1987

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William F. Trench, *Trinity University*
Leon Y. Bahar

Int. J. Non-Linear Mechanics, Vol. 22, No. 3, pp. 217-226, 1987.
Printed in Great Britain

0020-7462/87 $3.00 + 0.00

FIRST INTEGRALS FOR EQUATIONS WITH NON-LINEARITIES OF THE EMDEN–FOWLER TYPE

WILLIAM F. TRENCH*

Department of Mathematics, Trinity University, San Antonio, TX 78284, U.S.A.

and

LEON Y. BAHAR

Department of Mechanical Engineering and Engineering Mechanics, Drexel University, Philadelphia, PA 19104, U.S.A.

(*Received 25 September 1986*)

Abstract—An *ad hoc* procedure is given for obtaining first integrals of second order differential equations in which the non-linear term is a power of the dependent variable, as in the Emden–Fowler equation. The main theorem is a considerable extension of previous results along these lines. A corollary implies several known examples.

1. INTRODUCTION

In a recent paper [1], Sarlet and Bahar presented a constructive method for finding first integrals for second order equations of the form

$$x'' + c(t)x' + b(t)x^m = 0 \qquad (m \neq -1), \tag{1.1}$$

in which the non-linearity is of the kind encountered in the Emden–Fowler equation [(3.1), below]. The method used in [1] is an *ad hoc* procedure, motivated by the idea of attempting to generalize the construction of energy integrals which are quadratic in x', by using an integrating factor of the form $h(t)x'$. On the one hand, using this form for the multiplier permits more flexibility than the conventional approach of simply multiplying by x', and, on the other, avoiding a multiplier of the more general form $h(t, x, x')$ preserves the Newtonian character of the equation of motion prior to the integration process. Moreover, the choice of this form for the multiplier is consistent with the presence of the purely time-dependent factor in the first integral of the damped harmonic oscillator (Example 9, below), as obtained by Logan [2] by means of Noether's theorem, or of the Lane–Emden equation, as obtained by Jones and Ames [3] (Example 7, below) through the introduction of similarity variables.

Although reasons for seeking first integrals for dynamical systems are given in [1], it is perhaps worthwhile to summarize them here:

(i) A complete set of first integrals determines the solution of the differential equation.

(ii) A first integral can be used to reduce the order of the differential equation.

(iii) A first integral may yield insight into the qualitative behavior of solutions of the equation.

(iv) A first integral may be used to construct a Lyapunov function by means of the method of Chetayev [4]. The latter yields information on the stability properties of the solutions via the second (or direct) method of Lyapunov [5]. For examples, see Djukic [6], Pozharitskii [7] and Risito [8].

In [1], after multiplication of the differential equation (1.1) by $h(t)x'$, all terms derivable as the derivatives of energy-like terms were collected, and the sum of the remaining terms was assumed to be the derivative of an expression of the form $g(t)xx'$, computed along the

* Correspondence address: Dept. of Mathematics, Trinity University, 715 Stadium Drive, San Antonio, TX 78284, U.S.A.

trajectories of the equation. Although this procedure produced non-trivial results, it was noted already in [1] that this restrictive assumption concerning the form of the remaining terms leads to difficulties even in the linear case. (Bahar and Kwatny [9] later extended the method to some linear and non-linear systems.) In the present paper, the sum of the remaining terms (i.e. those resulting from multiplication of the equation by $h(t)x'$ which are not the derivatives of energy-like quantities) are assumed to be the derivatives—along trajectories—of an expression of the more general form $f(t)x^2 + g(t)xx'$. Although this is still an *ad hoc* procedure, we will show below that it is applicable to a much wider class of problems than the procedure of [1].

Our main result (Theorem 1 of Section 2) deals with an equation of the form

$$Lx + a(t)x + b(t)|x|^m \operatorname{sgn} x = 0, \qquad t \in J \quad (m \neq -1) \tag{1.2}$$

with

$$Lx = (r(t)x')' + q(t)x, \tag{1.3}$$

where a, b, r and q satisfy suitable assumptions (given below) on the open interval J, and the equation

$$Ly = 0 \tag{1.4}$$

is assumed to have a solution y_0 with no zeros on J. It is, of course, well known that any second order linear equation can be transformed into the form

$$(r(t)x')' + a(t)x = 0,$$

which corresponds to the choice of $q = 0$ in (1.3); however, the possibility of including part of the multiplier of y in L (so long as (1.4) still has a non-oscillatory solution on J) lends additional generality and convenience to the method, as we will see in Section 3.

