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Characteristic Polynomials of Symmetric Rationally Generated Toeplitz Matrices

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Formulas are given for the characteristic polynomials $\{p_n(\lambda)\}$ and eigenvectors of the family $\{T_n\}$ of real symmetric Toeplitz matrices generated by a rational function $R(z)$ with real coefficients such that $R(z) = R(1/z)$. The formulas are in terms of the zeros of a fixed polynomial $P(w; \lambda)$ with coefficients which are simple functions of λ and the coefficients of $R(z)$. The representation for $p_n(\lambda)$ exhibits two factors such that the zeros of one have associated symmetric eigenvectors and the zeros of the other have associated skew-symmetric eigenvectors. In all of these formulas, n is a parameter; that is, the formula does not become more complicated as n increases.

1. INTRODUCTION

Let

$$A(z) = \sum_{j=0}^q a_j z^j$$

and

$$C(z) = \sum_{j=-p}^p c_j z^j,$$

where a_0, a_1, \dots, a_q and c_{-p}, \dots, c_p are real, $c_{-j} = c_j$ ($1 \leq j \leq p$), $(p+q)a_0a_qc_p \neq 0$, and no two of the polynomials $A(z)$, $z^q A(1/z)$, and $z^p C(z)$ have a zero in common. We consider the real symmetric Toeplitz matrices

$$T_n = (t_{j-i})_{i,j=1}^n \quad (n = 1, 2, \dots), \quad (1)$$

associated with the coefficients $\{t_j\}$ in the formal Laurent series

$$\frac{C(z)}{A(z)A(1/z)} \sim \sum_{j=-\infty}^{\infty} t_j z^j, \quad (2)$$

defined as follows. If $q > 0$, write

$$\frac{1}{A(z)} = \sum_{j=0}^{\infty} \alpha_j z^j, \quad |z| < R, \quad (3)$$

where

$$R = \sup\{\rho \mid A(z) \neq 0 \text{ if } |z| < \rho\},$$

and define the formal Laurent series on the right of (2) by

$$\sum_{j=-\infty}^{\infty} t_j z^j = C(z) \left[z^q f(z) \left(\sum_{j=0}^{\infty} \alpha_j z^j \right) + g(z) \sum_{j=0}^{\infty} \alpha_j z^{-j} \right], \quad (4)$$

where f and g are the unique polynomials of degree $< q$ such that

$$z^q f(z) A(1/z) + g(z) A(z) = 1. \quad (5)$$

(Recall that $A(z)$ and $z^q A(1/z)$ are relatively prime.) This series formally represents the rational function on the left of (2) in the sense that formal multiplication yields

$$A(z)A(1/z) \sum_{j=-\infty}^{\infty} t_j z^j = C(z),$$

because of (3), (4), and (5). If $q = 0$, then $t_j = c_j$ if $-p \leq j \leq p$ and $t_j = 0$ if $|j| > p$; thus, (1) is banded if $n > 2p$.

If $R > 1$, then we can replace " \sim " in (2) with " $=$ " for all z in the annulus $1/R < |z| < R$. The covariance matrices of real-valued autoregressive moving average time series are Toeplitz matrices generated in this way by rational functions of the form (2), where A has no zeros in $|z| \leq 1$, $C(z) = B(z)B(1/z)$, and

$$B(z) = \sum_{j=0}^p b_j z^j,$$

with b_0, \dots, b_p real.

In [2] we obtained formulas for the characteristic polynomials of Toeplitz matrices for which the $\{t_j\}$ in (2) are the coefficients of a formal Laurent series of an arbitrary rational function, so that T_n need not be symmetric. Here we start with the results of [2] specialized to the

symmetric case, and deduce from them new formulas which give additional insight into the symmetric case. Numerical experiments (discussed in [3]) seem to indicate that these formulas may provide an efficient method for computing the eigenvalues of these matrices.

We let

$$\max(p, q) = m \geq 1, \quad (6)$$

and define $\theta_{-p}, \dots, \theta_p$ by

$$[A(z)A(1/z)] = \sum_{j=-q}^q \theta_j z^j. \quad (7)$$

It is convenient to define $c_j = 0$ if $|j| > p$ and $\theta_j = 0$ if $|j| > q$.

