

**Trinity University**

---

**From the Selected Works of William F. Trench**

---

March, 1988

# Asymptotic integration of a perturbed constant coefficient differential equation under mild integral smallness conditions

William F. Trench, *Trinity University*



Available at: [https://works.bepress.com/william\\_trench/61/](https://works.bepress.com/william_trench/61/)

# ASYMPTOTIC INTEGRATION OF A PERTURBED CONSTANT COEFFICIENT DIFFERENTIAL EQUATION UNDER MILD INTEGRAL SMALLNESS CONDITIONS\*

WILLIAM F. TRENCH†

**Abstract.** The problem of asymptotic behavior of solutions of an  $n$ th order linear differential equation is reconsidered, and a result obtained by Hartman and Wintner under integral smallness conditions on the perturbing terms which require absolute integrability is shown to hold under weaker integrability conditions requiring only ordinary (perhaps conditional) convergence of some of the improper integrals that occur. The estimates of the asymptotic behavior of solutions of the perturbed equation are also sharper than in the classical result.

**Key words.** linear perturbations, asymptotic behavior, integral smallness conditions

**AMS(MOS) subject classifications.** 34C10, 34D10, 34E10

**1. Introduction.** We consider the scalar equation

$$(1.1) \quad x^{(n)} + a_1 x^{(n-1)} + \cdots + a_k x^{(n-k)} + P_1(t)x^{(n-1)} + \cdots + P_n(t)x = f(t), \quad t > 0,$$

where  $a_1, \dots, a_k$  are complex constants, with

$$(1.2) \quad 1 \leq k \leq n-1, \quad a_k \neq 0.$$

It is assumed through that  $P_1, \dots, P_n$ , and  $f$  are complex-valued and continuous on  $(0, \infty)$ . We give conditions on them which imply that (1.1) has a solution which behaves asymptotically like a given polynomial of degree  $< n-k$ .

The following theorem of Hartman and Wintner [1, Thm. 17.3, p. 316] addresses this question. (We use “ $o$ ” and “ $O$ ” in the standard way to denote behavior as  $t \rightarrow \infty$ .)

**THEOREM 1.** *Suppose that the polynomial*

$$Q(\lambda) = \lambda^k + a_1 \lambda^{k-1} + \cdots + a_k$$

*has no purely imaginary zeros, and that*

$$(1.3) \quad \int_0^\infty |P_j(t)| t^q dt < \infty, \quad 1 \leq j \leq k+1,$$

*and*

$$(1.4) \quad \int_0^\infty |P_j(t)| t^{j-k-1+q} dt < \infty, \quad k+2 \leq j \leq n,$$

*for some  $q \geq 0$ . Then, for each  $l = 0, 1, \dots, n-k-1$ , the equation*

$$(1.5) \quad x^{(n)} + a_1 x^{(n-1)} + \cdots + a_k x^{(n-k)} + P_1(t)x^{(n-1)} + \cdots + P_n(t)x = 0, \quad t > 0,$$

*has a solution  $x_l$  such that*

$$\left( x_l(t) - \frac{t^l}{l!} \right)^{(r)} = \begin{cases} o(t^{l-r-q}), & 0 \leq r \leq n-k-1, \\ o(t^{-n+l+k+1-q}), & n-k \leq r \leq n-1. \end{cases}$$

Prevatt [3] has obtained the conclusion of Theorem 1 under weaker integrability conditions on  $|P_1|, \dots, |P_n|$ , and Hartman [2] has recently extended Prevatt's results

\* Received by the editors March 31, 1986; accepted for publication (in revised form) December 22, 1986.

† Department of Mathematics, Trinity University, San Antonio, Texas 78284.

to the case where  $Q(\lambda)$  may have purely imaginary zeros. Here we retain the assumption that  $Q(\lambda)$  has no purely imaginary zeros, and we obtain results which imply the conclusion of Theorem 1 under integral smallness conditions on  $P_1, \dots, P_n$  and  $f$  which allow conditional convergence of some (in some cases all) of the improper integrals involved. We also give more precise estimates of the asymptotic behavior of the desired solutions.

The results obtained here are analogous to those obtained in [8] for the equation

$$(1.6) \quad x^{(n)} + P_1(t)x^{(n-1)} + \dots + P_n(t)x = f(t)$$

(see also [7]); however, the condition (1.2) necessitates a distinctly nontrivial extension of the methods used in [7] and [8]. (For example, see Lemma 1 and its proof.)

In work related to the present paper in the sense that the integrability conditions on  $P_1, \dots, P_n$ , and  $f$  are stated in terms of possibly conditional convergence, Šimša [4]–[6], has studied (1.1) with  $k = n$ , regarding it as a perturbation of the constant coefficient equation

$$x^{(n)} + a_1 x^{(n-1)} + \dots + a_n x = 0.$$

The author [9] has also considered this problem.

**2. The main theorem.** It is to be understood below that improper integrals appearing in hypotheses are assumed to converge, and that the convergence may be conditional unless, of course, the integrand is necessarily nonnegative.

It is convenient to collect some technical definitions in the following standing assumption, which holds throughout the paper.

*Assumption A.* Let

$$(2.1) \quad Q(\lambda) = (\lambda - \lambda_1)^{d_1} \dots (\lambda - \lambda_L)^{d_L}$$

where  $\lambda_l = \mu_l + i\nu_l$  are distinct,  $\mu_l \neq 0$  ( $1 \leq l \leq L$ ), and  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_L$ . Let  $N$  be the unique integer in  $\{1, \dots, L+1\}$  such that

$$(2.2) \quad \mu_l < 0 \quad \text{if } 1 \leq l \leq N-1$$

and  $\mu_l > 0$  if  $N \leq l \leq L$ . Suppose that  $p$  is a given polynomial of degree  $< n - k$ , and define

$$(2.3) \quad g = f - \sum_{j=k+1}^n P_j p^{(n-j)}.$$

Let  $m$  be an integer in  $\{0, \dots, n - k - 1\}$ . Let  $\phi$  be continuous, positive, and nonincreasing on  $[a, \infty)$  for some  $a > 0$ . If  $m \neq 0$ , suppose that  $t^\gamma \phi(t)$  is nondecreasing for some  $\gamma < 1$ . If  $N \geq 2$  (so that (2.2) is nonvacuous), let there be a number  $\alpha$  such that  $0 < \alpha < -\mu_{N-1}$  and  $e^{\alpha t} t^{-n+m+k+1} \phi(t)$  is nondecreasing on  $[a, \infty)$ .