Following Wong [10], we have chosen to write our basic equation in the form (1.2) rather than as

$$Lx + a(t)x + b(t)x^m = 0 \tag{1.5}$$

to avoid the difficulty that the function $w(x) = x^m$ has no branch which is real-valued for $x < 0$ unless $m = p/(2q + 1)$, where p and q are integers. If m has this form with p odd, then the real-valued branch of $w(x) = x^m$ is given by $w(x) = |x|^m \operatorname{sgn} x$ $(x \neq 0)$, so that (1.2) and (1.5) are equivalent; however, if p is even, then $w(x) = |x|^m$ $(x \neq 0)$, so that (1.2) and (1.5) are different. Nevertheless, the results given below can be adapted to (1.5) in this case by replacing "$|x|^m \operatorname{sgn} x$" by "$x^m$" in the differential equation, and "$|x|^{m+1}$" by "x^{m+1}" in the associated first integrals.

In Section 2 we first consider the simpler equation

$$x'' + a(t)x + b(t)|x|^m \operatorname{sgn} x = 0 \qquad (m \neq -1), \tag{1.6}$$

and then generalize the result obtained in this connection to obtain our main theorem, which concerns (1.2). Although it could be argued that this step is superfluous, since (1.2) can be transformed into the form (1.6) by an easy change of variables (given in Section 2), we believe that the result is more conveniently applicable if stated in terms of the given equation (1.2). Although we can claim only that our conditions are *sufficient* for the existence of first integrals of the form considered here, it seems reasonable to believe that our results are the most general that can be obtained by this particular *ad hoc* approach. In any case, they are sufficiently general so as to imply several previous results as very special cases, as will be shown by the examples in Section 3.

2. THE MAIN RESULT

We first seek sufficient conditions for (1.6) to have a first integral of the form

$$I(t,x,x') = h\left[(x')^2 + ax^2 + \frac{2b\,|x|^{m+1}}{m+1}\right] + fx^2 + gxx'. \tag{2.1}$$

We derive this result in a purely manipulative way, and then state it carefully in Lemma 1.

Obviously, a, b, f, g and h are related by the requirement that (2.1) be an invariant for (1.5). Although it may seem natural to start with a and b and then find f, g and h, we will show that all these functions can be most conveniently expressed in terms of h, which can be any function that is positive and has three continuous derivatives on J.

We start by observing that

$$(h(x')^2)' = 2hx'x'' + h'(x')^2, \tag{2.2}$$

$$(hax^2)' = 2haxx' + (ha)'x^2 \tag{2.3}$$

and

$$\frac{2}{m+1}(hb\,|x|^{m+1})' = 2hbx'\,|x|^m \operatorname{sgn} x + \frac{2}{m+1}|x|^{m+1}(hb)'. \tag{2.4}$$

If x satisfies (1.6), then

$$|x|^m \operatorname{sgn} x = -(x'' + ax)/b,$$

so (2.4) can be rewritten as

$$\frac{2}{m+1}(hb\,|x|^{m+1})' = 2hbx'\,|x|^m \operatorname{sgn} x - \frac{2x(x'' + ax)(hb)'}{(m+1)b}. \tag{2.5}$$

Notice that if x satisfies (1.6), then the sum of the first terms in (2.2), (2.3), and (2.5) is zero. Therefore, differentiating the right side of (2.1), using (2.2), (2.3), and (2.5), and equating the resulting multipliers of x^2, $(x')^2$, xx' and xx'' to zero shows that I in (2.1) is a first integral of (1.6) if

$$(ha)' - \frac{2a(hb)'}{(m+1)b} + f' = 0, \tag{2.6}$$

$$g = -h', \tag{2.7}$$

$$f = -\frac{g'}{2} = \frac{h''}{2}, \qquad \text{[cf. (2.7)]}, \tag{2.8}$$

and

$$\frac{-2(hb)'}{(m+1)b} = -g = h' \qquad \text{[cf. (2.7)]},$$

respectively. The last equation is equivalent to

$$\frac{b'}{b} = \frac{-(m+3)}{2}\frac{h'}{h},$$

so

$$b = \lambda h^{-(m+3)/2} \tag{2.9}$$

for some constant λ.