We start from the following lemma, which can be obtained by applying Theorem 2 and Corollary 1 of [2] under our present assumptions.

LEMMA Suppose that λ is such that

$$c_m - \lambda \theta_m \neq 0, \quad (8)$$

and the Laurent polynomial

$$Q(z; \lambda) = c_0 - \lambda \theta_0 + \sum_{j=1}^m (c_j - \lambda \theta_j)(z^j + z^{-j}), \quad (9)$$

has $2m$ distinct zeros

$$z_1, \dots, z_m, \frac{1}{z_1}, \dots, \frac{1}{z_m}. \quad (10)$$

For $n \geq 1$, let D_n be the $2m \times 2m$ determinant given in block form by

$$D_n = \begin{vmatrix} [z_s^{-r-1} A(z_s)] & [z_s^{-r+1} A(1/z_s)] \\ [z_s^{n+m+r-1} A(1/z_s)] & [z_s^{-n-m-r+1} A(z_s)] \end{vmatrix}, \quad (11)$$

where $1 \leq r, s \leq m$ in each of the four $m \times m$ matrices on the right. Let V be the $2m \times 2m$ determinant obtained by letting $n = 0$ and $A(z) = 1$ in (11); i.e.,

$$V = \begin{vmatrix} [z_s^{-r-1}] & [z_s^{-r+1}] \\ [z_s^{m+r-1}] & [z_s^{-m-r+1}] \end{vmatrix},$$

is the Vandermonde determinant of the zeros (10) of $Q(\cdot; \lambda)$. Then the value of the characteristic polynomial

$$p_n(\lambda) = \det(\lambda I_n - T_n), \quad (12)$$

is given by

$$p_n(\lambda) = K_n (c_m - \lambda \theta_m)^n D_n / V, \quad (13)$$

where K_n is a constant. Moreover, if λ is an eigenvalue of T_n and $\{G_1, \dots, G_m, H_1, \dots, H_m\}$ is a nontrivial solution of the $2m \times 2m$ system

$$(a) \sum_{s=1}^m [z_s^{-1} A(z_s) G_s + z_s^{-r+1} A(1/z_s) H_s] = 0, \quad 1 \leq r \leq m, \\ (b) \sum_{s=1}^m [z_s^{n+m+r-1} A(1/z_s) G_s + z_s^{-n-m-r+1} A(z_s) H_s] = 0, \quad 1 \leq r \leq m, \quad (14)$$

(which has the determinant D_n), then the vector

$$U = [u_1, \dots, u_n]^t, \quad (15)$$

with components

$$u_r = \sum_{s=1}^m A(z_s) A(1/z_s) [G_s z_s^{m+r-1} + H_s z_s^{-m-r+1}], \quad 1 \leq r \leq m, \quad (16)$$

is a λ -eigenvector of T_n .

The value of K_n is given explicitly in [2], but it is not important here.

2. THE MAIN THEOREM

Following Cantoni and Butler [1], we say that an n -vector (15) is symmetric if $u_r = u_{n-r+1}$ ($1 \leq r \leq n$), or skew-symmetric if $u_r = -u_{n-r+1}$ ($1 \leq r \leq n$). Cantoni and Butler have shown that if T_n is a real symmetric Toeplitz matrix of order n , then R^n has an orthonormal basis consisting of $[n/2]$ skew symmetric and $n - [n/2]$ symmetric eigenvectors of T_n . (Here $[x]$ is the integer part of x .)

The following is our main result.

