Asymptotic integration of $y^{(n)} + p(t)y^{\gamma} = f(t)$ under mild integral smallness conditions

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Asymptotic Integration of $y^{(n)} + P(t)y' = f(t)$ under Mild Integral Smallness Conditions

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This paper gives sufficient conditions for the equation

(1) $y^{(n)} + P(t)y' = f(t)$  \hspace{1cm} (n \geq 2)

to have solutions which behave like polynomials of degree $< n$ as $t \to \infty$. This question has been investigated by many authors, but, to our knowledge, always under integral smallness conditions on $P$ and $f$ which require absolute convergence of all improper integrals in question. Also, most authors have assumed that $\gamma > 0$. (For exceptions to this, see [1], [2], and [4].) Here we allow $\gamma$ to be any real number. Moreover, our integral smallness conditions require only ordinary (i.e., perhaps conditional) convergence, except for a condition on $P$ in Theorem 2 which does require absolute convergence, but is still considerably weaker than the usual condition.

We assume throughout that $P$ and $f$ are real-valued and continuous on $(0, \infty)$ and that $\gamma$ is real. When we say that an improper integral converges, we mean that it may converge conditionally, unless, of course, it is clear that the integrand is non-negative.

Theorem 1. Suppose $\nu$ and $m$ are integers, $0 \leq \nu \leq m \leq n - 1$, and let $\alpha$ be a non-negative number such that

(2) $\nu - \alpha < m$;

moreover, suppose that $\alpha < 1$ if $\nu \neq 0$. Assume that the integrals

(3) $\int_0^\infty t^{n-\nu+\alpha+m-1} P(t) dt$

and

(4) $\int_0^\infty t^{n-\nu+\alpha} f(t) dt$

converge. Let

(5) $q(t) = \sum_{j=0}^{m} a_j t^j$.
where \( a_r \) are given real constants, with \( a_m > 0 \). Then (1) has a solution \( y_0 \), defined for sufficiently large \( t \), such that

\[
(6) \quad y_0^\prime(t) = p(t) + o(t^{-r-n}), \quad 0 \leq r \leq n - 1.
\]

The following two lemmas will be useful in proving Theorem 1 as well as Theorem 2, which is stated below.

**Lemma 1.** Suppose \( u \in C[t_0, \infty) \) for some \( t_0 \geq 0 \) and let \( a \) and \( b \) be constants, \( 0 \leq a \leq b \). Suppose also that \( \int_{t_0}^\infty t^b u(t) dt \) converges, and define

\[
\rho_A(t) = \sup_{t \geq t_0} \left| \int_t^\infty s^a u(s) ds \right|.
\]

Then

\[
(7) \quad \left| \int_{t_0}^\infty (s - t)^a u(s) ds \right| \leq 2\rho_A(t)t^{a-b}, \quad t \geq t_0 \geq t_0.
\]

**Proof.** Let \( U(t) = \int_{t_0}^t s^b u(s) ds \), and note that

\[
|U(t)| \leq \rho_A(t).
\]

Now,

\[
(s - t)^a u(s) = -\left(1 - \frac{t}{s}\right)^a s^{a-b} U'(s),
\]

so integrating by parts yields

\[
(9) \quad \int_{t_0}^t (s - t)^a u(s) ds = -\left(1 - \frac{t}{s}\right)^a s^{a-b} U(s) \bigg|_t^{t_0} + \int_t^{t_0} U(s) \frac{d}{ds} \left[\left(1 - \frac{t}{s}\right)^a s^{a-b}\right] ds.
\]

But

\[
\frac{d}{ds} \left[\left(1 - \frac{t}{s}\right)^a s^{a-b}\right] = \left(1 - \frac{t}{s}\right)^a \frac{d}{ds} \left(s^{a-b}\right) + s^{a-b} \frac{d}{ds} \left[\left(1 - \frac{t}{s}\right)^a\right],
\]

where the first product is nonpositive and the second is positive if \( s > t \). This enables us to let \( t \to \infty \) in (9) and infer that

\[
\left| \int_{t_0}^\infty (s - t)^a u(s) ds \right| \leq \left(1 - \frac{t}{s}\right)^a t^{a-b} |U(t)| - \int_{t_0}^\infty \left(1 - \frac{t}{s}\right)^a |U(s)| \frac{d}{ds} (s^{a-b}) ds
\]

\[
+ \int_{t_0}^\infty |U(s)| s^{a-b} \frac{d}{ds} \left(1 - \frac{t}{s}\right)^a ds
\]

\[
\leq 2\rho_A(t)t^{a-b}.
\]
(see (8)). This completes the proof of Lemma 1.