With f and b as in (2.8) and (2.9), elementary manipulations show that (2.6) is equivalent to

$$(ah^2)' = -\frac{hh'''}{2}, \tag{2.10}$$

which is in turn equivalent to

$$a = \frac{\alpha}{h^2} - \frac{1}{2}\frac{h''}{h} + \frac{1}{4}\left(\frac{h'}{h}\right)^2, \tag{2.11}$$

where α is an arbitrary constant.

We summarize these results in the following lemma.

Lemma 1. Suppose that $h > 0$ and h''' is continuous on an open interval J, and let λ be an arbitrary constant. Then the equation

$$x'' + a(t)x + \lambda(h(t))^{-(m+3)/2}|x|^m \operatorname{sgn} x = 0, \qquad t \in J,$$

has the first integral

$$I(t, x, x') = h(x')^2 + \left(ha + \frac{h''}{2}\right)x^2 + \frac{2\lambda h^{-(m+1)/2}|x|^{m+1}}{(m+1)} - h'xx',$$

provided that a is a solution of (2.10), or, equivalently, is of the form (2.11) with α an arbitrary constant.

This lemma can be verified by simply substituting (2.7), (2.8), and (2.9) into (1.6) and (2.1).

We now generalize our results so as to make them applicable to the more general equation (1.2).

Theorem 1. Suppose that r'', q', and H''' are continuous and $r, H > 0$ on the interval J. Suppose also that (1.4) has a solution y_0 with no zeros on J, and that A is a solution of the equation

$$(AH^2)' = -\tfrac{1}{2}H(ry_0^2(ry_0^2H')')', \tag{2.12}$$

or, equivalently, that

$$A = \frac{\alpha}{H^2} - \frac{ry_0^2}{2H}(ry_0^2H')' + \frac{r^2y_0^4}{4}\left(\frac{H'}{H}\right)^2,$$

where α is a constant. Let λ be a constant. Then the equation

$$Lx + \frac{1}{r(t)}\left[\frac{A(t)}{y_0^4(t)}x + \lambda(H(t)y_0^2(t))^{-(m+3)/2}|x|^m \operatorname{sgn} x\right] = 0, \qquad t \in J, \tag{2.13}$$

has the first integral

$$I(t,x,x') = Hr^2y_0^4(X')^2 + [HA + \tfrac{1}{2}ry_0^2(ry_0^2H')']X^2 + \frac{2\lambda H^{-(m+1)/2}|X|^{m+1}}{(m+1)} - r^2y_0^4H'XX', \tag{2.14}$$

where $X(t) = x(t)/y_0(t)$.

Proof. Notice that our differentiability assumptions on r, q and H imply that the right side of (2.14) is defined. Let t_0 be an arbitrary point in J, and define the new independent variable

$$s = s(t) = \int_{t_0}^{t} \frac{d\tau}{ry_0^2}.$$

This transformation maps J monotonically onto an interval J_1. Denote the inverse transformation by $t = t(s)$, and note that

$$\frac{dt}{ds} = ry_0^2 \tag{2.15}$$

[where here, and in the following, all functions of t are evaluated at $t = t(s)$]. We will show that if x satisfies (2.13), then the function

$$u(s) = X(t) = \frac{x(t)}{y_0(t)} \tag{2.16}$$

satisfies the equation

$$\ddot{u} + A(t)u + \lambda(H(t))^{-(m+3)/2}|u|^m \operatorname{sgn} u = 0, \qquad t \in J_1, \tag{2.17}$$

where the dot indicates differentiation with respect to s. From (2.15), (2.16) and the chain rule,

$$\dot{u} = r(y_0x' - y_0'x);$$

therefore,

$$\ddot{u} = ry_0^2[y_0(rx')' - x(ry_0')']. \tag{2.18}$$

It is now routine to verify that if y_0 and x satisfy (1.4) and (2.13), respectively, then (2.18) implies (2.17).

