1985

On t_∞ quasi-similarity of linear systems

William F. Trench, *Trinity University*

TRENCH, WILLIAM F.
1985
Annali di Matematica pura ed applicata
(IV), Vol. CXLII, pp. 293-302

On t_∞ Quasi-Similarity of Linear Systems (*).

WILLIAM F. TRENCH

Summary. – *Conti defined t_∞ similarity of systems* (1) $x' = A(t)x$ *and* (2) $y' = B(t)y$, *and showed that it is an equivalence relation which preserves uniform and strict stability. Here the definition is weakened by imposing less stringent integrability conditions, some in terms of perhaps conditionally convergent improper integrals, on the matrix function relating A and B. The extended relation, t_∞ quasi-similarity, is not symmetric or transitive; however, it is shown that if* (2) *is t_∞ quasi-similar to* (1) *and* (1) *is uniformly, uniformly asymptotically, or strictly stable, then so is* (2). *Results are also given concerning linear asymptotic equilibrium of* (2) *in the case where* (1) *is stricly stable or has linear asymptotic equilibrium.*

Here we consider an old problem: If the linear homogeneous system

$$x' = A(t)x\,, \qquad t \geqslant a\,, \tag{1}$$

has certain properties in connection with stability and asymptotic behavior of solutions, then what conditions connecting the matrix functions A and B imply that a second system

$$y' = B(t)y\,, \qquad t \geqslant a\,, \tag{2}$$

has the same or related properties? This problem has been investigated extensively; nevertheless, we believe that the results given here are nontrivial improvements over those previously published, because we introduce an extension of CONTI's definition [2] of t_∞ similarity in which the integral conditions relating the matrices A and B require only ordinary (that is, perhaps conditional) convergence of some of the improper integrals that occur. Integrability conditions of this kind are unusual in the literature on systems; see, e.g., WINTNER [9], HALLAM [5], DOLLARD and FRIEDMAN [4], and the author [7], [8].

We assume throughout that A and B are continuous on $[a, \infty)$, and we let X and Y denote fundamental matrices for (1) and (2). We study uniform stability, uniform asymptotic stability, strict (restrictive) stability, and linear asymptotic equilibrium. These are standard terms, whose definitions can be found in [1] or [6]. For our purposes it is convenient to state necessary and sufficient conditions for a

(*) Entrata in Redazione il 2 gennaio 1985.

Indirizzo dell'A.: Department of Mathematics and Computer Science, Drexel University, Philadelphia, PA 19104, U.S.A.

system to possess each of these properties, as in the following lemma. For the proof, see [1] or [6].

LEMMA 1. – Let Z be a fundamental matrix for the system

$$z' = C(t)z\,, \qquad t \geqslant a\,, \tag{3}$$

where C is continuous on $[a, \infty)$. Then (3)

(*a*) is uniformly stable if and only if there is a constant M such that

$$\|Z(t)Z^{-1}(\tau)\| \leqslant M\,, \qquad a \leqslant \tau \leqslant t\,;$$

(*b*) is uniformly asymptotically stable if and only if there are constants M and $\nu > 0$ such that

$$\|Z(t)Z^{-1}(\tau)\| \leqslant M \exp\left[-\nu(t-\tau)\right]\,, \qquad a \leqslant \tau \leqslant t\,;$$

(*c*) is strictly stable if and only if Z and Z^{-1} are bounded on $[a, \infty)$, or, equivalently, there is a constant M such that

$$\|Z(t)Z^{-1}(\tau)\| \leqslant M\,, \qquad t, \tau \geqslant a\,;$$

(*d*) has linear asymptotic equilibrium if and only if $Z(\infty) = \lim\limits_{t\to\infty} Z(t)$ exists and is invertible.

We take the underlying field to be either the real or complex numbers. Let $\mathfrak{R}$ be the set of continuous $n \times n$ matrix functions C on $[a, \infty)$ such that $\int^{\infty} C(t)\,dt$ converges (perhaps conditionally), and let $\mathbb{S}$ be the set of $n \times n$ matrix functions S such that S' is continuous and S and S^{-1} are bounded on $[a, \infty)$. It is convenient to state the general form of our main assumption as follows.