**Lemma 2.** Suppose \( u \in C[t_0, \infty) \) for some \( t_0 \geq 0 \) and

\[
\int_0^\infty t^{n-\nu-1+s} u(t) dt
\]

converges, where \( \nu \) and \( \alpha \) are as in Theorem 1. Define

\[
w(t) = \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} u(s) ds \quad \text{if } \nu = 0,
\]

or

\[
w(t) = \int_{t_0}^t \frac{(t-\lambda)^{n-1}}{(n-1)!} d\lambda \int_0^\infty \frac{(\lambda-s)^{n-\nu-1}}{(n-\nu-1)!} u(s) ds \quad \text{if } 1 \leq \nu \leq n-1,
\]

and let

\[
\rho(t) = \sup_{t \geq s} \left| \int_s^\infty s^{n-\nu-1+s} u(s) ds \right|.
\]

Then \( w \in C^{(\nu)}[t_0, \infty) \),

\[
|w^{(r)}(t)| \leq \frac{2\rho(t)t^{s-r-\alpha}}{(n-r-1)!}, \quad \nu \leq r \leq n-1,
\]

and, if \( \nu \geq 1 \),

\[
|w^{(r)}(t)| \leq \frac{2\rho(t_0)t^{s-r-\alpha}}{(n-\nu-1)!} \frac{1}{\Gamma(j-r)}, \quad 0 \leq r \leq \nu - 1.
\]

Moreover,

\[
w^{(r)}(t) = o(t^{s-r-\alpha}), \quad 0 \leq r \leq n-1.
\]

**Proof.** From Lemma 1 with

\[
a = n-r-1, \quad b = n-\nu-1+\alpha, \quad t = t_0,
\]

(15)

\[
\left| \int_t^\infty (t-s)^{n-r-1} u(s) ds \right| \leq 2\rho(t)t^{s-r-\alpha}, \quad \nu \leq r \leq n-1.
\]

This implies that \( w \) as defined by (10) or (11) is in \( C^{(\nu)}[t_0, \infty) \) and satisfies (12) and (14) for \( \nu \leq r \leq n-1 \). Therefore, the proof is complete if \( \nu = 0 \). If \( 0 \leq r \leq \nu - 1 \), then

\[
|w^{(r)}(t)| \leq \frac{2}{(\nu-r-1)! (n-\nu-1)!} \int_{t_0}^t (t-\lambda)^{s-r-1} \lambda^{s-r-\alpha} \rho(\lambda) d\lambda
\]

from (11) and (15) (the latter with \( r=\nu \)). Since \( \rho \) is nonincreasing, we may replace
\( \rho(\lambda) \) by \( \rho(t_0) \) here, then replace \( t_0 \) by zero in the lower limit of integration (recall that \( \alpha \leq 1 \)), and integrate repeatedly by parts to obtain (13).

From (16),
\[
|w^{(r)}(t)| t^{-\nu+r+s} \leq \frac{2t^{s-1}}{(\nu-r-1)!(n-\nu-1)!} \int_{t_0}^t \rho(\lambda) \lambda^{-s} d\lambda,
\]
which implies (14) for \( 0 \leq r \leq \nu - 1 \). (If \( \int_0^\infty \rho(\lambda) \lambda^{-s} d\lambda < \infty \), this is obvious; if \( \int_0^\infty \rho(\lambda) \lambda^{-s} d\lambda = \infty \), it follows from 's Hospital's rule. Here again we have used the assumption that \( \alpha < 1 \).) This completes the proof of Lemma 2.

