Stability of a class of discrete minimum variance smoothing formulas

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STABILITY OF A CLASS OF DISCRETE MINIMUM VARIANCE
SMOOTHING FORMULAS

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Abstract. We study stability of midpoint smoothing formulas matched to discrete data consisting of equally spaced samples of an unknown polynomial of known maximal degree plus a random error with known spectral density. Stability is established for a class of minimum variance smoothing formulas which includes least squares and minimum $R_n$ smoothing formulas, previously shown to be stable by T. N. E. Greville.

1. Introduction. We consider the problem of smoothing a sequence of observations,

\[ u_r = f(r) + \varepsilon_r, \]

where $f$ is an unknown polynomial of degree not exceeding $2k$ and $\{\varepsilon_r\}$ is a sample sequence from a real-valued stationary time series with zero mean and continuous spectral density

\[ \Phi(\lambda) = \sum_{-\infty}^{\infty} \phi_r \cos r\lambda; \]

that is,

\[ E(\varepsilon_r\varepsilon_{r+s}) = \phi_r. \]

We apply to (1) the smoothing formula

\[ u_r = \sum_{s=-q}^{q} w_s v_{r-s}, \]

where the weighting coefficients $w_{-q}, \ldots, w_q$ are chosen to minimize

\[ Q(w_{-q}, \ldots, w_q) = \sum_{s=-q}^{q} \phi_{r-s} w_s w_s \]

subject to the constraints

\[ \sum_{s=-q}^{q} w_s \delta_r = \delta_{0r}, \quad 0 \leq r \leq 2k. \]

If $\{w_{-q}, \ldots, w_q\}$ is any solution of (3) and

\[ u_r^* = \sum_{s=-q}^{q} w_s v_{r-s}, \]

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then

\[ E u^*_k = f(r) \]

whenever \( f \) is a polynomial of degree not exceeding \( 2k \), and

\[ E(u^*_k - f(r))^2 = Q(w_{-q}, \ldots, w_q). \]

For these reasons we shall follow the convention introduced in [11], and refer to (2) as \( \text{MV}(q, k; \Phi) \), which stands for “minimum variance smoothing formula, with respect to \( \Phi \), of span \( 2q + 1 \) and degree \( 2k + 1 \).”

If \( \Phi \neq 0 \), the constrained minimum problem has a unique solution for every \( q \) and \( k \). Moreover, it happens that

\[ w_s = w_{-s}, \]

so that \( \text{MV}(q, k; \Phi) \) is symmetric, and (4) holds even if \( f \) is of degree \( 2k + 1 \), rather than \( 2k \).

If \( q \leq k \), then (3) has only the uninteresting solution

\[ w_0 = 1, \]
\[ w_s = 0, \quad s \neq 0; \]

therefore we shall assume that \( q > k \).

The characteristic function of \( \text{MV}(q, k; \Phi) \) is defined to be

\[ C(\lambda) = \sum_{-q}^{q} w_s \cos r. \]

It follows [6] from (3) and (5) that

\[ C(\lambda) = 1 + O(\lambda^{2k+2}), \quad \lambda \to 0. \]

Schoenberg [5] has shown that a symmetric smoothing formula is stable under repeated application if and only if

\[ |C(\lambda)| < 1, \quad 0 < |\lambda| \leq \pi. \]

(For a different interpretation of (7), see [11] and the footnote reference to Lanczos in [6].)

Results on stability of minimum variance smoothing formulas are quite limited. Greville [1] has shown that \( \text{MV}(q, k; \Phi) \) is stable for all \( q \geq k + 1 \geq 1 \) if

\[ \Phi(\lambda) = \sin^2(\lambda/2), \]

where \( m \) is a nonnegative integer. If \( m = 0 \), this is equivalent to least-squares smoothing, the stability of which had been conjectured by Schoenberg; if \( m \geq 1 \), it is equivalent to minimum \( R_m \) smoothing, as defined by Wolfenden [13]. Trench [11] has obtained the following result.