Assumption $\mathcal{A}_k(A, B, S)$: There is an S in $\mathbb{S}$ such that the function

$$F_0 = S' + SB - AS \tag{4}$$

is in $\mathfrak{R}$, and either

$$\int^{\infty} \|F_0(t)\|\,dt < \infty \tag{5}$$

or, for some $k \geqslant 1$, there are matrix functions $F_1, \ldots, F_k$, $G_1, \ldots, G_k$, $M_0, \ldots, M_k$,

$N_0, \ldots, N_k$ in $\mathfrak{R}$ which satisfy

$$F_0 = M_0 + N_0\,, \tag{6}$$

$$G_j(t) = \int_t^\infty N_{j-1}(s)\,ds\,, \qquad 1 \leqslant j \leqslant k\,, \tag{7}$$

$$F_j = G_j B - A G_j = M_j + N_j\,, \quad 1 \leqslant j \leqslant k\,, \tag{8}$$

such that

$$N_k = 0 \tag{9}$$

and

$$\int^\infty \left\| \sum_{j=0}^{k} M_j(t) \right\| dt < \infty\,. \tag{10}$$

If this assumption holds we say that B is t_∞ quasi-similar to A, and write

$$B \sim A\colon (k, S) \qquad (k = 0 \text{ if } (5) \text{ holds}) \tag{11}$$

if we wish to emphasize the role of k and S. If (11) holds for some k and S, we also say that the system (2) is t_∞ quasi-similar to (1).

When $k = 0$, t_∞ quasi-similarity of systems reduces to t_∞ similarity as defined by CONTI [2], who showed that the latter is an equivalence relation which preserves uniform and strict stability, but not stability. It can also be shown that it preserves uniform asymptotic stability, but not linear asymptotic equilibrium. CONTI addressed the latter question in [3], and showed that if $\mathcal{A}_0(A, B, S)$ holds with some S in S such that $\lim_{t\to\infty} S(t)$ and $\lim_{t\to\infty} S^{-1}(t)$ both exist, then (2) has linear asymptotic equilibrium if and only if (1) does.

Since the integrals in (7) may converge conditionally, t_∞ quasi-similarity is not symmetric or transitive if $k \geqslant 1$.

REMARK 1. – The partition of F_j into M_j and N_j in (6) and (8) is not unique. One possibility is that assumption $\mathcal{A}_k(A, B, S)$ holds with $M_0 = \ldots = M_{k-1} = 0$; then (7), (8), (9), and (10) take the simpler form

$$G_j(t) = \int_t^\infty F_{j-1}(s)\,ds\,, \qquad 1 \leqslant j \leqslant k\,,$$

$$F_j = G_j B - A G_j\,, \quad 1 \leqslant j \leqslant k\,,$$

and

$$\int^\infty \|F_k(t)\|\,dt < \infty\,.$$

Another possibility is to require that $M_0, \dots, M_k$ be separately absolutely integrable, which is stronger than (10); that is, to remove from F_{j-1} its absolutely integrable part before computing G_j. Our formulation of assumption $\mathcal{A}_k(A, B, S)$ includes these possibilities, but is also more general.

If assumption $\mathcal{A}_k(A, B, S)$ holds, let

$$\Gamma_0 = I\,; \qquad \Gamma_j = I + S^{-1}\sum_{i=1}^{j} G_i\,, \qquad 1 \leqslant j \leqslant k\,, \tag{12}$$

and

$$H_j = \sum_{i=0}^{j} M_i\,, \qquad 0 \leqslant j \leqslant k\,. \tag{13}$$

In terms of (13) we can combine (5) and (10) as

$$\int^{\infty} \|H_k(t)\|\,dt < \infty\,. \tag{14}$$

(See (6) and (9) with $k = 0$.)

The following lemma is basic for our results. The special case with $k = 0$ is essentially due to SANSONE and CONTI [6, p. 492].