The following is our main theorem.

**THEOREM 2.** Suppose that Assumption A holds, that

$$(2.4) \quad \int_t^\infty s^{n-m-k-1} g(s) ds = O(\phi(t)),$$

and that

$$(2.5) \quad I_j(t) = \int_t^\infty |P_j(s)| \phi(s) ds = o(\phi(t)), \quad 1 \leq j \leq k+1.$$

Suppose further that

$$(2.6) \quad F_j(t) = \int_t^\infty P_j(s) ds = o(t^{-j+k+1}), \quad k+2 \leq j \leq n,$$

and that

$$(2.7) \quad I_j(t) = \int_t^\infty |F_j(s)| s^{j-k-2} \phi(s) ds = o(\phi(t)), \quad k+2 \leq j \leq n.$$

Then (1.1) has a solution  $\hat{x}$  such that

$$(2.8) \quad \hat{x}^{(r)}(t) - p^{(r)}(t) = \begin{cases} O(\phi(t)t^{m-r}), & 0 \leq r \leq n-k-1, \\ O(\phi(t)t^{-n+m+k+1}), & n-k \leq r \leq n-1. \end{cases}$$

Moreover, if (2.4) can be replaced by

$$(2.9) \quad \int_t^\infty s^{n-m-k-1} g(s) ds = o(\phi(t)),$$

then (2.8) can be replaced by

$$(2.10) \quad \hat{x}^{(r)}(t) - p^{(r)}(t) = \begin{cases} o(\phi(t)t^{m-r}), & 0 \leq r \leq n-k-1, \\ o(\phi(t)t^{-n+m+k+1}), & n-k \leq r \leq n-1. \end{cases}$$

The proof uses the Banach contraction principle. It is convenient to introduce the new dependent variable  $h = x - p$ , which transforms (1.1) into

$$(2.11) \quad Q(D)h^{(n-k)} = g - Mh,$$

with  $g$  as in (2.3) and

$$(2.12) \quad Mh = \sum_{j=1}^n P_j h^{(n-j)}.$$

We will construct a transformation  $\mathcal{T}$  which, for  $t_0$  sufficiently large, is a contraction of the Banach space  $B(t_0)$  consisting of functions  $h$  in  $C^{(n-1)}[t_0, \infty)$  such that

$$(2.13) \quad h^{(r)}(t) = \begin{cases} O(\phi(t)t^{m-r}), & 0 \leq r \leq n-k-1, \\ O(\phi(t)t^{-n+m+k+1}), & n-k \leq r \leq n-1, \end{cases}$$

with norm

$$(2.14) \quad \|h\| = \sup_{t \geq t_0} (\phi(t))^{-1} \left[ \sum_{r=0}^{n-k-1} t^{r-m} |h^{(r)}(t)| + t^{n-m-k-1} \sum_{r=n-k}^{n-1} |h^{(r)}(t)| \right].$$

For reference below, we define

$$(2.15) \quad B_0(t_0) = \{h \in B(t_0) \mid (2.13) \text{ holds with "O" replaced by "o"}\}.$$

If  $\hat{h}$  is a solution of (2.11) which is in  $B(t_0)$ , then the function

$$(2.16) \quad \hat{x} = p + \hat{h}$$

is a solution of (1.1) on  $[t_0, \infty)$  (which can be continued over  $(0, \infty)$ ), and  $\hat{x}$  has the desired asymptotic behavior (2.8); moreover, if  $\hat{h} \in B_0(t_0)$ , then  $\hat{x}$  satisfies (2.10). Therefore, we wish to construct  $\mathcal{T}$  so that if  $\mathcal{T}\hat{h} = \hat{h}$ , then  $\hat{h}$  satisfies (2.11).

To this end, let  $A_1, \dots, A_L$  be the uniquely defined polynomials such that  $\deg A_l < d_l$  (see (2.1)) and

$$(2.17) \quad \sum_{l=1}^L [A_l(t) e^{\lambda_l t}]^{(r)} \Big|_{t=0} = \delta_{r, k-1}, \quad 0 \leq r \leq k-1.$$

If  $v \in C[t_0, \infty)$ , define  $\mathcal{L}_1 v$  formally by

$$(\mathcal{L}_1 v)(t) = \sum_{l=1}^{N-1} \int_{t_0}^t A_l(t-\tau) e^{\lambda_l(t-\tau)} v(\tau) d\tau - \sum_{l=N}^L \int_t^\infty A_l(t-\tau) e^{\lambda_l(t-\tau)} v(\tau) d\tau.$$

Then formal differentiation and (2.17) imply that

$$(2.18) \quad \begin{aligned} (\mathcal{L}_1 v)^{(r)}(t) &= \sum_{l=1}^{N-1} \int_{t_0}^t A_{lr}(t-\tau) e^{\lambda_l(t-\tau)} v(\tau) d\tau \\ &\quad - \sum_{l=N}^L \int_t^\infty A_{lr}(t-\tau) e^{\lambda_l(t-\tau)} v(\tau) d\tau, \quad 0 \leq r \leq k-1, \end{aligned}$$

and that

$$(2.19) \quad Q(D) \mathcal{L}_1 v = v,$$

where  $A_{lr}$  is the polynomial defined by

$$A_{lr}(t) = e^{-\lambda_l t} [A_l(t) e^{\lambda_l t}]^{(r)}, \quad 0 \leq r \leq k-1.$$

If  $w \in C[t_0, \infty)$ , define  $\mathcal{L}_2 w$  formally by

$$(2.20) \quad (\mathcal{L}_2 w)(t) = \int_t^\infty \frac{(t-s)^{n-k-1}}{(n-k-1)!} w(s) ds \quad \text{if } m=0$$

or by

$$(2.21) \quad (\mathcal{L}_2 w)(t) = \int_{t_0}^t \frac{(t-\lambda)^{m-1}}{(m-1)!} d\lambda \int_\lambda^\infty \frac{(\lambda-s)^{n-k-m-1}}{(n-k-m-1)!} w(s) ds \quad \text{if } 1 \leq m \leq n-k-1.$$

In either case,

$$(2.22) \quad (\mathcal{L}_2 w)^{(n-k)} = -w.$$

Now define

$$(2.23) \quad G = \mathcal{L}_2(\mathcal{L}_1 g) \quad (\text{see (2.3)}),$$

$$(2.24) \quad \mathcal{L}h = \mathcal{L}_2(\mathcal{L}_1(Mh)) \quad (\text{see (2.12)}),$$

and

$$(2.25) \quad \mathcal{T}h = -G + \mathcal{L}h.$$

Formal manipulations using (2.19) and (2.22) show that

$$Q(D)(\mathcal{T}h)^{(n-k)} = g - Mh;$$

therefore, a fixed point (function)  $\hat{h}$  of  $\mathcal{T}$  satisfies (2.11). We will show that  $\mathcal{T}$  is a contraction of  $B(t_0)$ , and therefore has a fixed point in  $B(t_0)$ , provided that  $t_0$  is sufficiently large. It is convenient to present the lengthy proof of this assertion in a series of lemmas.