Proof of Theorem 1. For \( t_0 \geq 0 \), let \( H(t_0) \) be the Banach space of functions \( h \) in \( C^{(n-1)}[t_0, \infty) \) such that
\[
h^{(r)}(t) = 0(t^{s-r} - t), \quad 0 \leq r \leq n-1,
\]
with norm
\[
\|h\| = \sup_{t \geq t_0} \left\{ t^{-s+r} \sum_{r=0}^{n-1} t^r |h^{(r)}(t)| \right\}.
\]
For \( M > 0 \), let
\[
H_M(t_0) = \{ h \in H(t_0) \mid \|h\| \leq M \}.
\]
Since \( \nu \leq m \) and \( a_m > 0 \) in (5), there are constants \( M, \lambda, \) and \( T_0 \) such that if
\[
t_0 \geq T_0 \quad \text{and} \quad h \in H_M(t_0),
\]
then
\[
q(t) + h(t) \geq \frac{1}{2} a_m \frac{t^m}{m!}
\]
and
\[
|Q^{(r)}(t) + h^{(r)}(t)| \leq \lambda t^{m-r}, \quad 0 \leq r \leq n-1,
\]
for all \( t \geq t_0 \). (From (20), \( q+h \) is defined and real-valued on \([t_0, \infty) \) if (19) holds.) We assume henceforth that \( h, h_1, \) and \( h_2 \) are in \( H_M(t_0) \) for some \( t_0 \geq T_0 \). The constants appearing in estimates that follow may depend upon \( T_0 \), but they do not depend upon \( t_0, h, h_1, h_2, \) etc. We assume that \( t \geq t_0 \) throughout.

We will show that the transformation
\[
\hat{h} = \mathcal{F} h
\]
defined by

\begin{equation}
\hat{h}(t) = \int_t^{\infty} \frac{(t-s)^{\nu-1}}{(n-1)!} \left[-f(s) + P(s)(q(s)+h(s))^\nu\right] ds \quad \text{if } \nu = 0
\end{equation}

or by

\begin{equation}
\hat{h}(t) = \int_{t_0}^t \frac{(t-s)^{\nu-1}}{(n-1)!} \int_0^1 \frac{(\lambda-s)^{\nu-1}}{(n-1)!} \left[-f(s) + P(s)(q(s)+h(s))^\nu\right] ds d\lambda
\end{equation}

if \(1 \leq \nu \leq n-1\),

is a contraction mapping of \(H_{\nu}(t_0)\) into itself if \(t_0\) is sufficiently large. To this end, we first study the integral

\begin{equation}
F(t; h) = \int_t^{\infty} s^{n-\nu-1} P(s)(q(s)+h(s))^\nu ds.
\end{equation}

The convergence of this integral follows easily from Dirichlet's test and the convergence of (3); nevertheless, we will write out the details of the proof, because they will be useful in obtaining estimates that we need below. Let

\begin{equation}
\Phi(t) = \int_t^{\infty} s^{n-\nu-1+\alpha + \nu^s} P(s) ds,
\end{equation}

which exists, because (3) converges. If \(\tau \geq t\), then

\begin{align}
\int_t^{\tau} s^{n-\nu-1+\alpha} P(s)(q(s)+h(s))^\nu ds
&= -\int_t^{\tau} \Phi'(s)[s^{-\alpha}(q(s)+h(s))]^\nu ds
&= -\Phi(s)[s^{-\alpha}(q(s)+h(s))]^\nu ds
&\quad + \gamma \int_t^{\tau} \Phi(s)[s^{-\alpha}(q(s)+h(s))]^\nu \left[(s^{-\alpha}q(s))' + (s^{-\alpha}h(s))'\right] ds.
\end{align}

Now,

\begin{equation}
(s^{-\alpha}q(s))' = 0(s^{-\alpha})
\end{equation}

and

\begin{equation}
|(s^{-\alpha}h(s))'| \leq (m+1)M2^{-m-1-\alpha}.
\end{equation}

(See (17) and (18).) Since \(\Phi(t) = o(1)\), the last two inequalities together with (2), (20) and (21) enable us to let \(\tau \to \infty\) in (26) to obtain

\begin{equation}
F(t; h) = \Phi(t)[t^{-m}(q(t)+h(t))]^\nu
\end{equation}

\begin{align}
&\quad + \gamma \int_t^{\infty} \Phi(s)[s^{-\alpha}(q(s)+h(s))]^\nu \left[(s^{-\alpha}q(s))' + (s^{-\alpha}h(s))'\right] ds,
\end{align}
where the integral on the right converges absolutely.

