l.

The assumption that $\tau_1, \dots, \tau_{n-1}$ are distinct was imposed for simplicity. Theorem 6 can easily be modified to cover the exceptional case where $\{\tau_1, \dots, \tau_{n-1}\}$ contains fewer than $n-1$ distinct elements.

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