LEMMA 1. Suppose that  $v$  is complex-valued and continuous on  $[t_0, \infty)$  with  $t_0 \geq a > 0$ , and that  $\int_0^\infty t^q v(t) dt$  converges for some nonnegative integer  $q$ . Let

$$(2.26) \quad \psi(t) = \sup_{\tau \geq t} \left| \int_\tau^\infty s^q v(s) ds \right|.$$

Let  $\lambda = \mu + i\nu$  be a complex number and  $X$  be a polynomial. Then

(i) If  $\mu > 0$ , the functions

$$(2.27) \quad f_1(t) = \int_t^\infty X(t-\tau) e^{\lambda(t-\tau)} v(\tau) d\tau$$

and

$$(2.28) \quad f_2(t) = \int_t^\infty s^q ds \int_s^\infty X(s-\tau) e^{\lambda(s-\tau)} v(\tau) d\tau = \int_t^\infty s^q f_1(s) ds$$

are defined on  $[t_0, \infty)$  and satisfy the inequalities

$$(2.29) \quad |f_1(t)| \leq K_1 t^{-q} \psi(t), \quad t \geq t_0,$$

$$(2.30) \quad |f_2(t)| \leq K_2 \psi(t), \quad t \geq t_0,$$

where  $K_1$  and  $K_2$  are constants which depend only on  $\lambda$  and  $X$ .

(ii) If  $\mu < 0$ , suppose that  $\psi(t) = 0(\phi(t))$ , where  $\phi$  is nonincreasing and continuous on  $[a, \infty)$ , and  $e^{\alpha t} t^{-q} \phi(t)$  is nondecreasing on  $[a, \infty)$  for some  $\alpha$  such that

$$(2.31) \quad 0 < \alpha < -\mu.$$

Let

$$(2.32) \quad \psi_1(t) = \sup_{\tau \geq t} \frac{\psi(\tau)}{\phi(\tau)},$$

and define

$$(2.33) \quad f_3(t) = \int_{t_0}^t X(t-\tau) e^{\lambda(t-\tau)} v(\tau) d\tau.$$

Then

$$(2.34) \quad |f_3(t)| \leq K_3 \psi_1(t_0) t^{-q} \phi(t), \quad t \geq t_0,$$

where  $K_3$  is a constant depending only on  $X$  and  $\lambda$ , and the function

$$(2.35) \quad f_4(t) = \int_t^\infty s^q ds \int_{t_0}^s X(s-\tau) e^{\lambda(s-\tau)} v(\tau) d\tau = \int_t^\infty s^q f_3(s) ds$$

is defined on  $[t_0, \infty)$ , and it satisfies the inequality

$$(2.36) \quad |f_4(t)| \leq K_4 \psi_1(t_0) \phi(t), \quad t \geq t_0,$$

where  $K_4$  is a constant depending only on  $X$  and  $\lambda$ .

Finally, if

$$(2.37) \quad \psi(t) = o(\phi(t)),$$

then

$$(2.38) \quad f_3(t) = o(t^{-q} \phi(t))$$

and

$$(2.39) \quad f_4(t) = o(\phi(t)).$$

The lengthy proof of this lemma is given in § 4.

(Note that because of (2.29) and (2.30), (2.37) implies that  $f_1(t) = o(t^{-q}\phi(t))$  and  $f_2(t) = o(\phi(t))$ .)

LEMMA 2. Suppose that  $\phi$  and  $m$  are as in Assumption A and  $w \in C[t_0, \infty)$  for some  $t_0 \geq a$ . Suppose also that  $\int_t^\infty t^{n-m-k-1}w(t) dt$  converges, and that

$$(2.40) \quad \int_t^\infty s^{n-m-k-1}w(s) ds = O(\phi(t)).$$

Define

$$(2.41) \quad \rho(t) = \sup_{\tau \geq t} (\phi(\tau))^{-1} \left| \int_\tau^\infty s^{n-k-m-1}w(s) ds \right|.$$

Then  $\mathcal{L}_2 w \in C^{(n-k)}[t_0, \infty)$  (see (2.20) and (2.21)), and there is a constant  $K$  which does not depend on  $w$  or  $t_0$  such that

$$(2.42) \quad |(\mathcal{L}_2 w)^{(r)}(t)| \leq K\rho(t_0)\phi(t)t^{m-r}, \quad t \geq t_0, \quad 0 \leq r \leq n-k-1.$$

Moreover, if

$$(2.43) \quad \lim_{t \rightarrow \infty} \rho(t) = 0,$$

then

$$(2.44) \quad (\mathcal{L}_2 w)^{(r)}(t) = o(\phi(t)t^{m-r}), \quad 0 \leq r \leq n-k-1.$$

This lemma follows immediately from Lemma 1 of [8]. Since its conclusion follows trivially from the assumption that

$$\int_t^\infty s^{n-k-m-1}|w(s)| ds < \infty,$$

it is important to emphasize here that the convergence in (2.40) may be conditional. (Note: The existence of the constant  $\gamma$  in Assumption A is required for this lemma.)