We will now show that $F(t; h)$ satisfies a Lipschitz condition with respect to $h$. Applying the mean value theorem to $G(u) = u^r$ and invoking (20) if $r < 0$ or (21) with $r = 0$ if $r > 0$ yields the inequality

$$
\begin{align*}
&|\left[ t^{-m}(q(t) + h_1(t)) \right]^r - \left[ t^{-m}(q(t) + h_2(t)) \right]^r | \\
&\leq A_4 t^{-m} |h_1(t) - h_2(t)| \\
&\leq A_4 t^{-m - a} \| h_1 - h_2 \| 
\end{align*}
$$

(30)

(see (17)) for some constant $A_4$. With $Q_j(s)(j = 1, 2)$ defined by

$$
Q_j(s) = [s^{-m}(q(s) + h_j(s))]^{-1} [(s^{-m}q(s))' + (s^{-m}h_j(s))'],
$$

applying the mean value theorem to $G(u, v) = u^{-1} v$ and invoking (20) and (21) yields

$$
|Q_1(s) - Q_2(s)| \leq A_5 s^{-m-1} |h_1(s) - h_2(s)| + A_3 \| s^{-m}h_1(s) \| - (s^{-m}h_2(s))' |
$$

(31)

for suitable constants $A_5$ and $A_3$. (Here we have also used (2), (27), and (28) to obtain the first term on the right.) From (17) and (31),

$$
|Q_1(s) - Q_2(s)| \leq A_4 \| h_1 - h_2 \| s^{-m-1-a}
$$

(32)

for some constant $A_4$.

From (2), (29), (30) and (32),

$$
|F(t; h_1) - F(t; h_2)| \leq A_6 \| h_1 - h_2 \| t^{-m-a} \phi(t),
$$

(33)

for some constant $A_6$, with

$$
\phi(t) = \sup_{T \geq t} \Phi(T) = o(1).
$$

(34)

Here we have used (2) again.

The convergence of (4) and (25) imply that the function

$$
G(t; h) = \int_t^\infty s^{u^{-1}a} \left[ -f(s) + P(s)(q(s) + h(s))' \right] ds
$$

is defined on $[t_0, \infty)$. Moreover,

$$
|G(t; h)| \leq |G(t; 0)| + |G(t; h) - G(t; 0)|
$$

$$
= |G(t; 0)| + |F(t; h) - F(t; 0)|,
$$

so that invoking (33) with $h_1 = h$ and $h_2 = 0$ (and recalling that $\| h \| \leq M$) yields

$$
|G(t; h)| \leq \sigma(t) = A_5 M t^{r - m - a} + \sup_{T \geq t} |G(T; 0)|.
$$
Now Lemma 2 with \( u = -f + P(q + h)^r \) implies that \( \hat{h} \) as defined by (10) or (11) is in \( H(t_0) \), and that
\[
\| \hat{h} \| \leq K \sigma(t_0) \tag{35}
\]
for a suitable constant \( K \). Moreover, if \( \hat{h}_i = \mathcal{T}h_i \, (i = 1, 2) \), we can apply Lemma 2 with
\[
u = P[(q + h_1)^r - (q + h_2)^r],
\]
and conclude from (33) that
\[
\| \hat{h}_1 - \hat{h}_2 \| \leq K A_2 t_0^{a-m-a} \sigma(t_0) \| h_1 - h_2 \|. \tag{36}
\]
Since \( \sigma \) and \( \phi \) both decrease to zero as \( t \to \infty \), we can choose \( t_0 \) so that
\[
K \sigma(t_0) \leq M \tag{37}
\]
and
\[
KA_2 t_0^{a-m-a} \sigma(t_0) < 1. \tag{38}
\]
Now (35) and (37) imply that \( \mathcal{T} \) maps \( H_\mu(t_0) \) into itself, and (36) and (38) imply that \( \mathcal{T} \) is a contraction mapping. Therefore there is a function \( h_0 \) in \( H_\mu(t_0) \) such that \( h_0 = \mathcal{T}h_0 \). Since (23) or (24) holds with \( \hat{h} = h = h_0 \), the function \( y_0 = q + h_0 \) satisfies (1). Moreover, Lemma 2 (specifically, (14)) with \( u = w = h_0 \) implies that
\[
h_0^{r} = o(t^{r-r}), \quad 0 \leq r \leq n-1,
\]
and this implies (6). This completes the proof of Theorem 1.