LEMMA 2. – If assumption $\mathcal{A}_k(A, B, S)$ holds, then there is an $a_1 \geqslant a$ such that

$$H(t) = \Gamma_k^{-1}(t)\,S^{-1}(t)\,X(t)\left[X^{-1}(\tau)\,S(\tau)\,\Gamma_k(\tau)\,Y(\tau) + \int_\tau^t X^{-1}(s)\,H_k(s)\,Y(s)\,ds\right], \qquad t,\ \tau \geqslant a_1\,. \tag{15}$$

PROOF. – We first observe from (12) and the boundedness of S^{-1} that Γ_k^{-1} exists on $[a_1, \infty)$ for some $a_1 \geqslant a$, and that

$$\lim_{t\to\infty} \Gamma_k^{-1}(t) = I\,. \tag{16}$$

Since $(X^{-1})' = -X^{-1}A$, (4) implies that $(X^{-1}SY)' = X^{-1}F_0Y$. Integrating this and multiplying the result by $S^{-1}X$ yields

$$Y(t) = S^{-1}(t)\,X(t)\left[X^{-1}(\tau)\,S(\tau)\,Y(\tau) + \int_\tau^t X^{-1}(s)\,F_0(s)\,Y(s)\,ds\right], \qquad t,\ \tau \geqslant a\,, \tag{17}$$

which is equivalent to (15) if $k = 0$. We now show by finite induction on j that if $k \geqslant 1$, then

$$\begin{aligned}\Gamma_j(t)\,Y(t) = S^{-1}(t)\,X(t)\Big[&X^{-1}(\tau)\,S(\tau)\,\Gamma_j(\tau)\,Y(\tau) + \\ &+ \int_\tau^t X^{-1}(s)\,H_j(s)\,Y(s)\,ds + \int_\tau^t X^{-1}(s)\,N_j(s)\,Y(s)\,ds\Big], \qquad 1 \leqslant j \leqslant k;\ t,\ \tau \geqslant a\,.\end{aligned} \tag{18}$$

Since $\Gamma_0 = I$, this is equivalent to (17) when $j = 0$, because of (6) and (13). With $j = k$ it implies (15), since $N_k = 0$. Suppose (18) holds for some j, $0 \leqslant j \leqslant k-1$. Then

$$\begin{aligned}(X^{-1}G_{j+1}Y)' &= X^{-1}(-N_j + G_{j+1}B - AG_{j+1})Y \qquad (\text{see } (7))\\ &= X^{-1}(-N_j + F_{j+1})Y \qquad (\text{see } (8))\\ &= X^{-1}(-N_j + M_{j+1} + N_{j+1})Y\,.\end{aligned}$$

Solving this for $X^{-1}N_jY$ and integrating yields

$$\int_\tau^t X^{-1}(s)N_j(s)Y(s)\,ds = X^{-1}(\tau)G_{j+1}(\tau)Y(\tau) - X^{-1}(t)G_{j+1}(t)Y(t) + \\ + \int_\tau^t X^{-1}(s)M_{j+1}(s)Y(s)\,ds + \int_\tau^t X^{-1}(s)N_{j+1}(s)Y(s)\,ds\,.$$

Substituting this into (18) and invoking (12) and (13) completes the finite induction.

THEOREM 1. – Suppose assumption $\mathcal{A}_k(A, B, S)$ holds and (1) is uniformly stable, uniformly asymptotically stable, or strictly stable. Then (2) has the same property.

PROOF. – It suffices to confine our attention to $[a_1, \infty)$, since the stability properties of (1) and (2) on this interval are the same as on $[a, \infty)$. Define

$$g_\nu(t,\tau) = \|X(t)X^{-1}(\tau)\| \exp[\nu(t-\tau)]\,, \qquad h_\nu(t,\tau) = \|Y(t)Y^{-1}(\tau)\| \exp[\nu(t-\tau)]$$

with $\nu \geqslant 0$. Multiplying (15) on the right by $[\exp \nu(t-\tau)]Y^{-1}(\tau)$ and using routine estimates which invoke the boundedness of Γ_k, S, Γ_k^{-1}, and S^{-1} yields the inequality