LEMMA 3. Suppose that  $v$  is complex-valued and continuous on  $[t_0, \infty)$  with  $t_0 \geq a$ , and that  $\int_t^\infty t^{n-m-k-1}v(t) dt$  converges. Let

$$(2.45) \quad \psi(t) = \sup_{\tau \geq t} \left| \int_\tau^\infty s^{n-m-k-1}v(s) ds \right| = O(\phi(t)),$$

and define  $\psi_1$  as in (2.32). Then the function

$$(2.46) \quad u = \mathcal{L}_2(\mathcal{L}_1 v)$$

is in  $B(t_0)$ , and

$$(2.47) \quad \|u\| \leq W\psi_1(t_0),$$

where  $W$  is a constant independent of  $t_0$  and  $v$ . Moreover, if

$$(2.48) \quad \psi(t) = o(\phi(t)),$$

then

$$(2.49) \quad u \in B_0(t_0) \quad (\text{see (2.15)}).$$

*Proof.* From (2.18),  $\mathcal{L}_1 v$  and its first  $k-1$  derivatives are linear combinations of integrals of the forms (2.27) and (2.33) with  $X = A_{lr}$ ; hence, Lemma 1 with  $q = n-m-k-1$  (specifically, (2.29), (2.32) and (2.34)) implies that  $\mathcal{L}_1 v \in C^{(k-1)}[t_0, \infty)$ , and that

$$(2.50) \quad |(\mathcal{L}_1 v)^{(j)}(t)| \leq \alpha_1 \psi_1(t_0) \phi(t) t^{-n+m+k+1}, \quad 0 \leq j \leq k-1,$$

where  $\alpha_1$  is a constant independent of  $t_0$  and  $v$ . Lemma 1 also implies that  $\int_t^\infty s^{n-m-k-1} (\mathcal{L}_1 v)(s) ds$  converges (since it is a linear combination of integrals of the forms (2.28) and (2.35), again with  $q = n-m-k-1$  and  $X = A_{lr}$ ), and that

$$(2.51) \quad \left| \int_t^\infty s^{n-m-k-1} (\mathcal{L}_1 v)(s) ds \right| \leq \alpha_2 \psi_1(t_0) \phi(t), \quad t \geq t_0,$$

where  $\alpha_2$  is a constant independent of  $t_0$  and  $v$ . (See (2.30), (2.32), and (2.36).)

Now we apply Lemma 2 with  $w = \mathcal{L}_1 v$ . Then (2.51) implies (2.40) and (2.41), with  $\rho(t) \leq \alpha_2 \psi_1(t_0)$ . Recalling (2.46), we now see from (2.42) with  $w = \mathcal{L}_1 v$  that

$$(2.52) \quad |u^{(r)}(t)| \leq K \alpha_2 \psi_1(t_0) \phi(t) t^{m-r}, \quad 0 \leq r \leq n-k-1.$$

Moreover, since

$$(2.53) \quad u^{(n-k+j)} = -(\mathcal{L}_1 v)^{(j)}, \quad 0 \leq j \leq k-1$$

(from (2.22) with  $w = \mathcal{L}_1 v$ ), (2.50) implies that

$$(2.54) \quad |u^{(r)}(t)| \leq \alpha_1 \psi_1(t_0) \phi(t) t^{-n+m+k+1}, \quad n-k \leq r \leq n-1.$$

Now (2.14), (2.52) and (2.54) imply (2.47), with  $W = \max\{\alpha_1, K\alpha_2\}$ .

It remains only to show that (2.48) implies (2.49). From the closing paragraph of Lemma 1, (2.48) implies that

$$(\mathcal{L}_1 v)^{(j)}(t) = o(\phi(t) t^{-n+m+k+1}), \quad 0 \leq j \leq k-1,$$

and therefore

$$(2.55) \quad u^{(r)}(t) = o(\phi(t) t^{-n+m+k+1}), \quad n-k \leq r \leq n-1,$$

because of (2.53). The closing paragraph of Lemma 1 also implies that

$$\int_t^\infty s^{n-m-k-1} (\mathcal{L}_1 v)(s) ds = o(\phi(t)),$$

because of (2.48). Therefore, (2.43) holds if  $w = \mathcal{L}_1 v$  in (2.41), and so

$$(2.56) \quad u^{(r)}(t) = o(\phi(t) t^{m-r}), \quad 0 \leq r \leq n-k-1,$$

from (2.44). Since (2.55) and (2.56) are equivalent to (2.49), this completes the proof of Lemma 3.

**LEMMA 4.** Suppose that the assumptions of Theorem 2 hold, and let  $\mathcal{L}$  be as defined in (2.24). Suppose also that  $h \in B(t_0)$  for some  $t_0 \geq a$ . Then  $\mathcal{L}h \in B_0(t_0)$  and

$$(2.57) \quad \|\mathcal{L}h\| \leq W\sigma(t_0)\|h\|,$$

where  $\sigma$  is defined on  $[a, \infty)$ ,

$$(2.58) \quad \lim_{t \rightarrow \infty} \sigma(t) = 0,$$

and  $W$  is as in (2.47).



*Proof.* We first consider the integral

$$(2.59) \quad \begin{aligned} J(t; h) &= \int_t^\infty s^{n-m-k-1} (Mh)(s) ds \\ &= \sum_{j=1}^n \int_t^\infty s^{n-m-k-1} P_j(s) h^{(n-j)}(s) ds. \end{aligned}$$

We will show that the integrals in this sum converge, and estimate them. We remind the reader that  $\|h\|$  is defined in (2.14).

From (2.5),

$$(2.60) \quad \left| \int_t^\infty s^{n-m-k-1} P_j(s) h^{(n-j)}(s) ds \right| \leq \|h\| I_j(t), \quad 1 \leq j \leq k+1.$$

If  $k+2 \leq j \leq n$ , then integration by parts yields

$$(2.61) \quad \begin{aligned} &\int_t^\infty s^{n-m-k-1} P_j(s) h^{(n-j)}(s) ds \\ &= t^{n-m-k-1} h^{(n-j)}(t) F_j(t) + \int_t^\infty F_j(s) [s^{n-m-k-1} h^{(n-j)}(s)]' ds \quad (\text{see (2.6)}). \end{aligned}$$

To justify this we first observe that

$$\lim_{T \rightarrow \infty} T^{n-m-k-1} h^{(n-j)}(T) F_j(T) = 0, \quad k+2 \leq j \leq n,$$

because of (2.6) and (2.13). Moreover, since (2.13) and (2.14) imply that

$$(2.62) \quad |[s^{n-m-k-1} h^{(n-j)}(s)]'| \leq (n-m-k) \|h\| \phi(s) s^{j-k-2}, \quad k+2 \leq j \leq n,$$

(2.7) implies that the integral on the right side of (2.61) converges (absolutely). We now conclude that  $J(t; h)$  exists on  $[t_0, \infty)$  if  $h \in B(t_0)$ ; moreover, (2.59), (2.60), (2.61) and (2.62) imply that  $|J(t; h)| \leq \|h\| \Gamma(t)$ , where

$$\Gamma(t) = \sum_{j=1}^{k+1} I_j(t) + (n-m-k) \sum_{j=k+1}^n I_j(t) + \phi(t) \sum_{j=k+2}^n t^{j-k-1} |F_j(t)|.$$

(See (2.6) and (2.7).) Now define  $\sigma(t) = \sup_{\tau \geq t} \Gamma(\tau) / \phi(\tau)$ , and note that  $\sigma$  satisfies (2.58), because of (2.5), (2.6) and (2.7).