**Theorem 1.** Suppose \( \text{MV}(q, k; \Phi) \) is stable for all \( q \geq k + 1 \geq 1 \), and let

\[ Q(x) = \frac{x^k \prod_{j=1}^{r} (1 + \theta_j x)}{\prod_{j=1}^{r} (1 - \gamma_j x)}, \]
where \( t \) is a nonnegative integer, \( \theta_i \geq 0 \), and \( 0 \leq \gamma_j < 1 \). Define

\[
\eta(\lambda) = Q(\sin^2(\lambda/2))\Phi(\lambda).
\]

Then \( \text{MV}(q, k; \eta) \) is stable for all \( q \geq k + 1 \geq 1 \).

Wilf [12], Lorch and Szegö [2], [3], Lorch, Muldoon and Szegö [4], and Trench [8], [9], [10] have considered related questions for continuous smoothing formulas.

In this paper we obtain sufficient conditions (Theorem 3) for stability of \( \text{MV}(q, k; \Phi_{\mu}) \), where

\[
\Phi_{\mu}(\lambda) = (\sin^2(\lambda/2))^{\mu}(\cos^2(\lambda/2))^v, \quad \mu, v > -1/2.
\]

These results are extended to more general spectral densities in Theorem 4.

2. Characteristic function of \( \text{MV}(q, k; \Phi_{\mu}) \). Throughout this paper

\[
(u)_k = u(u+1)\cdots(u+k-1)
\]

and

\[
(u)_k^m = u(u-1)\cdots(u-k+1).
\]

The following result reduces to Sheppard's formula for the characteristic function of minimum \( R_m \) smoothing [1], [7] when \( \mu = m \) and \( v = 0 \).

**Theorem 2.** The characteristic function of \( \text{MV}(q, k; \Phi_{\mu}) \) is

\[
C(\lambda) = 1 + \frac{(-1)^{k+1}}{k!} \sum_{s=k+1}^{q} \frac{(-q)_s(q+\mu+v+1)_s}{s(s-k-1)!k+1_3s} \sin^{2s}(\lambda/2).
\]

**Proof.** The variance of the output of \( \text{MV}(q, k; \Phi_{\mu}) \) is

\[
\sigma^2 = \sum_{r,s=-q}^{q} \phi_{r,s} w_r w_s,
\]

where \( \phi_{r,s} \) are the Fourier coefficients of \( \Phi_{\mu} \). This can be written as

\[
\sigma^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |C(\lambda)|^2 \Phi_{\mu}(\lambda) \, d\lambda.
\]

Since \( \cos r\lambda \) is a polynomial of degree \( |r| \) in

\[
x = \sin^2(\lambda/2),
\]

\( C(\lambda) \) is a polynomial of degree \( q \) in \( x \), which, from (6), is of the form

\[
C(\lambda) = P(x) = 1 - \sum_{s=k+1}^{q} b_s x^s.
\]

Substituting this into (9) and taking \( x \) as the new variable of integration yields

\[
\sigma^2 = \frac{1}{\pi} \int_{-1}^{1} \left( 1 - \sum_{s=k+1}^{q} b_s x^s \right)^2 x^{s-1/2}(1-x)^{-1/2} \, dx.
\]

Thus, \( P(x) \) (and therefore \( C(\lambda) \)) can be obtained by minimizing (11) with respect to

\[ b_{k+1}, \ldots, b_q. \]
We complete the proof of Theorem 2 with the following lemma.