For convenience, let us rewrite (2.17) as

$$\ddot{u} + a(s)u + \lambda(h(s))^{-(m+3)/2}|u|^m \operatorname{sgn} u = 0, \tag{2.19}$$

where

$$a(s) = A(t(s)) \quad \text{and} \quad h(s) = H(t(s)). \tag{2.20}$$

Lemma 1 implies that if

$$\frac{d}{ds}(ah^2) = \frac{h\dddot{h}}{2} \qquad [\text{cf. } (2.10)], \tag{2.21}$$

then (2.19) has the first integral

$$\tilde{I}(s, u, \dot{u}) = h\dot{u}^2 + \left(ha + \frac{\ddot{h}}{2}\right)u^2 + \frac{2\lambda h^{-(m+1)/2}|u|^{m+1}}{m+1} - \dot{h}u\dot{u}. \tag{2.22}$$

However, (2.15), (2.16), (2.20) and the chain rule imply that (2.12) and (2.21) are equivalent, and that

$$\tilde{I}(s, u(s), \dot{u}(s)) = I(t, x(t), x'(t))$$

[cf. (2.14) and (2.22)]. This completes the proof.

Theorem 1 seems to characterize the class of equations to which our *ad hoc* procedure can be successfully applied. Although this is a restricted class of equations and our approach can be described as somewhat artificial (after all, we have started with the multiplier H and found the equations for which it is appropriate!), the fact is that the end result is really comparatively general; in fact, we will show in Section 3 that many known examples are consequences of the following corollary of Theorem 1.

Corollary 1. Let q, r and H satisfy the assumptions of Theorem 1, and, in addition, suppose that

$$(ry_0^2(ry_0^2H')')' = 0. \tag{2.23}$$

Let λ and β be arbitrary constants. Then the equation

$$Lx + \frac{1}{r(t)}[\beta(H(t)y_0^2(t))^{-2}x + \lambda(H(t)y_0^2(t))^{-(m+3)/2}|x|^m \operatorname{sgn} x] = 0, \qquad t \in J, \tag{2.24}$$

has the first integral (2.14), with $A = \beta/H^2$.

3. EXAMPLES

The Emden–Fowler equation

$$(t^k x')' + \lambda t^l |x|^m \operatorname{sgn} x = 0 \qquad (\lambda = \text{constant}) \tag{3.1}$$

has many physical and mathematical applications. For a study of its qualitative properties and an extensive list of references on this equation and its generalizations, see the review paper by Wong [10]. Ames and Adams [11] have studied boundary value and eigenvalue problems for (3.1), and Rosenau [12] has given sufficient conditions (relating k and l) for (3.1) to have a first integral of the kind studied here. We will obtain his conclusions as special cases of our results on the more general equation

$$(t^k x')' + qt^{k-2}x + \lambda t^l |x|^m \operatorname{sgn} x = 0, \tag{3.2}$$

where q and λ are constants, and

$$(k-1)^2 > 4q. \tag{3.3}$$

Equation (3.2) is of the form (1.2), with

$$Lx = (t^k x')' + qt^{k-2}x \qquad [\text{cf. } (1.3)]. \tag{3.4}$$

The associated equation $Ly = 0$ is equivalent to the Euler equation

$$x'' + kt^{-1}x' + qt^{-2}x = 0,$$

which has solutions of the form $y_0 = t^\nu$, where

$$\nu = \frac{1 - k \pm [(k-1)^2 - 4q]^{1/2}}{2}; \tag{3.5}$$

hence, (3.3) implies that the possible choices for $y_0 = t^\nu$ are distinct, real-valued, and non-zero on $(0, \infty)$. In (3.4), $r(t) = t^k$; hence, $r(t)y_0^2(t) = t^{k+2\nu}$, and condition (2.23) becomes

$$[t^{k+2\nu}(t^{k+2\nu}H')']' = 0.$$

Since $k + 2\nu \neq 1$ (because of (3.3) and (3.5)), this has the general solution

$$H(t) = c_0 + c_1 t^{-k-2\nu+1} + c_2 t^{-2k-4\nu+2}.$$

To obtain specific results, we consider three natural special choices for H, summarized as follows. (In the following, q, β, and λ are constants, (3.3) is assumed to hold, and ν is defined by (3.5) with a specific—but arbitrary—choice of "$\pm$". The differential equation in each case is obtained from (2.13) with L as in (3.4), while the first integral is obtained from (2.14). We omit the routine—but burdensome—algebra. The reader who wishes to verify these results should bear in mind that $r(t) = t^k$, $y_0(t) = t^\nu$, $A = \beta/H^2$, and ν is defined by (3.5).)

