1988

# Mixed sublinear, superlinear, and singular systems of functional differential equations

William F. Trench, *Trinity University*

Proceedings of the International Conference on Theory and Applications of Differential Equations, Ohio University, 1988), pp. 448-453

# MIXED SUBLINEAR, SUPERLINEAR, AND SINGULAR SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

William F. Trench

*Trinity University, San Antonio, TX USA*

We consider systems of the form

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}(t)(x_j(g(t)))^{\gamma_j}, \quad t > t_0, \quad 1 \le i \le n, \tag{1}$$

where $a_{ij}\colon [t_0,\infty) \to R$ and $g\colon [t_0,\infty) \to R$ are continuous, and $\gamma_1,\dots,\gamma_n$ are nonzero rational numbers with odd denominators, so that the quantity $x^{\gamma_j}$ is real-valued whenever $x$ is real. However, this restriction is for notational convenience only; with trivial modifications our results are valid for the system

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}(t)|x_j(g(t)))|^{\gamma_j} sgn(x_j(g(t)), \quad t > t_0, \quad 1 \le i \le n.$$

The asymptotic behavior of systems of functional differential equations has recently begun to receive attention (see, e.g., [1]–[9]). Here we give condi-

AMS(MOS) Subject Classifications: 34K15, 34K25.

tions which imply that (1) has solutions on the half-line $[t_0, \infty)$ that approach a given constant vector $C$ as $t \to \infty$. Since there are no assumptions on the deviating argument $g$ other than continuity, we must allow for the possibility that $g(t) < t_0$ for some $t > t_0$. For this reason we introduce the following definition.

DEFINITION. *If $-\infty < t_0 < \infty$, then $C_n(t_0)$ is the space of continuous $n$-vector functions on $(-\infty, \infty)$ which are constant on $(-\infty, t_0]$, with the topology induced by the following definition of convergence: $X_j \to X$ as $j \to \infty$ if $\|X_j(t) - X(t)\| \to 0$ uniformly as $j \to \infty$ on every half-line $(-\infty, b]$.*

We say that a function $X$ in $C_n(t_0)$ is a solution of (1) if $X$ is differentiable and satisfies (1) on $(t_0, \infty)$. We give conditions which guarantee the existence of a solution of (1) such that $\lim_{t\to\infty} x_i(t) = c_i,\ 1 \le i \le n$, for given $c_1, \dots, c_n$. For convenice, we will abbreviate (1) as $x_i'(t) = f_i(t; X),\ 1 \le i \le n$, or in system form as $X'(t) = F(t; X)$. We obtain our results by applying the Schauder–Tychonoff theorem to the transformation $Y = TX$ defined by

$$Y(t) = \begin{cases} C - \int_t^\infty F(s; X)\, ds, & t \ge t_0, \\ C - \int_{t_0}^\infty F(s; X)\, ds, & t < t_0. \end{cases} \tag{2}$$

The system (1) will be said to be *linear, superlinear, sublinear,* or *singular* with respect to $x_i$ if, respectively, $\gamma_i = 1,\ \gamma_i > 1,\ 0 < \gamma_i < 1,$ or $\gamma_i < 0$. In the following $\mathcal{A} = \{i \mid 1 \le i \le n \text{ and } \gamma_i > 0\}$, and $\mathcal{B} = \{i \mid 1 \le i \le n \text{ and } \gamma_i < 0\}$. For a given constant vector $C$, let $\mathcal{N} = \{i \mid 1 \le i \le n \text{ and } c_i \ne 0\}$ and $\mathcal{Z} = \{i \mid 1 \le i \le n \text{ and } c_i = 0\}$. Any of the sets $\mathcal{A}$ $\mathcal{B}$, $\mathcal{N}$, and $\mathcal{Z}$ may be empty.

We impose the following integrability conditions on the coefficient functions $\{a_{ij}\}$ in (1). It should be understood that this assumption applies throughout the remainder of the paper.