We now consider the case where \( m = \nu \) and \( \alpha = 0 \), so that (2) does not hold; that is, we will give sufficient conditions for (1) to have a solution \( y_0 \) which satisfies
\[
y_0^{r} = \begin{cases} \frac{(a_0 + o(1))t^{r-r}/(v-r)!}{o(t^{r-r})}, & 0 \leq r \leq v, \\ o(t^{r-r}), & v+1 \leq r \leq n-1. \end{cases} \tag{39}
\]
A digression is needed to formulate this condition.

**Lemma 3.** Suppose \( u \in C[t_0, \infty) \) for some \( t_0 \geq 0 \) and \( \int_{t_0}^{\infty} t^{r-1}u(t)dt \) converges. Define
\[
I_0(t; u) = u(t)
\]
and
\[
I_j(t; u) = \int_{t}^{\infty} \frac{u(s)}{(j-1)!} (s-t)^{j-1} ds, \quad 1 \leq j \leq k. \tag{40}
\]
Then the integrals (40) converge and satisfy the inequalities

\begin{equation}
| I_j(t; u) | \leq \frac{2\delta(t) t^{j-k}}{(j-1)!}, \quad 1 \leq j \leq k,
\end{equation}

where

\[ \delta(t) = \sup_{r \geq t} \left| \int_r^s s^{k-1} u(s) ds \right|. \]

The integrals

\begin{equation}
\int_0^\infty t^{k-j-1} I_j(t; u) dt, \quad 0 \leq j \leq k-1,
\end{equation}

all converge, and if this convergence is absolute for some \( j_0 \) in \( \{0, 1, \ldots, k-1\} \), then it is absolute for \( j_0 \leq j \leq k-1 \).

\textbf{Proof.} The convergence of the integrals (40) and inequality (41) follow from Lemma 1. Since

\begin{equation}
I_j(t; u) = -I_{j-1}(t; u), \quad 1 \leq j \leq k-1,
\end{equation}

integration by parts yields

\[
\int_{t_1}^{t_2} t^{k-j-1} I_j(t; u) dt = \frac{t^{k-j}}{k-j} I_j(t; u) \bigg|_{t_1}^{t_2} + \frac{1}{k-j} \int_{t_1}^{t_2} t^{k-j} I_{j-1}(t; u) dt,
\]

so (41) and the assumed convergence of

\[ \int_0^\infty t^{k-1} I_0(t; u) dt = \int_0^\infty t^{k-1} u(t) dt \]

imply that (42) converges, by finite induction. If

\begin{equation}
\int_0^\infty t^{k-j-1} | I_j(t; u) | dt < \infty
\end{equation}

for some \( j < k-1 \), then

\begin{equation}
\int_0^\infty | I_j(s; u) | ds = o(t^{-k+j+1}),
\end{equation}

and

\[
\int_{t_1}^{t_2} t^{k-j-1} \left( \int_0^\infty | I_j(s; u) | ds \right) dt
= \frac{t^{k-j-1}}{k-j-1} \int_{t_1}^{t_2} | I_j(s; u) | ds \bigg|_{t_1}^{t_2} + \frac{1}{k-j-1} \int_{t_1}^{t_2} t^{k-j-1} | I_j(t; u) | dt.
\]
Now (44) and (45) imply that
\[
\int_{0}^{\infty} t^{n-\gamma-1} \left( \int_{t}^{\infty} |I_{j}(s; u)| ds \right) < \infty,
\]
which in turn implies that
\[
\int_{0}^{\infty} t^{n-\gamma-1} |I_{j+1}(t; u)| dt < \infty,
\]
since
\[
|I_{j+1}(t; u)| \leq \int_{t}^{\infty} |I_{j}(s; u)| ds.
\]
(See (43) with \(j\) replaced by \(j+1\).) This completes the proof of Lemma 1.

If \(1 \leq j_{0} \leq k-1\), there are functions \(u\) such that
\[
\int_{0}^{\infty} t^{n-\gamma-1} |I_{j}(t; u)| dt \begin{cases} = \infty & \text{if } 0 \leq j \leq j_{0} - 1, \\ < \infty & \text{if } j_{0} \leq j \leq k-1. \end{cases}
\]
For example, the function
\[
u(t) = t^{-k} \sin t
\]
satisfies this condition with \(j_{0} = 1\). A rather tedious argument involving repeated integration by parts shows that the function
\[
u(t) = t^{-k} \cos ((\log t)^{\gamma+1})
\]
satisfies (46) if \(j_{0}^{-1} < \alpha < (j_{0} - 1)^{-1}\).