Case 1. ($H(t) = 1$.) If

$$(t^k x')' + [qt^{k-2} + \beta t^{-k-4\nu}]x + \lambda t^{-k-\nu(m+3)}|x|^m \operatorname{sgn} x = 0,$$

then

$$t^{2(k+\nu)}(x')^2 + [\beta t^{-2\nu} + \nu^2 t^{2(k+\nu-1)}]x^2 - 2\nu t^{2k+2\nu-1}xx' + \frac{2}{m+1}\lambda t^{-(m+1)\nu}|x|^{m+1} = c.$$

Case 2. ($H(t) = t^{-2\nu-k+1}$.) If

$$(t^k x')' + (\beta + q)t^{k-2}x + \lambda t^{[(m+1)k-(m+3)]/2}|x|^m \operatorname{sgn} x = 0, \tag{3.6}$$

then

$$t^{k+1}(x')^2 + (\beta + q)t^{k-1}x^2 + (k-1)t^k xx' + \frac{2}{m+1}\lambda t^{(m+1)(k-1)/2}|x|^{m+1} = c. \tag{3.7}$$

[Notice that we may take $\alpha = \beta + q$ to be the arbitrary parameter in (3.6) and (3.7). In verifying the coefficient of x^2 in (3.7), notice that $-\nu(\nu + k - 1) = q$, from (3.5).]

Case 3. ($H(t) = t^{-4\nu-2k+2}$.) If

$$(t^k x')' + [qt^{k-2} + \beta t^{3k+4\nu-4}] + \lambda t^{[(m+2)k+(m+3)(\nu-1)]}|x|^m \operatorname{sgn} x = 0,$$

then

$$t^{2-2\nu}(x')^2 + [\beta t^{2k+2\nu-2} + (k+\nu-1)^2 t^{-2\nu}]x^2 + 2(k+\nu-1)t^{-2\nu+1}xx' + \frac{2}{m+1}\lambda t^{(m+1)(k+\nu-1)}|x|^{m+1} = c.$$

Remark. Changing the choice of "$\pm$" in ν [cf. (3.5)] is equivalent to replacing ν by $1 - k - \nu$, which merely interchanges Cases 1 and 3.

Taking $\beta = q = \nu = 0$ in Cases 1, 2 and 3 yields three classes of Emden–Fowler equations with first integrals, as follows. (Recall that $(k-1)(m+1) \neq 0$.)

Example 1. If

$$(t^k x')' + \lambda t^{-k}|x|^m \operatorname{sgn} x = 0,$$

then

$$t^{2k}(x')^2 + \frac{2}{m+1}\lambda|x|^{m+1} = c.$$

Example 2. If

$$(t^k x')' + \lambda t^{[(m+1)k-(m+3)]/2}|x|^m \operatorname{sgn} x = 0, \tag{3.8}$$

then

$$t^{k+1}(x')^2 + (k-1)t^k x x' + \frac{2}{m+1}\lambda t^{(m+1)(k-1)/2}|x|^{m+1} = c. \tag{3.9}$$

Example 3. If

$$(t^k x')' + \lambda t^{[(m+2)k-(m+3)]}|x|^m \operatorname{sgn} x = 0, \tag{3.10}$$

then

$$t^2(x')^2 + (k-1)^2x^2 + 2(k-1)txx' + \frac{2}{m+1}\lambda t^{(m+1)(k-1)}|x|^{m+1} = c.$$