$$h_\nu(t,\tau) \leqslant c_1 g_\nu(t,\tau) + c_2\left|\int_\tau^t g_\nu(t,s)\|H_k(s)\|h_\nu(s,\tau)\,ds\right|, \qquad t,\ \tau \geqslant a_1\,, \tag{19}$$

where c_1 and c_2 are constants. The remainder of the proof follows SANSONE and CONTI [6], who considered uniform and strict (not uniform asymptotic) stability with $k = 0$. From Lemma 1 (*a*), (*b*), there is a constant M such that

$$g_\nu(t,\tau) \leqslant M\,, \qquad a_1 \leqslant \tau \leqslant t\,, \tag{20}$$

with $\nu = 0$ if (1) is uniformly stable, or with some $\nu > 0$ if (1) is uniformly asymptotically stable. From (19) and (20),

$$h_\nu(t,\tau) \leqslant M\left[c_1 + c_2\left|\int_\tau^t \|H_k(s)\|h_\nu(s,\tau)\,ds\right|\right], \tag{21}$$

if $a_1 \leqslant \tau \leqslant t$. Applying Gronwall's inequality with respect to t yields

$$h_\nu(t, \tau) \leqslant Mc_1\left[\exp Mc_2\left|\int_\tau^t \|H_k(s)\|\, ds\right|\right] \tag{22}$$

if $a_1 \leqslant \tau \leqslant t$. Therefore, from (14),

$$h_\nu(t, \tau) \leqslant N = Mc_1 \exp\left[Mc_2\int_a^\infty \|H_k(s)\|\, ds\right] \tag{23}$$

if $a_1 \leqslant \tau \leqslant t$, and this proves that (2) is uniformly stable if $\nu = 0$, or asymptotically uniformly stable if $\nu > 0$, again by Lemma 1 (a), (b).

Now suppose (1) is strictly stable. From Lemma 1 (c), there is an M such that $g_0(t, \tau) \leqslant M$ if $t, \tau \geqslant a_1$. Therefore, (21), (22), and (23) hold with $\nu = 0$ and $t, \tau \geqslant a_1$; hence (2) is strictly stable, by Lemma 1 (c). (Note: When $a \leqslant t \leqslant \tau$, an easy variation of the standard Gronwall lemma—necessitated by the fact that t is in the lower limit when the absolute value bars are deleted on the right in (21)—is required to infer (22) from (21).) This completes the proof.

COROLLARY 1. – Suppose (1) is strictly stable and $B^* \sim A^*$: (k, S) (where $*$ = conjugate transpose) for some integer $k \geqslant 0$ and S in $\mathcal{S}$. Then (2) is strictly stable.

PROOF. – Since a system is strictly stable if and only if its adjoint is, it suffices to apply Theorem 1 to the adjoint systems $x' = -A^*(t)x$, $y' = -B^*(t)y$.

THEOREM 2. – Suppose (1) is strictly stable and assumption $\mathcal{A}_k(A, B, S)$ holds for some $k \geqslant 0$ and S in $\mathcal{S}$ such that

$$\lim_{t\to\infty} S^{-1}(t)X(t) \tag{24}$$

exists. Then (2) has linear asymptotic equilibrium.

PROOF. – Theorem 1 implies that (2) is strictly stable; hence Y^{-1} is bounded (Lemma 1 (c)). Moreover,

$$Y(\infty) = \lim_{t\to\infty} Y(t) \tag{25}$$

exists because of (14), (15), (16), and the existence of (24), and is invertible because Y^{-1} is bounded. Now Lemma 1 (d) implies the conclusion.

In this general form Theorem 2 is of limited usefulness, since it requires that we know a fundamental matrix for (1). However, the following corollaries of Theorem 2 are more readily applicable.

COROLLARY 2. – Suppose assumption $\mathcal{A}_k(A, B, S)$ holds for some $k \geqslant 0$ and S in $\mathcal{S}$, with

$$A = S'S^{-1}, \tag{26}$$

so that (4) and (8) reduce to $F_0 = SB$ and $F_j = G_jB - S'S^{-1}G_j = M_j + N_j$. Then (2) has linear asymptotic equilibrium.