**Theorem 2.** Let \(\nu\) be an integer in \(\{0, 1, \ldots, n-1\}\) and suppose the integrals
\[
\int_{0}^{\infty} t^{n-1+\nu(t-1)} P(t) dt
\]
and
\[
\int_{0}^{\infty} t^{n-\gamma-1} f(t) dt
\]
converge. Suppose also that
\[
\int_{0}^{\infty} t^{\gamma(t-1)} |I_{n-1}(t; P)| dt < \infty \quad \text{if } \gamma \geq 1,
\]
or that
\[
\int_{t_0}^{\infty} |I_{n-1}(t; Q)| \, dt < \infty \quad \text{if } \tau < 1,
\]
where
\[
Q(t) = t^{\nu(\tau - 1)} P(t).
\]

Let \(a_\nu\) be an arbitrary positive constant. Then (1) has a solution \(y_0\) which is defined for sufficiently large \(t\) and satisfies (39).

Proof. For \(t_0 \geq 0\), let \(H(t_0)\) be the Banach space of functions \(h\) in \(C^{(n-1)}(t_0, \infty)\) such that
\[
h^{(r)}(t) = 0(t^{r-\tau}), \quad 0 \leq r \leq n-1,
\]
with norm
\[
\|h\| = \sup_{t \geq t_0} \left\{ \sum_{r=0}^{n-1} t^{r-\nu} |h^{(r)}(t)| \right\},
\]
and let \(H_{\nu}(t_0)\) be as in (18). It is convenient here to write
\[
u(t) = \frac{a_\nu}{\nu!} t^{\tau} + h(t), \quad h \in H_{\nu}(t_0).
\]
Since \(a_\nu > 0\), there are constants \(M, \lambda\) and \(T_0\) such that
\[
u(t) \geq \frac{1}{2} \frac{a_\nu}{\nu!} t^{\nu}
\]
and
\[
|\nu^{(r)}(t)| \leq \lambda t^{r-\tau}, \quad 0 \leq r \leq n-1,
\]
if (51) holds and \(t \geq t_0 \geq T_0\), which we assume henceforth. As in the proof of Theorem 1, we will show that \(\mathcal{J}\) as defined by (22) and (23) or (24) is a contraction mapping of \(H_{\nu}(t_0)\) into itself if \(t_0\) is sufficiently large; therefore, we first consider the integral
\[
F(t; h) = \int_{t_0}^{\infty} s^{n-\nu-1} P(s)(\nu(s))^\tau \, ds
\]
(recall (51)), which is the appropriate analog of (25). We must consider two cases, depending upon \(\tau\).

Case 1. Suppose \(\tau \geq 1\). Then (43) and repeated integration by parts yields
\[
\int_{t_0}^{t} s^{n-\nu-1} P(s)(\nu(s))^\tau \, ds
\]
\[
= -\sum_{j=1}^{n-1} I_j(s; P)[s^{n-\nu-1}(\nu(s))^\tau]^j - \frac{\partial}{\partial t} \int_{t_0}^{t} I_{n-1}(s; P)[s^{n-\nu-1}(\nu(s))^\tau]^{(n-1)} \, ds.
\]
Asymptotic Integration

From the formula of Faa di Bruno [3] for the derivatives of a composite function,

\begin{equation}
\frac{d^l}{ds^l} u^r = \sum_{k=0}^{l} (\gamma)^{(k)} u^{r-k} \sum_{i_1 + \cdots + i_k = k} \frac{k!}{i_1! \cdots i_k!} \left( \frac{u}{1!} \right)^{i_1} \left( \frac{u'}{2!} \right)^{i_2} \cdots \left( \frac{u^{(l)}}{l!} \right)^{i_l}
\end{equation}

if \( l = 1, 2, \ldots \), where

\( (\gamma)^{(k)} = \gamma(\gamma - 1) \cdots (\gamma - k + 1) \)

and \( \sum_1 \) is over all partitions of \( k \) as a sum of nonnegative integers,

\begin{equation}
k_1 + k_2 + \cdots + k_i = k
\end{equation}

such that

\begin{equation}k_1 + 2k_2 + \cdots + ik_i = l.\end{equation}

From Leibniz's formula for the derivatives of a product

\begin{equation}[(s^{n-v-1} u(s))^{(j-1)}] = \sum_{\ell=0}^{j-1} \binom{j-1}{l} (s^{n-v-1})^{(j-1-l)} [u(s)']^{(l)},\end{equation}

