Mixed sublinear, superlinear, and singular systems of functional differential equations

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We consider systems of the form

\[ x'_i(t) = \sum_{j=1}^{n} a_{ij}(t)(x_j(g(t)))^{\gamma_j}, \quad t > t_0, \quad 1 \leq i \leq n, \quad (1) \]

where \( a_{ij} : [t_0, \infty) \to \mathbb{R} \) and \( g : [t_0, \infty) \to \mathbb{R} \) are continuous, and \( \gamma_1, \ldots, \gamma_n \) are nonzero rational numbers with odd denominators, so that the quantity \( x^{\gamma_j} \) is real-valued whenever \( x \) is real. However, this restriction is for notational convenience only; with trivial modifications our results are valid for the system

\[ x'_i(t) = \sum_{j=1}^{n} a_{ij}(t)|x_j(g(t))|^{\gamma_j} \text{sgn}(x_j(g(t))), \quad t > t_0, \quad 1 \leq i \leq n. \]

The asymptotic behavior of systems of functional differential equations has recently begun to receive attention (see, e.g., [1]–[9]). Here we give condi-
tions which imply that (1) has solutions on the half-line \([t_0, \infty)\) that approach a given constant vector \(C\) as \(t \to \infty\). Since there are no assumptions on the deviating argument \(g\) other than continuity, we must allow for the possibility that \(g(t) < t_0\) for some \(t > t_0\). For this reason we introduce the following definition.

**Definition.** If \(-\infty < t_0 < \infty\), then \(C_{n}(t_0)\) is the space of continuous \(n\)-vector functions on \((-\infty, \infty)\) which are constant on \((-\infty, t_0]\), with the topology induced by the following definition of convergence: \(X_j \to X\) as \(j \to \infty\) if \(\|X_j(t) - X(t)\| \to 0\) uniformly as \(j \to \infty\) on every half-line \((-\infty, b]\).

We say that a function \(X\) in \(C_{n}(t_0)\) is a solution of (1) if \(X\) is differentiable and satisfies (1) on \((t_0, \infty)\). We give conditions which guarantee the existence of a solution of (1) such that \(\lim_{t \to \infty} x_i(t) = c_i\), \(1 \leq i \leq n\), for given \(c_1, \ldots, c_n\). For convenice, we will abbreviate (1) as \(x_i'(t) = f_i(t; X)\), \(1 \leq i \leq n\), or in system form as \(X'(t) = F(t; X)\). We obtain our results by applying the Schauder–Tychonoff theorem to the transformation \(Y = TX\) defined by

\[
Y(t) = \begin{cases} 
C - \int_t^\infty F(s; X) \, ds, & t \geq t_0, \\
C - \int_{t_0}^\infty F(s; X) \, ds, & t < t_0.
\end{cases}
\] (2)

The system (1) will be said to be **linear**, **superlinear**, **sublinear**, or **singular** with respect to \(x_i\) if, respectively, \(\gamma_i = 1\), \(\gamma_i > 1\), \(0 < \gamma_i < 1\), or \(\gamma_i < 0\). In the following \(\mathcal{A} = \{i \mid 1 \leq i \leq n \text{ and } \gamma_i > 0\}\), and \(\mathcal{B} = \{i \mid 1 \leq i \leq n \text{ and } \gamma_i < 0\}\). For a given constant vector \(C\), let \(\mathcal{N} = \{i \mid 1 \leq i \leq n \text{ and } c_i \neq 0\}\) and \(\mathcal{Z} = \{i \mid 1 \leq i \leq n \text{ and } c_i = 0\}\). Any of the sets \(\mathcal{A}, \mathcal{B}, \mathcal{N},\) and \(\mathcal{Z}\) may be empty.
We impose the following integrability conditions on the coefficient functions \{a_{ij}\} in (1). It should be understood that this assumption applies throughout the remainder of the paper.

**Assumption A.** Let \(\gamma_i > 0\) if \(i \in \mathbb{Z}\). Let \(\varphi_1 \ldots, \varphi_n\) be positive, nonincreasing and continuous on \((-\infty, \infty)\), with \(\varphi_i(t) = 1, t \leq t_0\). Suppose that the integrals \(\int_{t}^{\infty} a_{ij}(t) dt \ (1 \leq i, j \leq n)\) converge (perhaps conditionally) and that for \(1 \leq i \leq n\) and \(t \geq t_0\),

\[
\alpha_{ij}(t) = |\int_{t}^{\infty} a_{ij}(s) \, ds| = O(\varphi_i(t)), \quad j \in \mathcal{N},
\]

\[
\beta_{ij}(t) = |\int_{t}^{\infty} |a_{ij}(s)| \varphi_j(g(s)) \, ds = O(\varphi_i(t)), \quad j \in \mathcal{N},
\]

and

\[
\sigma_{ij}(t) = \int_{t}^{\infty} |a_{ij}(s)| (\varphi_j(g(s)))^{\gamma_j} \, ds = O(\varphi_i(t)), \quad j \in \mathbb{Z}.
\]