PROOF. – With A as in (26), we can take $X = S$, so that (24) exists and (1) is strictly stable, because of Lemma 1 (*c*) and the definition of $\mathcal{S}$. This implies the conclusion.

With $k = 0$ and $S = I$, this yields the best known sufficient condition for linear asymptotic equilibrium of (2), as follows.

COROLLARY 3. – If $\int^{\infty} \|B(t)\|\, dt < \infty$, then (2) has linear asymptotic equilibrium.

WINTNER [9] attributed this result to Bôcher, and improved on it as in the next corollary, which follows from Corollary 2 on taking $k \geqslant 1$, $S = I$, and $M_0 = \dots$ $\dots M_{k-1} = 0$.

COROLLARY 4. – Suppose that for some integer $k \geqslant 1$ the integrals

$$G_j(t) = \int_t^{\infty} G_{j-1}(s)B(s)\, ds\,, \quad 1 \leqslant j \leqslant k \ \ (G_0 = I)\,, \tag{27}$$

converge, and

$$\int^{\infty} \|G_k(t)B(t)\|\, dt < \infty\,. \tag{28}$$

Then (2) has linear asymptotic equilibrium.

If S is in $\mathcal{S}$ and S' is in $\mathcal{R}$, then $S(\infty) = \lim_{t\to\infty} S(t)$ exists and is invertible. Therefore, the same is true of $S^{-1}(\infty) = \lim_{t\to\infty} S^{-1}(t)$. Moreover, if (1) has linear asymptotic equilibrium, then it is strictly stable. This yields the following corollary of Theorem 2.

COROLLARY 5. – Suppose (1) has linear asymptotic equilibrium and assumption $\mathcal{A}_k(A, B, S)$ holds for some $k \geqslant 0$ and S in $\mathcal{S}$ such that S' is in $\mathcal{R}$. Then (2) has linear asymptotic equilibrium.

We will use Corollary 5 in the proof of the next theorem. Here it is convenient to rewrite (2) as

$$y' = \big(A(t) + C(t)\big)y\,; \tag{29}$$

thus

$$C = B - A\,. \tag{30}$$

THEOREM 3. – Suppose

$$\int^\infty \|A(t)\|\, dt < \infty \tag{31}$$

and there is an S in $\mathcal{S}$ such that the function

$$N_0 = S' + SC \tag{32}$$

is in $\mathcal{R}$. Suppose also that for some $k \geq 1$ there are functions $K_1, \ldots, K_k$, $N_1, \ldots, N_k$ in $\mathcal{R}$ such that

$$\left(\int_t^\infty N_{j-1}(s)\, ds\right) C(t) = K_j(t) + N_j(t)\,, \qquad 1 \leq j \leq k\,, \tag{33}$$

where $N_k = 0$ and

$$\int^\infty \left\|\sum_{j=1}^k K_j(t)\right\| dt < \infty\,. \tag{34}$$

Then (29) is strictly stable; moreover, if S' is in $\mathcal{R}$, then (29) has linear asymptotic equilibrium.

PROOF. – We first show that

$$A + C \sim A\colon (k, S)\,. \tag{35}$$

With $B = A + C$, (4) becomes

$$F_0 = SA - AS + S' + SC = M_0 + N_0\,, \tag{36}$$

with $M_0 = SA - AS$ and N_0 as in (32). Because of (31),

$$\int^\infty \|M_0(t)\|\, dt < \infty\,. \tag{37}$$

With $N_0, \ldots, N_k$ as defined in our present assumptions, let G_j be given by (7), and define

$$F_j = G_j C + G_j A - AG_j\,, \tag{38}$$

which is equivalent to the first equality in (8), with $B = A + C$. From (7), (33), and (38), we can rewrite (38) as $F_j = M_j + N_j$ (see the second equality in (8)),

where N_j is as in our present assumption, and

$$M_j = K_j + G_j A - A G_j\,, \quad 1 \leqslant j \leqslant k\,. \tag{39}$$

Since (31), (34), (37), and (39) imply (10), this proves (35).