THEOREM 1 With $\theta_{-p}, \dots, \theta_p$ as in (7), let

$$P(w; \lambda) = c_0 - \lambda \theta_0 + 2 \sum_{j=1}^m (c_j - \lambda \theta_j) \tau_j(w), \quad (17)$$

where τ_1, \dots, τ_m are the Chebychev polynomials; i.e., $\tau_n(\cos t) = \cos nt$. Let λ be such that (8) holds and $P(\cdot; \lambda)$ has m distinct zeros w_1, \dots, w_m such that

$$w_j \neq 1, -1, \quad 1 \leq j \leq m. \quad (18)$$

Now let

$$\gamma_j = \frac{1}{2} \cos^{-1} w_j, \quad 0 \leq \operatorname{Re}(\gamma_j) \leq \frac{\pi}{2}, \quad (19)$$

and define

$$C_{rn}(\gamma_j) = \sum_{j=0}^q a_j \cos(n + 2r - 2j - 1)\gamma_j, \quad (20)$$

and

$$S_{rn}(\gamma_j) = \sum_{j=0}^q a_j \sin(n + 2r - 2j - 1)\gamma_j. \quad (21)$$

Then the value of the characteristic polynomial (12) is given by

$$p_n(\lambda) = K_n(c_m - \lambda \theta_m)^n F_{0n}(\lambda) F_{1n}(\lambda), \quad (22)$$

where

$$F_{0n}(\lambda) = \frac{\det[C_{rn}(\gamma_s)]_{r,s=1}^m}{\det[\cos(2r - 1)\gamma_s]_{r,s=1}^m}, \quad (23)$$

and

$$F_{1n}(\lambda) = \frac{\det[S_{rn}(\gamma_s)]_{r,s=1}^m}{\det[\sin(2r - 1)\gamma_s]_{r,s=1}^m}. \quad (24)$$

Moreover, if $F_{1n}(\lambda) = 0$, then T_n has a λ -eigenvector which is symmetric if $l = 0$, or skew-symmetric if $l = 1$.

Proof Define

$$z_r = w_r + \sqrt{w_r^2 - 1}, \quad 1 \leq r \leq m, \quad (25)$$

so that

$$w_r = \frac{1}{2}(z_r + 1/z_r).$$

From the defining property of the Chebychev polynomials,

$$\tau_j(w_r) = \frac{1}{2}(z_r^j + z_r^{-j});$$

therefore, z_r and $1/z_r$ are zeros of $Q(z; \lambda)$ (cf. (9) and (17)). Moreover, since w_1, \dots, w_m are distinct and satisfy (18), the quantities (10) are distinct, and the hypotheses of Lemma 1 are verified.

We now perform manipulations on D_n , as defined in (11). For $1 \leq s \leq m$, divide column s and multiply column $m + s$ by $z_s^{(n+2m-1)/2}$ to obtain

$$D_n = \begin{vmatrix} [z_s^{-(n+2m-2r+1)/2} A(z_s)] & [z_s^{(n+2m-2r+1)/2} A(1/z_s)] \\ [z_s^{(n+2r-1)/2} A(1/z_s)] & [z_s^{-(n+2r-1)/2} A(z_s)] \end{vmatrix}.$$

Now it is convenient to reverse the order of the first m rows to obtain

$$D_n = \pm \begin{vmatrix} [z_s^{-(n+2r-1)/2} A(z_s)] & [z_s^{(n+2r-1)/2} A(1/z_s)] \\ [z_s^{(n+2r-1)/2} A(1/z_s)] & [z_s^{-(n+2r-1)/2} A(z_s)] \end{vmatrix}. \quad (26)$$

(For typographical reasons we are not specific about the " \pm ", which will turn out to be irrelevant to our final result.)

Subtracting row r from row $m+r$ ($1 \leq r \leq m$) in (26) yields

$$D_n = \pm \begin{vmatrix} [z_s^{-(n+2r-1)/2} A(z_s)] & [z_s^{(n+2r-1)/2} A(1/z_s)] \\ E_n & -E_n \end{vmatrix}, \quad (27)$$

with

$$E_n = [z_s^{(n+2r-1)/2} A(1/z_s) - z_s^{-(n+2r-1)/2} A(z_s)]_{r,s=1}^m. \quad (28)$$

Now adding column s to column $m+s$ ($1 \leq s \leq n$) in (27) yields

$$D_n = \pm \begin{vmatrix} [z_s^{-(n+2r-1)/2} A(z_s)] & F_n \\ E_n & O_n \end{vmatrix}, \quad (29)$$

where O_n is the $n \times n$ zero matrix and

$$F_n = [z_s^{(n+2r-1)/2} A(1/z_s) + z_s^{-(n+2r-1)/2} A(z_s)]_{r,s=1}^m. \quad (30)$$