ASSUMPTION A. *Let $\gamma_i > 0$ if $i \in \mathcal{Z}$. Let $\varphi_1 \ldots, \varphi_n$ be positive, nonincreasing and continuous on $(-\infty, \infty)$, with $\varphi_i(t) = 1, t \leq t_0$. Suppose that the integrals $\int^\infty a_{ij}(t)dt$ $(1 \leq i, j \leq n)$ converge (perhaps conditionally) and that for $1 \leq i \leq n$ and $t \geq t_0$,*

$$\alpha_{ij}(t) = |\int_t^\infty a_{ij}(s)\ ds| = O(\varphi_i(t)), \quad j \in \mathcal{N}, \tag{3}$$

$$\beta_{ij}(t) = |\int_t^\infty |a_{ij}(s)|\varphi_j(g(s))\ ds = O(\varphi_i(t)), \quad j \in \mathcal{N}, \tag{4}$$

*and*

$$\sigma_{ij}(t) = \int_t^\infty |a_{ij}(s)|(\varphi_j(g(s)))^{\gamma_j}\ ds = O(\varphi_i(t)), \quad j \in \mathcal{Z}. \tag{5}$$

For convenience below we define

$$\overline{\alpha}_{ij} = \sup_{t \geq t_0} \alpha_{ij}(t)/\varphi_i(t),\ \ j \in \mathcal{N}, \tag{6}$$

$$\overline{\beta}_{ij} = \sup_{t \geq t_0} \beta_{ij}(t)/\varphi_i(t),\ \ j \in \mathcal{N}, \tag{7}$$

$$\overline{\sigma}_{ij} = \sup_{t \geq t_0} \sigma_{ij}(t)/\varphi_i(t),\ \ j \in \mathcal{Z}, \tag{8}$$

and

$$M_{ij} = \overline{\alpha}_{ij} + \theta(1 \pm \theta)^{\gamma_j - 1}|\gamma_j|\overline{\beta}_{ij}, \tag{9}$$

where $\theta$ is a given number in $(0,1)$ and the "$\pm$" is "$+$" if $\gamma_j \geq 1$ or "$-$" if $\gamma_j < 1$. It is also convenient here to define the functions $\lambda_i(t), 1 \leq i \leq n$, by

$$\lambda_i(t) = \sum_{j \in \mathcal{Z}} r_j^{\gamma_j} \sigma_{ij}(t) + \sum_{j \in \mathcal{N}} |c_j|^{\gamma_j} [\alpha_{ij}(t) + \theta|\gamma_j|(1 \pm \theta)^{\gamma_j - 1} \beta_{ij}(t)] \qquad (10)$$

if $t \geq t_0$ and $\lambda_i(t) = \lambda_i(t_0)$ if $t < t_0$.

THEOREM 1. *If $r_i$ $(i \in \mathcal{Z})$ and $c_i$ $(i \in \mathcal{N})$ are constants such that*

$$\sum_{j \in \mathcal{Z}} \overline{\sigma}_{ij} r_j^{\gamma_j} + \sum_{j \in \mathcal{N}} M_{ij} |c_j|^{\gamma_j} \leq \begin{cases} \theta|c_i|, & i \in \mathcal{N}, \\ r_i, & i \in \mathcal{Z}, \end{cases} \qquad (11)$$

*then (1) has a solution $\hat{X}$ such that*

$$|\hat{x}_i(t) - c_i| \leq \lambda_i(t) \leq \theta|c_i|\varphi_i(t) \ (i \in \mathcal{N}), \quad -\infty < t < \infty, \qquad (12)$$

*and*

$$|\hat{x}_i(t)| \leq \lambda_i(t) \leq r_i \varphi_i(t) \ (i \in \mathcal{Z}), \quad -\infty < t < \infty. \qquad (13)$$

PROOF. We apply the Schauder–Tychonoff theorem to show that $\hat{X} = T\hat{X}$ (cf.(2)) for some $\hat{X}$ in the closed convex subset $\mathcal{S}$ consisting of functions $X$ in $\mathcal{C}_n(t_0)$ such that

$$|x_i(t) - c_i| \leq \theta|c_i|\varphi_i(t) \ (i \in \mathcal{N}), \quad -\infty < t < \infty, \qquad (14)$$

and

$$|x_i(t)| \leq r_i \varphi_i(t) \ (i \in \mathcal{Z}), \quad -\infty < t < \infty. \qquad (15)$$