**Lemma 1.** Suppose \( \alpha, \beta > -1, \) \( p \) is a positive integer, and \( n \) is a nonnegative integer. Then the minimum value of

\[
\int_0^1 (F(x))^2 x^\alpha (1-x)^\beta \, dx
\]

for \( F(x) \) of the form

\[
F(x) = 1 - x^n \sum_{s=0}^n a_s x^s,
\]

is attained with

\[
F(x) = 1 - \frac{(-1)^n n! \sum_{s=0}^n (-n-p)(n+p+\alpha+\beta+2)_s}{(p-1)! \sum_{s=0}^n s(s-p)(p+\alpha+1)_s} x^n.
\]

**Proof.** Differentiating

\[
\int_0^1 \left(1 - x^n \sum_{s=0}^n a_s x^s\right)^2 x^\alpha (1-x)^\beta \, dx
\]

with respect to \( a_0, \ldots, a_n \) and equating the results to zero yields

\[
\int_0^1 x^r x^\alpha (1-x)^\beta \, dx = \sum_{s=0}^n a_s \int_0^1 x^{2r+s+\alpha} (1-x)^\beta \, dx, \quad 0 \leq r \leq n.
\]

From the properties of the beta function,

\[
\int_0^1 x^\alpha (1-x)^\beta \, dx = \frac{\Gamma(\xi + 1)\Gamma(\eta + 1)}{\Gamma(\xi + \eta + 2)}, \quad \xi, \eta > -1.
\]

Applying this to (14) and cancelling common factors yields

\[
\frac{(p+r+\alpha+\beta+2)_p}{(p+r+\alpha+1)_p} = \sum_{s=0}^n \frac{(2p+r+\alpha+1)_s}{(2p+r+\alpha+\beta+2)_s} a_s, \quad 0 \leq r \leq n.
\]

Subtracting the \( r \)th equation from the \((r+1)\)st and using the relationship

\[
\frac{(x+1)_j}{(y+1)_j} - \frac{(x)_j}{(y)_j} = \frac{(x+1)_j}{(y)_j} (y-x)
\]

yields

\[
\frac{(p+r+\alpha+\beta+3)_{p+1}}{(p+r+\alpha+1)_{p+1}} = \sum_{s=0}^{n-1} \frac{(2p+r+\alpha+2)_s}{(2p+r+\alpha+\beta+2)_s} \left[ \frac{(s+1)a_{s+1}}{p} \right] - \frac{(s+1)a_{s+1}}{p},\quad 0 \leq r \leq n-1.
\]

Denote the solution of (15) more precisely by \( a_{n, \alpha, \beta, p} \); writing (15) for \( \alpha = 1, \beta + 1, p + 1 \) and \( n - 1, \) and comparing the result with (16) yields

\[
a_{n, \alpha, \beta, p} = \frac{-p}{s} a_{s-1, \alpha-1, \beta+1, p+1}, \quad 1 \leq s \leq n.
\]
Given $a_{0,n-1}, \ldots, a_{n-1}$, for all $\alpha, \beta$ and $p$, this yields $a_{1,n}, \ldots, a_{n}$ but not $a_{\alpha\beta}$; hence we need another recursion formula. Multiplying (15) by $(2p + r + \alpha + \beta + 2)_n/(2p + r + \alpha + 1)_n$, yields, after some manipulation,

$$
\frac{(p + r + \alpha + \beta + 2)}{(p + r + \alpha + 1)} a_{n+p} = \sum_{s=0}^{n} \frac{(2p + r + s + \alpha + \beta + 2)}{(2p + r + s + \alpha + 1)} a_{n-s} a_{n}(\alpha, \beta, p), \quad 0 \leq r \leq n.
$$

Subtracting the $r$th equation from the $(r+1)$st yields

$$
\frac{(p + r + \alpha + \beta + 3)}{(p + r + \alpha + 1)} a_{n+p-1} = \sum_{s=0}^{n-1} \frac{(2p + r + s + \alpha + \beta + 3)}{(2p + r + s + \alpha + 1)} a_{n-s-1} - \frac{n-s}{n-p} a_{n}(\alpha, \beta, p), \quad 0 \leq r \leq n-1.
$$

Comparing this with (18) for $\alpha, \beta + 1, p$ and $n-1$ yields

$$
a_{n}(\alpha, \beta, p) = \frac{n + p}{n} a_{n}(\alpha, \beta + 1, p), \quad 0 \leq s \leq n-1.
$$

Starting from (15) with $n = 0$, induction on $n$ using (17) and (19) implies that

$$
a_{n}(\alpha, \beta, p) = \frac{(n + p + \alpha + \beta + 3)}{(p + 1)(n + p + \alpha + \beta + 2)} a_{n}, \quad 0 \leq s \leq n,
$$

which yields (12).