We now apply Lemma 3 with  $v = Mh$ ; then the function  $\psi$  defined in (2.45) satisfies the inequality

$$\psi(t) \leq \|h\| \Gamma(t) = o(\phi(t)).$$

Since  $u = \mathcal{L}h$  in (2.46) when  $v = Mh$  (see (2.24)), we conclude that  $\mathcal{L}h \in B_0(t_0)$ , and (2.47) with  $\psi_1 = \sigma \|h\|$  implies (2.57). This proves Lemma 4.

We can now complete the proof of Theorem 2. From (2.4) and Lemma 3 (with  $v = g$ ),  $G$  as defined in (2.23) is also in  $B(t_0)$ . Now (2.25) and Lemma 4 imply that  $\mathcal{T}(B(t_0)) \subset B(t_0)$ ; moreover,  $\mathcal{T}$  is obviously a contraction if  $\mathcal{L}$  is, and the latter is so if

$$(2.63) \quad \sigma(t_0) < 1/W \quad (\text{see (2.57)}).$$

From (2.58), we can choose  $t_0$  to satisfy (2.63). By the contraction mapping principle, there is an  $\hat{h}$  in  $B(t_0)$  such that  $\mathcal{T}\hat{h} = \hat{h}$ ; hence,  $\hat{x}$  as defined in (2.16) satisfies (1.1) and (2.8).

Now suppose that (2.9) holds. Then Lemma 3 implies that  $G \in B_0(t_0)$ . Since  $\mathcal{L}\hat{h} \in B_0(t_0)$  (by Lemma 4) and

$$\hat{h} = -G + \mathcal{L}\hat{h} \quad (\text{see (2.25)}),$$

it now follows that  $\hat{h} \in B_0(t_0)$ . This and (2.16) imply (2.10), which completes the proof of Theorem 2.

**3. Corollaries.** If  $P \in C[a, \infty)$  and  $\int_t^\infty |P(t)| dt < \infty$ , then obviously

$$\int_t^\infty |P(s)|\phi(s) ds = o(\phi(t))$$

if  $\phi$  is nonincreasing. Also, if  $\int_t^\infty t^\alpha |P(t)| dt < \infty$  for some  $\alpha > 0$ , then

$$\int_t^\infty P(s) ds = o(t^{-\alpha}) \quad \text{and} \quad \int_t^\infty t^{\alpha-1} \left| \int_t^\infty P(\tau) d\tau \right| dt < \infty$$

(see Corollary 3 of [8]), which in turn implies that

$$\int_t^\infty s^{\alpha-1} \left| \int_s^\infty P(\tau) d\tau \right| \phi(s) ds = o(\phi(t))$$

if  $\phi$  is nonincreasing. Since the converses of these statements are false, the integrability conditions (1.3) and (1.4)—even with  $q = 0$ —are stronger than (2.5), (2.6) and (2.7).

We remind the reader that Assumption A is still in force.

**COROLLARY 1.** Let  $l$  be an integer in  $\{0, 1, \dots, n-k-1\}$ , and suppose that

$$(3.1) \quad \int_t^\infty s^{j-k-1} P_j(s) ds = O(t^{-q}), \quad n-l \leq j \leq n,$$

for some  $q \geq 0$  such that

$$(3.2) \quad q \neq 0, 1, \dots, l.$$

Let

$$(3.3) \quad \int_t^\infty P_j(s) ds = o(t^{-j+k+1}), \quad k+2 \leq j \leq n-l-1,$$

and define

$$(3.4) \quad \beta = \begin{cases} q - [q] & \text{if } l \geq [q] \quad (= \text{integer part of } q), \\ q - l & \text{if } l < [q]. \end{cases}$$

Finally, suppose that

$$(3.5) \quad \int_t^\infty |P_j(s)| s^{-\beta} ds = o(t^{-\beta}), \quad 1 \leq j \leq k+1.$$

Then (1.5) has a solution  $x_l$  such that

$$(3.6) \quad \left( x_l(t) - \frac{t^l}{l!} \right)^{(r)} = \begin{cases} O(t^{l-r-q}), & 0 \leq r \leq n-k-1, \\ O(t^{l-n+k+1-q}), & n-k \leq r \leq n-1. \end{cases}$$

Moreover, if (3.1) can be replaced by

$$(3.7) \quad \int_t^\infty P_j(s) ds = o(t^{-j+k+1-q}), \quad n-l \leq j \leq n,$$

then (3.6) can be replaced by

$$(3.8) \quad \left( x_l(t) - \frac{t^l}{l!} \right)^{(r)} = \begin{cases} o(t^{l-r-q}), & 0 \leq r \leq n-k-1, \\ o(t^{l-n+k+1-q}), & n-k \leq r \leq n-1. \end{cases}$$

*Proof.* We start by observing that if  $0 < a < b$  and

$$(3.9) \quad \int_t^\infty P(s) ds = O(t^{-b}),$$

then

$$(3.10) \quad \int_t^\infty s^a P(s) ds = O(t^{a-b}).$$

If  $f = 0$  (as in (1.5)) and  $p(t) = t^l/l!$ , then

$$(3.11) \quad g(t) = - \sum_{j=n-l}^n P_j(t) \frac{t^{l-n+j}}{(l-n+j)!}$$

(see (2.3)); hence, (3.1) implies that

$$\int_t^\infty s^{n-m-k-1} g(s) ds = O(t^{l-m-q})$$

if  $l-m < q$ . We now apply Theorem 2 with

$$(3.12) \quad m = \max \{0, l - [q]\}$$

and

$$(3.13) \quad \phi(t) = O(t^{l-m-q}) = O(t^{-\beta}),$$

with  $\beta$  as in (3.4). (Note that if  $m > 0$ , then  $\beta = q - [q] < 1$ , and  $\phi$  as in (3.13) satisfies Assumption A with  $\gamma = \beta$ .)

Now (3.1), (3.2) and (3.3) imply (2.6), and (3.5) is the same as (2.5) with  $\phi(t) = t^{-\beta}$ . Since (3.2) and (3.4) imply that  $\beta > 0$ , (2.6) automatically implies (2.7) with  $\phi$  as in (3.13), without any absolute integrability assumptions on  $P_{k+2}, \dots, P_n$ . Now Theorem 2 implies that (1.5) has a solution  $x_l$  such that

$$\left( x_l(t) - \frac{t^l}{l!} \right)^{(r)} = \begin{cases} O(t^{m-r-\beta}), & 0 \leq r \leq n-k-1, \\ O(t^{m-n+k+1-\beta}), & n-k \leq r \leq n-1, \end{cases}$$

which, in view of (3.4) and (3.12), is equivalent to (3.6).