Since

$$0 < (1-\theta)|c_i| \leq |x_i(\tau)| \leq (1+\theta)|c_i| \ (i \in \mathcal{N}), \quad \infty < \tau < \infty, \qquad (16)$$

the continuity of the $\{a_{ij}\}$ implies that the functions

$$f_i(t;X) = \sum_{i=1}^{n} a_{ij}(t)(x_j(g(t)))^{\gamma_j}, \quad 1 \le i \le n, \quad X \in \mathcal{S},$$

are continuous on $[t_0, \infty)$. Moreover,

$$|\int_t^\infty f_i(s;X)\, ds| \le |\int_t^\infty f_i(s;C)\, ds| + \int_t^\infty |f_i(s;X) - f_i(s;C)|\, ds \tag{17}$$

if the integrals on the right converge, which we will now verify. From (3),

$$|\int_t^\infty f_i(s;C)\, ds| \le \sum_{j\epsilon\mathcal{N}} |c_j|^{\gamma_j} \alpha_{ij}(t). \tag{18}$$

Now consider

$$\begin{aligned} f_i(t;X) - f_i(t;C) = & \sum_{j\epsilon\mathcal{Z}} a_{ij}(t)(x_j(g(t)))^{\gamma_j} \\ & + \sum_{j\epsilon\mathcal{N}} a_{ij}(t)[(x_j(g(t)))^{\gamma_j} - c_j^{\gamma_j}]. \end{aligned} \tag{19}$$

From the mean value theorem, $|x^\gamma - c^\gamma| \le |\gamma||\hat{x}|^{\gamma_j - 1}|x - c|$ with $\hat{x}$ between $x$ and $c$, provided that $x$ and $c$ $(\neq 0)$ have the same sign. Therefore, from (16) with $\tau$ replaced by $g(t)$,

$$|(x_j(g(t)))^{\gamma_j} - c_j^{\gamma_j}| \le |\gamma_j||\hat{x}_j|^{\gamma_j - 1}|(x_j(g(t)))^{\gamma_j} - c_j^{\gamma_j}|, \quad j \in \mathcal{N}, \tag{20}$$

where

$$(1-\theta)|c_j| < |\hat{x}_j| < (1+\theta)|c_j|, \quad j \in \mathcal{N}. \tag{21}$$

Now (14), (20), and (21) imply that

$$|(x_j(g(t)))^{\gamma_j} - c_j^{\gamma_j}| \leq \theta|\gamma_j|(1 \pm \theta)^{\gamma_j - 1}|c_j|^{\gamma_j}\varphi_j(g(t))\ (j \in \mathcal{N})\ ,\ X \in \mathcal{S},$$

where the "$\pm$" is "$+$" if $\gamma_j \geq 1$, "$-$" if $\gamma_j < 1$. Hence, (4), (5), (15), and (19) imply that

$$\int_t^\infty |f_i(s;X) - f_i(s;C)|ds \leq \sum_{j \epsilon \mathcal{Z}} r_j^{\gamma_j}\sigma_{ij}(t) + \theta \sum_{j \in \mathcal{N}} |\gamma_j|(1 \pm \theta)^{\gamma_j - 1}|c_j|^{\gamma_j}\beta_{ij}(t),$$

which, together with (17) and (18) yields the inequalities

$$|\int_t^\infty f_i(s;X)ds| \leq \lambda_i(t), \quad 1 \leq i \leq n,$$

with $\lambda_i$ as defined in (10). Therefore, (6), (7), (8), (9), and (11) imply that if $Y = TX$, then

$|y_i(t) - c_i| \leq \lambda_i(t) \leq \theta|c_i|\varphi_i(t), (i \in \mathcal{N})$ and $|y_i(t)| \leq \lambda_i(t) \leq r_i\varphi_i(t), (i \in \mathcal{Z})$,

for all $t$. Hence, $T(\mathcal{S}) \subset \mathcal{S}$. Since it is routine to verify that $T$ is continuous and $T(\mathcal{S})$ has compact closure, the Schauder–Tychonoff theorem now implies

that $T\hat{X} = \hat{X}$ for some $\hat{X}$ in $\mathcal{S}$ with components which satisfy (12) and (13). This completes the proof.