Comparing (11) and (13) shows that $P(x)$ can be obtained by setting

$$
p = k + 1, \quad n = q - k - 1, \quad \alpha = \mu - 1/2, \quad \beta = \nu - 1/2
$$

in (12). This and (10) yield (8), which completes the proof of Theorem 2.

3. Main results. From (10), MV(q, k; $\Phi_{n}$) is stable if and only if

$$|P(x)| < 1, \quad 0 < x \leq 1;
$$

however, it is convenient to consider the polynomial $F(x)$ defined by (12).

**Lemma 2.** If

$$|F(1)| < 1,
$$

then

$$|F(x)| < 1, \quad 0 < x \leq 1.
$$

Therefore, MV(q, k; $\Phi_{n}$) is stable if and only if (21) holds, with parameters $n, p, \alpha$ and $\beta$ given by (20).

**Proof.** From a result of Greville (see the proof of Lemma 2 of [1]), $(F(x))^2$ is interpolated at $x = 0$, at the relative extrema of $F(x)$ in $(0, 1)$, and at $x = 1$ by the polynomial

$$f(x) = 1 + \int_{0}^{1} t^{-p+1} q(t)(F(t))^2 \, dt,
$$
where

\[ q(x) = \sum_{s=0}^{p-1} d_s r(s, x)x^s, \]

\[ d_s = \frac{-(p-1)\gamma(p+\beta+1)p^{-s-1}}{s!(n+p+s)(n+p+\beta+2)p^{-s}} \]

and

\[ r(s, x) = (3p + 2x - 3s)(1 - x) - (2\beta + 1)x, \quad 0 \leq s \leq p - 1. \]

Clearly

\[ r(s, 0) > 0, \quad \alpha > -1, \quad 0 \leq s \leq p - 1; \]

consequently, since \( d_s < 0 \), \( q(x) \) is negative near \( x = 0 \). Moreover,

\[ r(s, x) < 0, \quad \alpha, \beta > -1, \quad 0 \leq s \leq p - 1, \]

so that \( q(x) \) is monotone increasing. Hence \( q(x) \) either remains negative for all \( x \) on \((0, 1)\) from (23), this is true if and only if \(-1 < \beta \leq -1/2\) or changes sign exactly once, from negative to positive. In either case,

\[ f(1) < 1 \]

implies

\[ f(x) < 1, \quad 0 < x \leq 1. \]

From the manner in which \( f(x) \) interpolates \((F(x))^2\), it now follows that (21) implies (22), which completes the proof of Lemma 2.

The next theorem is our main result on stability of \( \text{MV}(q, k; \Phi_{\mu}). \)

**Theorem 3.** (a) \( \text{MV}(q, 0; \Phi_{\mu}) \) is stable if and only if \(-1/2 < \nu < \mu + 1\).

(b) If \(-1/2 < \nu \leq 1/2, \) then \( \text{MV}(q, k; \Phi_{\mu}) \) is stable for all \( q \geq k + 1 \geq 1 \) and \( \mu > -1/2. \)

(c) For each \( k, \mu \) and \( \tau, \) \( \text{MV}(q, k; \Phi_{\mu}) \) is stable for all \( q \) sufficiently large if \(-1/2 < \nu < \mu + 1), \) or unstable for all \( q \) sufficiently large if \( \tau > \mu + 1. \)