If (3.9) holds with “ $O$ ” replaced by “ $o$ ”, then so does (3.10). From Theorem 2, this also implies the assertion concerning (3.7) and (3.8).

We now consider the exceptional cases where (3.2) is not satisfied.

**COROLLARY 2.** Let  $q$  and  $l$  be integers, with  $0 \leq q \leq l \leq n-k-1$ . Suppose that the integrals

$$(3.14) \quad \int_t^\infty t^{j-k-1+q} P_j(t) dt, \quad n-l \leq j \leq n,$$

converge, that (3.3) holds, that

$$(3.15) \quad \int_t^\infty |P_j(t)| dt < \infty, \quad 1 \leq j \leq k+1,$$

and that

$$(3.16) \quad \int_t^\infty t^{j-k-2} |F_j(t)| dt < \infty$$

for  $k+2 \leq j \leq n-l-1$ . Then (1.5) has a solution  $x_l$  which satisfies (3.8) if  $q > 0$ . If  $q = 0$ , the same conclusion is valid if the above assumptions hold and, in addition, (3.16) also holds for  $n-l \leq j \leq n$ .

*Proof.* Now (3.14) implies that  $\int_t^\infty t^{n-m-k-1} g(t) dt$  converges (see (3.11)) with  $m = l - q$ , and that (3.7) holds. We apply Theorem 2 with  $\phi = 1$ , in which case (2.5) is equivalent to (3.15). Also, (2.7) is then equivalent to (3.16) for  $k+2 \leq j \leq n$ . Since the convergence of (3.14) implies (3.7), which automatically implies (3.16) for  $n-l \leq j \leq n$ , the proof is complete.

Corollaries 1 and 2 imply the following corollary, which in turn implies Theorem 1.

COROLLARY 3. Suppose that the integrals

$$(3.17) \quad \int_t^\infty t^{j-k-1+q} P_j(t) dt, \quad k+1 \leq j \leq n,$$

converge for some  $q \geq 0$ . Suppose also that

$$(3.18) \quad \int_t^\infty |P_j(s)| s^{[q]-q} ds = o(t^{[q]-q}), \quad 1 \leq j \leq k+1.$$

Then the conclusions of Theorem 1 hold if either (i)  $q > 0$ ; or (ii)  $q = 0$  and

$$(3.19) \quad \int_t^\infty s^{j-k-2} \left| \int_s^\infty P_j(\tau) d\tau \right| ds < \infty, \quad k+2 \leq j \leq n.$$

It is important to note that (3.17) implies (3.19) if  $q > 0$ ; therefore, it is not necessary to assume (3.19) in this case.

Corollary 1 implies that if

$$(3.20) \quad \int_t^\infty s^{j-k-1} P_j(s) ds = O(\phi(t)), \quad k+1 \leq j \leq n,$$

where  $\phi(t) \rightarrow 0$  as  $t \rightarrow \infty$  like some positive power of  $1/t$ , then (1.5) has solutions  $x_l(t) \sim t^l/l!$  ( $0 \leq l \leq n-k-1$ ), without any further integrability assumptions on  $P_{k+1}, \dots, P_n$ , provided that  $P_1, \dots, P_k$  satisfy (2.5). The following corollary shows that this conclusion remains valid even if  $\phi$  decays more slowly than this.

COROLLARY 4. Suppose that  $\phi$  is as in Assumption A, and also that  $t^\gamma \phi(t)$  is eventually nondecreasing for  $\gamma < 1$ , and

$$(3.21) \quad \int_t^\infty \frac{\phi^2(s)}{s} ds = o(\phi(t)).$$

Suppose also that (2.5) and (3.20) hold. Then (1.5) has solutions  $x_0, \dots, x_{n-k-1}$  such that

$$(3.22) \quad \left( x_l(t) - \frac{t^l}{l!} \right)^{(r)} = \begin{cases} O(\phi(t)t^{l-r}), & 0 \leq r \leq n-k-1, \\ O(\phi(t)t^{-n+l+k+1}), & n-k \leq r \leq n-1. \end{cases}$$

Moreover, if (3.20) holds with "O" replaced by "o," then so does (3.22).

*Proof.* For each  $l = 0, 1, \dots, n-k-1$ , we apply Theorem 2 with  $p(t) = t^l/l!$  and  $m = l$ . The assumption (3.20) implies (2.4) and also that

$$F_j(t) = \int_t^\infty P_j(s) ds = O(\phi(t)t^{-j+k+1}), \quad k+2 \leq j \leq n.$$

This implies (2.6) and (2.7) (the latter because of (3.21)), which completes the proof.

A similar argument yields the following related corollary.

COROLLARY 5. Let  $\phi$  be as in Corollary 4, except that (3.21) is replaced by

$$\int_t^\infty \frac{\phi^2(s)}{s} ds = O(\phi(t)).$$

Suppose also that (2.5) holds, and that

$$\int_t^\infty s^{j-k-1} P_j(s) ds = o(\phi(t)), \quad k+1 \leq j \leq n.$$

Then (1.5) has solutions  $x_0, \dots, x_{n-k-1}$  which satisfy (3.22) with "O" replaced by "o."

Remark 1. Although we have assumed (1.2) throughout, our results are also valid for (1.6), which corresponds to the case where  $k=0$ , provided that obviously vacuous conditions (i.e., those involving  $0 \leq j \leq k-1$  and  $n-k \leq r \leq n-1$ ) are ignored. To see this, one has only to modify (in fact, simplify) the arguments as follows:

- (a) Omit the now vacuous assumptions on the zeros of  $Q(\lambda)$ .
- (b) Let  $\mathcal{L}_1$  be the identity operator; i.e.,  $\mathcal{L}_1 v = v$ .
- (c) Omit Lemma 1.

Viewed in this way, the present results improve on those in [7] and [8], since we assumed in those papers that  $\int_0^\infty |P_1(t)| dt < \infty$ , which is stronger than (2.5), (3.5) and (3.18) with  $k=0$  and  $j=1$ .