Now let $\mathcal{A}_0 = \{j \in \mathcal{N} \mid \gamma_j > 0\}$ and recall that $\mathcal{B} = \{j \mid 1 \le j \le n$ and $\gamma_j < 0\} \subset \mathcal{N}$.

COROLLARY 1. *Suppose that $r_i$ $(i \in \mathcal{Z})$ and $c_i$ $(i \in \mathcal{A}_0)$ are such that*

$$\sum_{j\in\mathcal{Z}} \bar{\sigma}_{ij} r_j^{\gamma_j} + \sum_{j\in\mathcal{A}_0} M_{ij}|c_j|^{\gamma_j} < \begin{cases} \theta|c_i|, & i \in \mathcal{A}_0 \\ r_i, & i \in \mathcal{Z}. \end{cases} \tag{22}$$

*Then the conclusions of Theorem 1 hold if $|c_i|$ is sufficiently large for $i \in \mathcal{B}$.*

PROOF. Clearly (22) implies (11) if $|c_i|$ $(i \in \mathcal{B})$ are sufficiently large.

COROLLARY 2. *The conclusions of Theorem 1 hold if either:*

*(i) $\gamma_i > 1$ for all $i$ in $\mathcal{A}$ (i.e., the nonsingular part of (1) is purely superlinear), provided that $r_i$ is sufficiently small for $i$ in $\mathcal{Z}$, $|c_i|$ is sufficiently small for $i$ in $\mathcal{A}_0$, and $|c_i|$ is sufficiently large for $i$ in $\mathcal{B}$.*

*(ii) $\gamma_i < 1$ for all $i$ in $\mathcal{A}$ (i.e., the nonsingular part of (1) is purely sublinear), and the constants $|c_i|$ and $r_i$ are all sufficiently large.*

## REFERENCES


1. I. Foltynska and J. Werbowski, *On the oscillatory behavior of solutions of systems of differential equations with deviating arguments,* Colloq. Math. Soc. János Bolyai 30: *Qualitative Theory of Differential Equations* (M. Farkas, Ed.), Vol. 1, 243–256, Szeged, 1981.

2. I. Györi, *On existence of the limits of solutions of functional differential equations,* Colloquia Mathematica Societatis Janos Bolyai 30. Qualitative Theory of Differential Equations, Szeged (Hungary), 1979. (North Holland, Amsterdam-Oxford-New York, 1980, 297–315.)

3. Y. Kitamura and T. Kusano, *Oscillation and a class of nonlinear differential systems with general deviating arguments,* Nonlinear Anal. 2 (1978), 537–551.

4. P. Marušiak, *Oscillatory properties of solutions of nonlinear differential systems with deviating arguments,* Czechoslovak Math. J. 36 (1986), 223–231.

5. V. Šeda, *On nonlinear differential systems with deviating arguments,* Czechoslovak Math. J. 36 (1986), 450–466.

6. V.N. Shevelo, *Oscillation of solutions of differential equations with deviating argument,* Naukova Dumka, Kiev, 1978. (Russian)

7. V.N. Shevelo, N.V. Varch and A.G. Gritsai, *Oscillatory properties of solutions of systems of differential equations with deviating argument,* Preprint 85.10, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev, 1985. (Russian)

8. W.F. Trench, *Asymptotics of differential systems with deviating arguments,* Proc. Amer. Math. Soc. 92 (1984), 219–224.

9. W.F. Trench and T. Kusano, *Systems of functional differential equations with asymptotically constant solutions,* Proc. Amer. Math. Soc.

(to appear).