**Proof.** From (12),

\[ F(1) = 1 - \frac{(-1)^p \sum_{s=p}^{n+p} (-n+p)_{s}(n+p+\alpha+\beta+2)_{s}}{(p-1)! s!(n+p+\alpha+1)_{s}}, \]

which can be rewritten (see the Appendix) as

\[ F(1) = \frac{(-1)^{n+p}(n+\beta+1)^{n+p+1}}{(n+p+\alpha+1)^{n+p+1}} \sum_{s=0}^{p-1} (-n+p+s)(n+p+\alpha+\beta+2)_{s} \]

If \( p = 1, \) then

\[ |F(1)| = \frac{(n+\beta+1)^{n+1}}{(n+\alpha+2)^{n+1}}; \]

hence \( |F(1)| < 1 \) if and only if \( \beta < \alpha + 1, \) and (a) follows from (20) and Lemma 2.
If \( \beta \leq 0 \), then (24) implies
\[
|F(1)| \leq \frac{(n + 1)!}{(n + p + \alpha + 1)^{p-1}} \sum_{s=0}^{p-1} \frac{(n + 1)_s}{s!}
\]
\[
= \frac{(n + 1)(n + 2)^{p-1}}{(n + p + \alpha + 1)^{p-1}(p - 1)!}
\]
\[
= \frac{(n + p)^{(p-1)}}{(n + p + \alpha + 1)^{(p-1)}(p - 1)!} < 1 \text{ if } \alpha > -1;
\]
hence (b) follows from (20) and Lemma 2.

(The first equality in (25) can be obtained from the identity \( \sum_{s=0}^{p-1} (u)_s/r! = (u+1/q)^r \).

To prove (c), we rewrite (24) as
\[
|F(1)| = \frac{(n + \beta + 1)^{(p-2)}}{(n + \alpha + 2)^{(p-2)}} \cdot \left[ \frac{(\beta + p - 1)^{(p-1)}}{(n + p + \alpha + 1)^{p-1}} \sum_{s=0}^{p-1} \frac{(n + 1)_s(n + p + \alpha + \beta + 2)_s}{s!(n + p + \alpha + 2)_s} \right].
\]
The expression in brackets approaches \((\beta + p - 1)^{(p-1)}(p - 1)!\) as \(n\) approaches infinity, and
\[
\lim_{n \to \infty} \frac{(n + \beta + 1)^{(p-2)}}{(n + \alpha + 2)^{(p-2)}} \cdot \left[ \frac{(\beta + p - 1)^{(p-1)}}{(n + p + \alpha + 1)^{p-1}} \sum_{s=0}^{p-1} \frac{(n + 1)_s(n + p + \alpha + \beta + 2)_s}{s!(n + p + \alpha + 2)_s} \right] = \begin{cases} 0 & \text{if } \beta < \alpha + 1, \\ \infty & \text{if } \beta > \alpha + 1; \end{cases}
\]
hence (c) follows from (20) and Lemma 2.

Parts (a) and (b) of the next theorem follow from Theorems 1 and 3. Part (c) requires a minor modification of Theorem 1: namely, replacement of the phrase “for all \( q \geq k + 1 \geq 1 \)” by “for each fixed \( k \) and sufficiently large \( q \).” This modified version of Theorem 1 also follows from the proof given in [11].

**Theorem 4.**

Let
\[
Q(x) = \frac{x^\mu(1 - x)^\nu \prod_{i=1}^{r} (1 + \theta_i x)}{\prod_{j=1}^{s} (1 - \gamma_j x)},
\]
where \( \mu, \nu > -1/2, \theta_i \geq 0 \) and \( 0 \leq \gamma_j < 1 \). Define
\[
\Phi(\lambda) = Q(\sin^2(\lambda/2)).
\]

Then
(a) \( MV(q, 0; \Phi) \) is stable if \( \nu < \mu + 1 \).

(b) If \(-1/2 < \nu \leq 1/2\), then \( MV(q, k; \Phi) \) is stable for all \( k \geq k + 1 \geq 1 \) and \( \mu > -1/2 \).