Because of (31) and Corollary 3 (applied to (1)), (1) has linear asymptotic equilibrium and is therefore strictly stable. Hence, (35) implies the stated conclusions, by Theorem 1 and Corollary 4.

Theorem 3 implies results obtained by DOLLARD and FRIEDMAN [4; Thm. 1 (*a*), (*b*)] with $S = I$.

REMARK 2. – If A satisfies (31) and

$$\int^{\infty} \|S'(t) + S(t)\,C(t)\|\,dt < \infty\,,$$

then $\int^{\infty} \|F_0(t)\|\,dt < \infty$ (recall (36)), so (2) is t_∞ similar to (1), and the conclusion of Theorem 3 also holds.

REMARK 3. – A system has linear asymptotic equilibrium if and only if its adjoint system does; hence, other conditions implying linear asymptotic equilibrium of (1) can be obtained by applying the above results to the adjoint systems $x' = -A^*(t)x$ and $y' = -B^*(t)y$. For example, doing this and taking conjugate transposes shows that Corollary 4 also holds if (27) and (28) are replaced by

$$G_j(t) = \int_t^{\infty} B(s)\,G_{j-1}(s)\,ds\,, \quad 1 \leqslant j \leqslant k \;\; (G_0 = I)$$

and

$$\int^{\infty} \|B(t)\,G_k(t)\|\,dt < \infty\,.$$

(See WINTNER [9].)

REMARK 4. – CONTI [3] showed that (2) has linear asymptotic equilibrium if and only if there is a matrix S in $\mathcal{S}$ such that S' is in R and either

$$\int^{\infty} \|S'(t) + S(t)B(t)\|\,dt < \infty \tag{40}$$

or

$$\int^{\infty} \|S'(t) - B(t)S(t)\|\,dt < \infty\,. \tag{41}$$

Corollary 5 with $k = 0$ and $A = 0$ also implies the sufficiency of (40). This and the argument advanced in Remark 3 imply the sufficiency of (41).

REMARK 5. – By using the (possibly conditional) integrability assumptions imposed on the matrices of the systems in question in such a way as to exhibit an appropriate matrix S, CONTI showed that the results of DOLLARD and FRIEDMAN [4] and WINTNER [9] follow from the main theorem in [3].

Acknowledgment. – The author thanks Professor CONTI for bringing [3] to his attention.

BIBLIOGRAPHY

[1] L. CESARI, *Asymptotic behavior and stability problems in ordinary differential equations*, Ergebn. d. Math. Vol. 16 (2nd ed.), Academic Press, New York, 1963.

[2] R. CONTI, *Sulla t-similitudine tra matrici e la stabilità dei sistemi differenziali lineari*, Atti Acc. Naz. Lincei, Rend. Cl. Fis. Mat. Nat., (8) **49** (1955), pp. 247-250.

[3] R. CONTI, *Equazioni differenziali lineari asintoticamente equivalenti a* $\dot{x} = 0$, Riv. Mat. Univ. Parma, (4) **5** (1979), pp. 847-853.

[4] J. D. DOLLARD - C. N. FRIEDMAN, *Asymptotic behavior of solutions of linear ordinary differential equations*, J. Math. Anal. Appl., **66** (1978), pp. 394-398.

[5] T. G. HALLAM, *Asymptotic integration of a nonhomogeneous differential equation with integrable coefficients*, Czechoslovak Math. J., **24** (1971), pp. 661-671.

[6] G. SANSONE - R. CONTI, *Nonlinear differential equations*, MacMillan, New York, 1964.

[7] W. F. TRENCH, *Systems of differential equations subject to mild integral conditions*, Proc. Amer. Math. Soc., **87** (1983), pp. 263-270.

[8] W. F. TRENCH, *Asymptotics of differential systems with deviating arguments*, Proc. Amer. Math. Soc., **92** (1984), pp. 219-224.

[9] A. WINTNER, *On a theorem of Bôcher in the theory of ordinary linear differential equations*, Amer. J. Math., **76** (1954), pp. 183-190.