Now Laplace's expansion of (29) yields

$$D_n = \pm \det(E_n) \det(F_n). \quad (31)$$

By taking $A(z) = 1$ and $n = 0$, we infer from this result that

$$V = \pm \det(E) \det(F), \quad (32)$$

where

$$E = [z_s^{(2r-1)/2} - z_s^{-(2r-1)/2}]_{r,s=1}^m, \quad (33)$$

and

$$F = [z_s^{(2r-1)/2} + z_s^{-(2r-1)/2}]_{r,s=1}^m, \quad (34)$$

and the \pm in (32) is the same as in (31). Since V is the Vandermonde determinant of the distinct points (10), (32) implies that $\det(E) \neq 0$ and $\det(F) \neq 0$.

Now (13), (31), and (32) imply (22), with

$$F_{0n}(\lambda) = \frac{\det(F_n)}{\det(F)}, \quad (35)$$

and

$$F_{1n}(\lambda) = \frac{\det(E_n)}{\det(E)}. \quad (36)$$

With γ_j as in (19), $w_j = \cos 2\gamma_j$, and (25) implies that $z_j = e^{2i\gamma_j}$. Substituting this into (28), (30), (33), and (34) shows that (35) and (36) can be rewritten as (23) and (24), respectively. (Recall (20) and (21).)

If $F_n(\lambda) = 0$ ($l = 0$ or 1), then the system

$$\sum_{s=1}^m [A(z_s)z_s^{-(n+2r-1)/2} + (-1)^l A(1/z_s)z_s^{(n+2r-1)/2}] P_s = 0, \quad 1 \leq r \leq m,$$

has a nontrivial solution P_1, \dots, P_m . From this it is straightforward to verify that the system (14) has the nontrivial solution

$$H_s = z_s^{(n+2m-1)/2} P_s, \quad G_s = (-1)^l z_s^{-(n+2m-1)/2} P_s, \quad 1 \leq s \leq m. \quad (37)$$

(To see this, it is convenient to replace r by $m-r+1$ in (14a).) Substituting (37) into (16) yields the eigenvector (15), with

$$u_r = \sum_{s=1}^m A(z_s) A(1/z_s) P_s [z_s^{(n-2r+1)/2} + (-1)^l z_s^{-(n-2r+1)/2}]. \quad (38)$$

It is easy to verify that $u_{n-r+1} = (-1)^l u_r$, which completes the proof of Theorem 1.

3. A REMARK

It is generally agreed that attempting to compute the eigenvalues of a high order matrix by any method involving the application of root finding techniques to its characteristic polynomial is wildly impractical. However, Theorem 1 essentially reduces the solution of the eigenvalue problem for T_n to finding those values of λ for which either

$$\det[C_{rn}(\gamma_s)]_{r,s=1}^m = 0 \quad \text{or} \quad \det[S_{rn}(\gamma_s)]_{r,s=1}^m = 0. \quad (39)$$

The computations to determine $\gamma_1, \dots, \gamma_m$ for a given λ are independent of n , and n enters into these determinants only as a parameter. Moreover, elementary manipulations make it possible to delete complex terms and factor out exponential factors which occur in the determinants in (39) if some of the quantities $\gamma_1, \dots, \gamma_m$ are not real-valued. Therefore, the determinants in (39) can be replaced by "well-scaled" functions which have the same zeros as $p_n(\lambda)$, but do not vary wildly, as $p_n(\lambda)$ does if n is large; in fact, they are bounded for all n . Numerical experiments already performed (and reported in [3]) show

that this approach can be used to obtain *all* eigenvalues of high order rationally generated symmetric Toeplitz matrices such that $m = 1, 2$, or 3 in (6), and at a cost per eigenvalue which is *essentially* independent of the order n . There seems to be no theoretical reason to preclude the extension of the method so as to deal with larger values of m .

References

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