Remark 2. While preparing this paper the author discovered errors in [7] and [8], caused by his overlooking the need for special treatment of the exceptional cases where (3.2) does not hold. Theorem 2 of [7] requires the additional assumption that

$$\int_0^\infty \left| \int_t^\infty p_l(\tau) d\tau \right| t^{l-2} dt < \infty, \quad 2 \leq l \leq n-r-1$$

(the notation here is that of [7]). Our present Corollary 2 (with  $k=0$ ) extends this corrected result. Example 1 of [8] requires the additional assumption that  $\alpha \neq 1, \dots, n-1$ . (The notation here is that of [8].) Corollary 1 (with  $k=0$ ) extends this corrected result, and Corollary 2 deals with the excluded cases where  $\alpha = 1, \dots, n-1$ .

4. Appendix. Proof of Lemma 1. We assume throughout that  $t \geq t_0 \geq a$ . If

$$(4.1) \quad V_r(t) = \int_t^\infty s^r v(s) ds, \quad 0 \leq r \leq q,$$

then (2.26) implies that

$$(4.2) \quad |V_q(t)| \leq \psi(t),$$

and by writing  $s^r v(s) = s^{r-q} (s^q v(s))$ , integrating by parts, and again invoking (2.26), we find that

$$(4.3) \quad |V_r(t)| \leq 2t^{r-q} \psi(t), \quad 0 \leq r \leq q-1.$$

Proof of (i). Now suppose that  $\mu > 0$ . If  $t_0 \leq s \leq T$ , then integration by parts shows that

$$\int_T^\infty X(s-\tau) e^{\lambda(s-\tau)} v(\tau) d\tau = X(s-T) e^{\lambda(s-T)} V_0(T) + \int_T^\infty V_0(\tau) [X(s-\tau) e^{\lambda(s-\tau)}]' d\tau.$$

Hence, (4.3) implies that

$$(4.4) \quad \left| \int_T^\infty X(s-\tau) e^{\lambda(s-\tau)} v(\tau) d\tau \right| \leq T^{-q} \psi(T) \hat{X}(T-s) e^{\mu(s-T)}, \quad t_0 \leq s \leq T,$$

where  $\hat{X}$  is a polynomial with nonnegative coefficients determined by  $X$  and  $\lambda$ . Setting  $s = T = t$  in (4.4) yields (2.29), with  $K_1 = \hat{X}(0)$ .

To prove (2.30), we consider the integral

$$(4.5) \quad I(t, T) = \int_t^T s^q f_1(s) ds = I_1(t, T) + I_2(t, T),$$

where

$$(4.6) \quad I_1(t, T) = \int_t^T s^q ds \int_s^T X(s - \tau) e^{\lambda(s - \tau)} v(\tau) d\tau$$

and

$$(4.7) \quad I_2(t, T) = \int_t^T s^q ds \int_T^\infty X(s - \tau) e^{\lambda(s - \tau)} v(\tau) d\tau.$$

From (4.4) and (4.7),

$$\begin{aligned} |I_2(t, T)| &\leq \psi(T) \int_t^T \hat{X}(T - s) e^{\mu(s - T)} ds \\ &< \psi(T) \int_0^\infty \hat{X}(\eta) e^{-\mu\eta} d\eta; \end{aligned}$$

hence,

$$(4.8) \quad \lim_{T \rightarrow \infty} I_2(t, T) = 0.$$

Changing the order of integration in (4.6) yields

$$(4.9) \quad I_1(t, T) = \int_t^T \Gamma(t, \tau) v(\tau) d\tau,$$

where

$$(4.10) \quad \Gamma(t, \tau) = \int_t^\tau s^q X(s - \tau) e^{\lambda(s - \tau)} ds.$$

Repeated integration by parts shows that

$$(4.11) \quad \Gamma(t, \tau) = \sum_{r=0}^q [X_r(0) \tau^r - t^r X_r(t - \tau) e^{\lambda(t - \tau)}],$$

where  $X_0, \dots, X_q$  are polynomials determined by  $X$  and  $\lambda$ . Substituting (4.11) into (4.9), we obtain  $I(t, T)$  in terms of integrals which converge as  $T \rightarrow \infty$ ; therefore, (4.5), (4.8) and (4.11) imply that the integral  $f_2(t) = I(t, \infty)$  (see (2.28)) converges, and that

$$(4.12) \quad f_2(t) = \sum_{r=0}^q X_r(0) V_r(t) - \sum_{r=0}^q t^r \int_t^\infty X_r(t - \tau) e^{\lambda(t - \tau)} v(\tau) d\tau$$

(see (4.1)). Replacing  $X$  by  $X_r$  in (4.4) and letting  $s = T = t$  shows that

$$\left| \int_t^\infty X_r(t - \tau) e^{\lambda(t - \tau)} v(\tau) d\tau \right| \leq K_{1,r} t^{-q} \psi(t),$$

where  $K_{1,r}$  is a constant determined by  $\lambda$  and  $X_r$  (and therefore by  $X$ ). This, (4.2), (4.3) and (4.12) imply (2.30) for suitable  $K_2$ , a constant determined by  $X$  and  $\lambda$ . This proves (i).

*Proof of (ii).* Now suppose that  $\mu < 0$ . Integrating (2.33) by parts yields

$$f_3(t) = V_0(t_0)X(t-t_0)e^{\lambda(t-t_0)} - X(0)V_0(t) + \int_{t_0}^t V_0(\tau)[X(t-\tau)e^{\lambda(t-\tau)}]' d\tau.$$

Therefore, from (4.3) with  $r=0$ ,

$$(4.13) \quad |f_3(t)| \leq 2t_0^{-q}\psi(t_0)|X(t-t_0)|e^{\mu(t-t_0)} + 2|X(0)|\psi(t)t^{-q} \\ + 2 \int_{t_0}^t \psi(\tau)\tau^{-q}\tilde{X}(t-\tau)e^{\mu(t-\tau)} d\tau$$

where  $\tilde{X}$  is a polynomial with positive coefficients determined by  $X$  and  $\lambda$ . Now (2.32) and our assumptions on  $\alpha$  and  $\phi$  imply that

$$\psi(\tau)\tau^{-q} \leq \psi_1(t_0)e^{\alpha(t-\tau)}\phi(t)t^{-q}, \quad t \geq \tau \geq t_0.$$

This and (4.13) imply (2.34), with

$$K_3 = 2 \left[ |X(0)| + \sup_{\xi \geq 0} |X(\xi)|e^{(\alpha+\mu)\xi} + \int_0^\infty \tilde{X}(\eta)e^{(\alpha+\mu)\eta} d\eta \right],$$

which is finite, because of (2.31).