From (52), (53), (56) and (59), it can be shown that

\begin{equation}|[s^{n-v-1} u(s)]^{(j-1)}| \leq B_i s^{n-j+1} v, \quad 1 \leq j \leq n-1,
\end{equation}

for some constant \( B_i \). (To verify this it is important to invoke (57) and (58).) However, from Lemma 1 and the convergence of (47),

\begin{equation}|I_j(s; P)| \leq \frac{2\delta(s)s^{n-j-1}}{(j-1)!}, \quad 1 \leq j \leq n,
\end{equation}

where

\begin{equation}\delta(t) = \sup_{T \geq t} \left| \int_0^T s^{n-1+v-1} P(s)ds \right|.
\end{equation}

From (60) and (61), we can let \( T \to \infty \) in (55) to obtain

\begin{equation}F(t; h) = \sum_{j=1}^{n-1} I_j(t; P)[t^{n-v-1}(u(t))]^{(j-1)} + \int_t^\infty I_{n-1}(s; P)[s^{n-v-1}(u(s))]^{(n-1)} ds,
\end{equation}

where the integral on the right converges absolutely because of (48) and (60) with \( j = n \).

Now suppose
By applying the mean value theorem to the function
\[ G_i(x_0, x_1, \cdots, x_i) = \sum_{k=1}^i (t)_k x_0^{i-k} \sum_{k} \frac{k!}{k_1! \cdots k_i!} \left( \frac{x_0^{k_1}}{1!} \right) \left( \frac{x_0^{k_2}}{2!} \right) \cdots \left( \frac{x_0^{k_i}}{i!} \right) \]
(see (56), (57) and (58)), and then using estimates similar to those which led to (60), it can be shown that
\[ |s^{n-\nu-1}[(u_i(s))^r - (u_i(s))^r]^{(j-1)}| \leq C_j \|h_1 - h_2\| s^{n-j+q(r-1)}, \quad 1 \leq j \leq n, \]
where \( C_1, \cdots, C_n \) are constants. This, (61) and (63) imply that
\[ |F(t; h_1) - F(t; h_2)| \leq \|h_1 - h_2\| \left( K_0 \delta(t) + C_n \int_t^\infty s^{q(r-1)} |I_{n-1}(s; P)| ds \right) \]
where \( K_0 \) is a constant.

Case 2. Suppose \( \tau < 1 \). Then we rewrite (54) as
\[ F(t; h) = \int_t^\infty s^{n-q-1} Q(s)(u(s))^r ds \]
(see (50)), and proceed as in Case 1, to obtain
\[ F(t; h) = \sum_{j=1}^{n-1} I_j(t; Q)[t^{n-\nu-1}(u(t))^r]^{(j-1)} + \int_t^\infty I_{n-1}(s; Q)[s^{n-q-1}(u(s))^r]^{(n-1)} ds, \]
where the integral on the right converges absolutely because of (49), and
\[ |F(t; h_1) - F(t; h_2)| \leq \|h_1 - h_2\| \left( \hat{K}_0 \delta(t) + \hat{C}_n \int_t^\infty |I_{n-1}(s; Q)| ds \right), \]
where \( \hat{K}_0 \) and \( \hat{C}_n \) are constants, and \( \delta \) is as in (62).

Now that we have shown that \( F(\cdot; h) \) satisfies a Lipschitz condition with respect to \( h \) for all real \( \tau \), the rest of the proof is similar to the part of the proof of Theorem 1 which follows (34).

Remark. If \( \tau \) is rational with odd denominator, so that \( y^\tau \) is real-valued for \( y < 0 \), then only trivial modifications of the proofs given above show that the conclusions of Theorems 1 and 2 are also valid if \( a_m < 0 \) or \( a_s < 0 \), respectively. A similar comment applies if (1) is replaced by
\[ y^{(m)} + P(t)|y|^{\tau} \text{ sgn } y = f(t), \]
without restrictions on (real) \( \tau \).
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References


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