(c) For each \( k \), \( \mu \) and \( \nu \), \( MV(q, k; \Phi) \) is stable for all \( q \) sufficiently large if \(-1/2 < \nu \leq \mu + 1 \).
Appendix. The purpose of this Appendix is to verify (24).

**Lemma A.1.** The following is an identity in \( u \) and \( v \):

\[
\sum_{s=0}^{m} (-1)^{s} \binom{m}{s} \frac{(u + v)_s}{(v)_s} = \frac{(u)^m}{(-v)^m};
\]

(A.1)

**Proof.** If \( v \neq -m + 1, \ldots, -1, 0 \), the left side of (A.1) is a polynomial of degree \( m \) in \( u \). Call it \( Q(u) \). If \( r = 0, 1, \ldots, m - 1 \), then

\[
Q(r) = \sum_{s=0}^{m} (-1)^{s} \binom{m}{s} \frac{(v + r)_s}{(v)_s} = \frac{(v + r)_m}{(v)_m} \sum_{s=0}^{m} (-1)^{s} \binom{m}{s} (v + s)_r = 0,
\]

since the last sum is the \( m \)th difference of a polynomial of degree less than \( m \). As a polynomial in \( u \), the right side of (A.1) has the same zeros as the left; moreover, both sides equal 1 when \( u = -v \). Hence (A.1) is an identity.

**Lemma A.2.** The following is an identity in \( x \) and \( y \):

\[
1 - \frac{(-1)^{p} \binom{n+p}{p} \binom{n-p}{(x+y)}_x}{(p-1)! \binom{n+p}{n-p} \binom{n-p}{y+y}} = \frac{(x)^{n+1} \sum_{s=0}^{p-1} (n+1)_s (x+y)_s}{(-y)^{n+1} \sum_{s=0}^{p-1} (n+1)_s (y+n+1)_s}.
\]

(A.2)

**Proof.** For a fixed \( y \neq -n - p + 1, \ldots, -1, 0 \), the left side of (A.2) is a polynomial of degree \( n + p \) in \( x \). Call it \( P(x) \). Then

\[
P(-y - r) = 1, \quad 0 \leq r \leq p - 1.
\]

(A.3)

Also, \( P(x) \) can be rewritten as

\[
P(x) = \frac{(-1)^{p-1} \binom{n+p}{n-p} \binom{n-p}{x+y} (x+y)_x}{(y)^{n+1} \sum_{s=0}^{p-1} (n+1)_s (x+y)_s}.
\]

If \( r = 0, 1, \ldots, n \), then

\[
P(r) = \frac{(-1)^{p-1} \binom{n+p}{n-p} \binom{n-p}{r+y} \sum_{s=0}^{p-1} (-1)^{s} \binom{n+p}{s} \binom{s-1}{r}}{(y)^{n+1} \sum_{s=0}^{p-1} (n+1)_s (r+y)_s},
\]

which is the \( (n + p) \)th difference of a polynomial of degree less than \( n + p \); hence

\[
P(r) = 0, \quad r = 0, 1, \ldots, n.
\]

The right side of (A.2) also vanishes at \( x = 0, 1, \ldots, n \). Because of (A.3), the proof will be complete if we show that the right side of (A.2) equals 1 when \( x = -y - r, r = 0, 1, \ldots, p - 1 \); that is, we must show that

\[
\frac{(-y)^{n+1}}{(-y)^{n+1}} \sum_{s=0}^{r} \frac{(n+1)_s (-r)_s}{s!(y+n+1)_s} = 1.
\]

(A.4)
This is accomplished by rewriting the left side of (A.4) as
\[
\frac{(-y-r)^{(n+1)}}{(-y)^{(n+1)}} \sum_{k=0}^r (-1)^k \binom{r}{k} \frac{y^{(n+1)}}{(y+n+1)^k},
\]
and invoking (A.1) with \( m = r, u = -y, \) and \( v = y+n+1. \)

Now (24) can be obtained by setting \( x = n+\beta+1 \) and \( y = p+\alpha+1 \) in (A.2).

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