To see that (2.37) implies (2.38), observe from (2.33) that if  $t_0 \leq t_1 \leq t$ , then

$$(4.14) \quad f_3(t) = \int_{t_0}^{t_1} X(t-\tau)e^{\lambda(t-\tau)}v(\tau) d\tau + \int_{t_1}^t X(t-\tau)e^{\lambda(t-\tau)}v(\tau) d\tau.$$

For fixed  $t_1$ ,

$$(4.15) \quad \left| \int_{t_0}^{t_1} X(t-\tau)e^{\lambda(t-\tau)}v(\tau) d\tau \right| \leq M(t_0, t_1)Y(t)e^{\mu t},$$

where

$$M(t_0, t_1) = \max \{ |v(\tau)| \mid t_0 \leq \tau \leq t_1 \}$$

and  $Y$  is a polynomial, while

$$(4.16) \quad \left| \int_{t_1}^t X(t-\tau)e^{\lambda(t-\tau)}v(\tau) d\tau \right| \leq K_3\psi_1(t_1)t^{-q}\phi(t), \quad t \geq t_1.$$

(Since this integral has the same form as  $f_3(t)$  in (2.33), with  $t_0$  replaced by  $t_1$ , (4.16) is obtained as was (2.34).)

Our assumptions on  $\phi$  and  $\alpha$  imply that

$$Y(t)e^{\mu t} = o(t^{-q}\phi(t)).$$

Therefore, (4.14), (4.15) and (4.16) imply that

$$\overline{\lim}_{t \rightarrow \infty} (t^{-q}\phi(t))^{-1}|f_3(t)| \leq K_3\psi_1(t_1), \quad t_1 \geq t_0.$$

Since (2.32) and (2.37) imply that  $\lim_{t \rightarrow \infty} \psi_1(t_1) = 0$ , we now see that (2.37) implies (2.38).

For reference below, we observe here that

$$(4.17) \quad \lim_{t \rightarrow \infty} t^q f_3(t) = 0$$

in any case. This is obvious from (2.34) if  $\phi(t) = o(1)$ . If  $\lim_{t \rightarrow \infty} \phi(t) > 0$ , then (2.26) implies (2.37), which implies (2.38) and, therefore, (4.17).

To prove (2.36), consider the integral

$$J(t, T) = \int_t^T s^q f_3(s) ds = \int_t^T s^q ds \int_{t_0}^s X(s-\tau) e^{\lambda(s-\tau)} v(\tau) d\tau.$$

Reversing the order of integration yields

$$J(t, T) = \int_{t_0}^t v(\tau) d\tau \int_t^T s^q X(s-\tau) e^{\lambda(s-\tau)} ds + \int_t^T v(\tau) d\tau \int_\tau^T s^q X(s-\tau) e^{\lambda(s-\tau)} ds.$$

Manipulating the limits of integration here shows that

$$J(t, T) = H(t) - H(T).$$

where

$$H(t) = \int_{t_0}^t \Gamma(t, \tau) v(\tau) d\tau \quad (\text{see (4.10)}).$$

Therefore, from (4.11),

$$(4.18) \quad H(t) = \sum_{r=0}^q X_r(0) \int_{t_0}^t \tau^r v(\tau) d\tau - \sum_{r=0}^q t^r \int_{t_0}^t X_r(t-\tau) e^{\lambda(t-\tau)} v(\tau) d\tau.$$

The integrals on the right converge as  $t \rightarrow \infty$ . Moreover, since the integrals in the second sum are of the same form as  $f_3(t)$  (see (2.33)), (4.17) implies that this sum approaches zero as  $t \rightarrow \infty$ ; hence,

$$(4.19) \quad H(\infty) = \lim_{T \rightarrow \infty} H(T) = \sum_{r=0}^q X_r(0) \int_{t_0}^{\infty} \tau^r v(\tau) d\tau.$$

Since  $f_4(t) = H(\infty) - H(t)$ , we now see from (4.1), (4.18) and (4.19) that

$$(4.20) \quad f_4(t) = \sum_{r=0}^q X_r(0) V_r(t) + \sum_{r=0}^q t^r \int_{t_0}^t X_r(t-\tau) e^{\lambda(t-\tau)} v(\tau) d\tau.$$

Recalling (2.32), (4.2), (4.3) and, again, that the integrals in the second sum here are of the same form as  $f_3(t)$ , and therefore satisfy an inequality like (2.34), we see from (4.20) that (2.36) holds for a suitable constant  $K_4$  which is ultimately determined by  $X$  and  $\lambda$ .

Finally, suppose that (2.37) holds. Then obviously  $V_r(t) = o(\phi(t))$ , from (4.2) and (4.3). Moreover,

$$\int_{t_0}^t X_r(t-\tau) e^{\lambda(t-\tau)} v(\tau) d\tau = o(t^{-q} \phi(t)),$$

as can be seen by replacing  $X$  with  $X_r$  in (2.33) and applying the argument that led to (2.38). Now (4.20) implies (2.39), which completes the proof of Lemma 1.

#### REFERENCES

- [1] P. HARTMAN, *Ordinary Differential Equations*, John Wiley, New York, 1964.
- [2] ———, *Asymptotic integration of ordinary differential equations*, this Journal, 14 (1983), pp. 772–779.
- [3] T. W. PREVATT, *Application of exponential dichotomies to asymptotic integration and the spectral theory of ordinary differential equations*, J. Differential Equations, 17 (1975), pp. 444–460.



- [4] J. ŠIMŠA, *Asymptotic integration of perturbed linear differential equations under conditions involving ordinary integral convergence*, this Journal, 15 (1984), pp. 116-123.
  - [5] ———, *The second order differential equation with oscillatory coefficient*, Arch. Math. (Brno), 18 (1982), pp. 95-100.
  - [6] ———, *The condition of ordinary integral convergence in the asymptotic theory of linear differential equations with almost constant coefficients*, this Journal, 16 (1985), pp. 757-769.
  - [7] W. F. TRENCH, *Asymptotic integration of linear differential equations subject to integral smallness conditions involving ordinary convergence*, this Journal, 7 (1976), pp. 213-221.
  - [8] ———, *Asymptotic integration of linear differential equations subject to mild integral conditions*, this Journal, 15 (1984), pp. 932-942.
  - [9] ———, *Linear perturbations of a constant coefficient differential equation subject to mild integral smallness conditions*, Czechoslovak Math J., 36 (1986), pp. 623-633.
-