Example 4. Rosenau [12] has shown that the equation

$$(t^{\mu+\alpha}x')' + \lambda t^{\mu}|x|^m \operatorname{sgn} x = 0 \tag{3.11}$$

has a first integral if either

$$(\mu + \alpha - 1)m = 3 - \alpha + \mu \tag{3.12}$$

or

$$(\mu + \alpha - 1)m = 3 - 2\alpha - \mu. \tag{3.13}$$

The conclusion concerning condition (3.12) can be obtained by setting

$$k = \mu + \alpha \quad \text{and} \quad \mu = [(m+1)k - (m+3)]/2 \tag{3.14}$$

in (3.8). It is easily verified that (3.12) and (3.14) are equivalent. Similarly, setting

$$k = \mu + \alpha \quad \text{and} \quad \mu = (m+2)k - (m+3)$$

in (3.10) yields (3.13). Example 1 with $k = \mu + \alpha$ and $\mu = -k$ provides another case—apparently missed by Rosenau—in which (3.11) has a first integral: $\mu + \alpha - 1 \neq 0$, $2\mu + \alpha = 0$, $m \neq -1$.

Example 5. Rosenau [12] also showed that the equation

$$tx'' + (1 + c)x' + \lambda t^{\mu}|x|^m \operatorname{sgn} x = 0 \tag{3.15}$$

has the first integral

$$I(t, x, x') = t^{c+2}(x')^2 + ct^{c+1}xx' + \frac{2}{m+1}\lambda t^{c(m+1)/2}|x|^{m+1} \tag{3.16}$$

if

$$2\mu = c(m-1) - 2, \tag{3.17}$$

thus generalizing the example of Djukic [13] with $c = 1$. To deduce this result of Rosenau from Example 2, rewrite (3.15) as

$$(t^{c+1}x')' + \lambda t^{\mu+c}|x|^m \operatorname{sgn} x = 0,$$

and let $\beta + q = 0$,

$$k = c + 1 \quad \text{and} \quad \frac{(m+1)k - (m+3)}{2} = \mu + c. \tag{3.18}$$

It is easy to verify that (3.17) and (3.18) are equivalent, and that (3.9) then reduces to (3.16).

Example 6. Taking $k = 0$, $\lambda = -1$ and $m = -2$ in Example 2 yields a result of Vujanowic [14]; namely, that if

$$x'' - t^{-1/2}x^{-2} \operatorname{sgn} x = 0,$$

then

$$t(x')^2 - xx' + 2t^{1/2}|x|^{-1} = c.$$

Example 7. Taking $\lambda = 1$, $k = 2$, and $m = 5$ in Example 2 shows that the Lane–Emden equation

$$(t^2x')' + t^2x^5 = 0$$

has the first integral

$$I(t, x, x') = t^3(x')^2 + t^2xx' + \tfrac{1}{3}t^3x^6.$$

This is equivalent to a result of Jones and Ames [3].

Example 8. Lewis [15] has shown that the equation

$$x'' + \omega^2(t)x = 0$$

has first integrals of the form

$$I(t, x, x') = r^{-2}x^2 + (rx' - r'x)^2,$$

where r is any solution of

$$r'' + \omega^2(t)r = r^{-3}.$$

Lemma 1 provides a more direct result in some special cases. Let

$$h(t) = c_0 + c_1t + c_2t^2 > 0, \qquad t \in J,$$

and suppose that

$$x'' + \lambda(h(t))^{-2}x = 0, \qquad t \in J.$$

Then Lemma 1 with $a = 0$ and $m = 1$ implies that

$$h(x')^2 + \left(\frac{h''}{2} + \frac{\lambda}{h}\right)x^2 - h'xx' = c.$$

A special case of this (with $\lambda = 1$ and $h(t) = t^2$) has been given previously by Lewis [15] and by Eliezer and Gray [16].

Example 9. Consider the constant coefficient homogeneous linear equation

$$x'' + \alpha x' + \lambda x = 0, \tag{3.19}$$

which is of the form governing the damped harmonic oscillator, and can be rewritten as

$$(e^{\alpha t}x')' + \lambda e^{\alpha t}x = 0.$$

This is of the form (2.24) with $m = 1$, $y_0 = 1$, $r(t) = e^{\alpha t}$, $\beta = 0$ and $H(t) = e^{-\alpha t}$, so that (2.23) holds. Substitution into (2.14) (with $A = 0$) yields the elementary conclusion that

$$e^{\alpha t}[(x')^2 + \alpha xx' + \lambda x^2] = c$$

for any solution of (3.19).

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