For convenience below we define

\[
\overline{\alpha}_{ij} = \sup_{t \geq t_0} \alpha_{ij}(t) / \varphi_i(t), \quad j \in \mathcal{N},
\]

\[
\overline{\beta}_{ij} = \sup_{t \geq t_0} \beta_{ij}(t) / \varphi_i(t), \quad j \in \mathcal{N},
\]

\[
\overline{\sigma}_{ij} = \sup_{t \geq t_0} \sigma_{ij}(t) / \varphi_i(t), \quad j \in \mathbb{Z},
\]

and

\[
M_{ij} = \overline{\alpha}_{ij} + \theta(1 \pm \theta)^{\gamma_j^{-1}} |\gamma_j| \overline{\beta}_{ij},
\]
where $\theta$ is a given number in $(0, 1)$ and the “±” is “+” if $\gamma_j \geq 1$ or “−” if $\gamma_j < 1$. It is also convenient here to define the functions $\lambda_i(t), 1 \leq i \leq n$, by

$$
\lambda_i(t) = \sum_{j \in Z} r_j^\gamma_i \sigma_{ij}(t) + \sum_{j \in N} |c_j|^\gamma_i [\alpha_{ij}(t) + \theta |\gamma_j|(1 \pm \theta)^{\gamma_j - 1} \beta_{ij}(t)]$$

(10)

if $t \geq t_0$ and $\lambda_i(t) = \lambda_i(t_0)$ if $t < t_0$.

**Theorem 1.** If $r_i$ ($i \in Z$) and $c_i$ ($i \in N$) are constants such that

$$
\sum_{j \in Z} \sigma_{ij} r_j^\gamma_i + \sum_{j \in N} M_{ij} |c_j|^\gamma_j \leq \left\{ \begin{array}{ll}
\theta |c_i|, & i \in N, \\
r_i, & i \in Z,
\end{array} \right.
$$

(11)

then (1) has a solution $\hat{X}$ such that

$$
|\hat{x}_i(t) - c_i| \leq \lambda_i(t) \leq \theta |c_i| \varphi_i(t) \ (i \in N), \quad -\infty < t < \infty,
$$

(12)

and

$$
|\hat{x}_i(t)| \leq \lambda_i(t) \leq r_i \varphi_i(t) \ (i \in Z), \quad -\infty < t < \infty.
$$

(13)

**Proof.** We apply the Schauder–Tychonoff theorem to show that $\hat{X} = T \hat{X}$ (cf.(2))) for some $\hat{X}$ in the closed convex subset $S$ consisting of functions $X$ in $C_n(t_0)$ such that

$$
|x_i(t) - c_i| \leq \theta |c_i| \varphi_i(t) \ (i \in N), \quad -\infty < t < \infty,
$$

(14)

and

$$
|x_i(t)| \leq r_i \varphi_i(t) \ (i \in Z), \quad -\infty < t < \infty.
$$

(15)

Since

$$
0 < (1 - \theta)|c_i| \leq |x_i(\tau)| \leq (1 + \theta)|c_i| \ (i \in N), \quad \infty < \tau < \infty,
$$

(16)
the continuity of the $\{a_{ij}\}$ implies that the functions

$$f_i(t; X) = \sum_{i=1}^{n} a_{ij}(t)(x_j(g(t)))^{\gamma_j}, \quad 1 \leq i \leq n, \quad X \in S,$$

are continuous on $[t_0, \infty)$. Moreover,

$$|\int_t^{\infty} f_i(s; X) \, ds| \leq |\int_t^{\infty} f_i(s; C) \, ds| + \int_t^{\infty} |f_i(s; X) - f_i(s; C)| \, ds \quad (17)$$

if the integrals on the right converge, which we will now verify. From (3),

$$|\int_t^{\infty} f_i(s; C) \, ds| \leq \sum_{j \in \mathcal{N}} |c_j|^{\gamma_j} \alpha_{ij}(t). \quad (18)$$

Now consider

$$f_i(t; X) - f_i(t; C) = \sum_{j \in \mathcal{J}} a_{ij}(t)(x_j(g(t)))^{\gamma_j}$$

$$+ \sum_{j \in \mathcal{N}} a_{ij}(t)[(x_j(g(t)))^{\gamma_j} - c_j^{\gamma_j}]. \quad (19)$$

From the mean value theorem, $|x^{\gamma} - c^{\gamma}| \leq |\gamma| |\hat{x}|^{\gamma-1} |x - c|$ with $\hat{x}$ between $x$ and $c$, provided that $x$ and $c$ ($\neq 0$) have the same sign. Therefore, from (16) with $\tau$ replaced by $g(t)$,

$$|(x_j(g(t)))^{\gamma_j} - c_j^{\gamma_j}| \leq |\gamma_j| |\hat{x}_j|^{\gamma_j-1} |(x_j(g(t)))^{\gamma_j} - c_j^{\gamma_j}|, \quad j \in \mathcal{N}, \quad (20)$$
where

\[(1 - \theta)|c_j| < |\hat{x}_j| < (1 + \theta)|c_j|, \quad j \in \mathcal{N}. \quad (21)\]

Now (14), (20), and (21) imply that

\[|(x_j(g(t)))^{\gamma_j} - c_j^{\gamma_j}| \leq \theta|\gamma_j|(1 \pm \theta)^{\gamma_j-1}|c_j|^{\gamma_j} \varphi_j(g(t)) \quad (j \in \mathcal{N}), \quad X \in \mathcal{S},\]

where the “±” is “+” if \(\gamma_j \geq 1\), “−” if \(\gamma_j < 1\). Hence, (4), (5), (15), and (19) imply that

\[\int_{t}^{\infty} |f_i(s; X) - f_i(s; C)|ds \leq \sum_{j \in \mathcal{Z}} r_j^{\gamma_j} \sigma_{ij}(t) + \theta \sum_{j \in \mathcal{N}} |\gamma_j|(1 \pm \theta)^{\gamma_j-1}|c_j|^{\gamma_j} \beta_{ij}(t),\]

which, together with (17) and (18) yields the inequalities

\[|\int_{t}^{\infty} f_i(s; X)ds | \leq \lambda_i(t), \quad 1 \leq i \leq n,\]

with \(\lambda_i\) as defined in (10). Therefore, (6), (7), (8), (9), and (11) imply that if \(Y = TX\), then

\[|y_i(t) - c_i| \leq \lambda_i(t) \leq \theta|\varphi_i(t)|, \quad (i \in \mathcal{N}) \quad \text{and} \quad |y_i(t)| \leq \lambda_i(t) \leq r_i \varphi_i(t), \quad (i \in \mathcal{Z}),\]

for all \(t\). Hence, \(T(S) \subset \mathcal{S}\). Since it is routine to verify that \(T\) is continuous and \(T(S)\) has compact closure, the Schauder–Tychonoff theorem now implies
that \( T\dot{X} = \dot{X} \) for some \( \dot{X} \) in \( \mathcal{S} \) with components which satisfy (12) and (13). This completes the proof.

Now let \( A_0 = \{ j \in \mathcal{N} \mid \gamma_j > 0 \} \) and recall that \( B = \{ j \mid 1 \leq j \leq n \text{ and } \gamma_j < 0 \} \subset \mathcal{N} \).

**Corollary 1.** Suppose that \( r_i \) \((i \in \mathcal{Z})\) and \( c_i \) \((i \in A_0)\) are such that

\[
\sum_{j \in \mathcal{Z}} \bar{\sigma}_{ij} r_j^\gamma_j + \sum_{j \in A_0} M_{ij} |c_j|^\gamma_j < \begin{cases} \theta |c_i|, & i \in A_0 \\ r_i, & i \in \mathcal{Z}. \end{cases} \tag{22}
\]

Then the conclusions of Theorem 1 hold if \(|c_i|\) is sufficiently large for \( i \in B \).

**Proof.** Clearly (22) implies (11) if \(|c_i|\) \((i \in B)\) are sufficiently large.

**Corollary 2.** The conclusions of Theorem 1 hold if either:

(i) \( \gamma_i > 1 \) for all \( i \in A \) (i.e., the nonsingular part of (1) is purely superlinear), provided that \( r_i \) is sufficiently small for \( i \in \mathcal{Z} \), \(|c_i|\) is sufficiently small for \( i \in A_0 \), and \(|c_i|\) is sufficiently large for \( i \in B \).

(ii) \( \gamma_i < 1 \) for all \( i \in A \) (i.e., the nonsingular part of (1) is purely sublinear), and the constants \(|c_i|\) and \( r_i \) are all sufficiently large.

**REFERENCES**


(to appear).