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An Unnoticed Consequence of Szegö's Distribution Theorem

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Abstract

Suppose $f = \sum_{\ell=1}^{k} w_\ell f_\ell$, where $w_1, w_2, \dots, w_k$ are positive constants, $\sum_{\ell=1}^{k} w_\ell = 1$, and $f_1, f_2, \dots, f_k$ are real-valued and Riemann integrable functions on $[-\pi, \pi]$. Let $\lambda_{1n}^{(\ell)} \le \lambda_{2n}^{(\ell)} \le \cdots \le \lambda_{nn}^{\ell)}$ be the eigenvalues of the Hermitian Toeplitz matrices $T_n^{(\ell)} = [t_{r-s}^{(\ell)}]_{r,s=1}^{n}$, where $t_r^{(\ell)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-irx} f_\ell(x)\,dx$, and let $\lambda_{1n} \le \lambda_{2n} \le \cdots \le \lambda_{nn}$ be the eigenvalues of $T_n = \sum_{\ell=1}^{n} w_\ell T_n^{(\ell)}$. We give conditions implying that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} |\lambda_{rn} - w_1\lambda_{rn}^{(1)} - w_2\lambda_{rn}^{(2)} - \cdots - w_k\lambda_{rn}^{(k)}| = 0.$$

A *Toeplitz* matrix has the form $T = [t_{r-s}]_{r,s=1}^{n}$. Let $f_1, f_2, \dots f_k$ be real-valued and Riemann integrable on $[-\pi, \pi]$, and consider the Hermitian Toeplitz matrices

$$T_n^{(\ell)} = [t_{r-s}^{(\ell)}]_{r,s=1}^{n}, \quad \text{where} \quad t_r^{(\ell)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-irx} f_\ell(x)\,dx \;\text{ for }\; r = 0, \pm1, \pm2, \dots$$

and $1 \le \ell \le k$. In standard termology, f_ℓ is the *symbol* of the family $\{T_n^{(\ell)}\}_{n=1}^{\infty}$. Let

$$T_n = w_1 T_n^{(1)} + w_2 T_n^{(2)} + \cdots + w_k T_n^{(k)},$$

where $w_1, w_2, \dots, w_k$ are real numbers. Thus, $\{T_n\}_{n=1}^{\infty}$ is a family of Hermitian Toeplitz matrices and

$$f = w_1 f_1 + w_2 f_2 + \cdots + w_k f_k \tag{1}$$

is its symbol.

Although there is in general no apparent relationship between the eigenvalues

$$\lambda_{1n}^{(\ell)} \le \lambda_{2n}^{(\ell)} \le \cdots \le \lambda_{nn}^{(\ell)} \tag{2}$$

of $T_n^{(\ell)}$, $1 \le \ell \le k$, and the eigenvalues

$$\lambda_{1n} \le \lambda_{2n} \le \cdots \le \lambda_{nn}, \tag{3}$$

of T_n, we will show that if

$$w_\ell > 0 \text{ for } 1 \le \ell \le k, \quad w_1 + w_2 + \cdots + w_k = 1, \tag{4}$$

and

$$\mu_{rn} = w_1 \lambda_{rn}^{(1)} + w_2 \lambda_{rn}^{(2)} + \cdots + w_k \lambda_{rn}^{(k)}, \tag{5}$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} |\lambda_{rn} - \mu_{rn}| = 0 \tag{6}$$

under suitable conditions on f_1, f_2, ..., f_k. Thus, a convex combination of the eigenvalues of $T_n^{(1)}$, $T_n^{(2)}$, ..., $T_n^{(k)}$ approximates the eigenvalues of the same convex combination of $T_n^{(1)}$, $T_n^{(2)}$ ..., $T_n^{(k)}$ sufficiently well that the average absolute error tends to zero as $n \to \infty$.

The motivation for the conditions on f_1, f_2, ..., f_k begins with Szegö's distribution theorem. Suppose h is real-valued and Riemann integrable on $[-\pi, \pi]$,

$$-\infty < a \le h(x) \le b < \infty \text{ for } -\pi \le x \le \pi,$$

and the sets

$$L_\epsilon = \{x \in [-\pi, \pi] \,:\, a < h(x) < a + \epsilon\}$$

and

$$U_\epsilon = \{x \in [-\pi, \pi] \,:\, b - \epsilon < h(x) < b\}$$

have positive Lebesgue measure for all $\epsilon > 0$; thus, a and b are the *essential* lower and upper bounds of h. We say that x_0 is an *essential minimum point* of h if $N(x_0) \cap [-\pi, \pi] \cap L_\epsilon$ has positive measure for every neighborhood $N(x_0)$ of x_0 and every $\epsilon > 0$, or an *essential maximum point* if $N(x_0) \cap [-\pi, \pi] \cap U_\epsilon$ has positive measure for every $N(x_0)$ and $\epsilon > 0$.

Let

$$\gamma_{1n} \le \gamma_{2n} \le \cdots \le \gamma_{nn}$$

be the eigenvalues of the Hermitian Toeplitz matrix

$$H_n = [h_{r-s}]_{r,s=1}^n, \text{ where } h_r = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-irx} h(x)\, dx \text{ for } r = 0, \pm 1, \pm 2, \dots.$$

From Szegö's distribution theorem [1, p. 62],

$$a \le \gamma_{1n} \le \gamma_{2n} \le \cdots \le \gamma_{nn} \le b \text{ for } n \ge 1,$$

$$\lim_{n\to\infty} \gamma_{1n} = a, \quad \lim_{n\to\infty} \gamma_{nn} = b,$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}G(\gamma_{rn})=\frac{1}{2\pi}\int_{-\pi}^{\pi}G(h(x))\,dx \ \text{ for all } \ G\in C[a,b].$$

Given Szegö's result and f as in (1), we assume that $f_1,\ f_2,\ \dots,\ f_k$ have the same essential mimimum and maximum a and b, so (1) and (4) imply that $a \le f(x) \le b$ for all x in $[-\pi,\pi]$. The essential minimum and maximum of f are a and b if and only $f_1,\ f_2,\ \dots,\ f_k$ have at least one essential minimum point and one essential maximum point in common, which we also assume.

Our final assumption stems from the following lemma, first stated in more general form in [2]. See [3] and [4] for expository discussions of this special case.

Lemma 1 *If*

$$-\infty < a \le \alpha_{1n} \le \alpha_{2n} \le \cdots \le \alpha_{nn} \le b < \infty$$

and

$$a \le \beta_{1n} \le \beta_{2n} \le \cdots \le \beta_{nn} \le b \ \text{ for } \ n \ge 1,$$

then the following statements are equivalent:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{n=1}^{\infty}(G(\alpha_{rn})-G(\beta_{rn}))=0 \ \text{ for all } G\in C[a,b];$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}|\alpha_{rn}-\beta_{rn}|=0;$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{n=1}^{\infty}|G(\alpha_{rn})-G(\beta_{rn})|=0 \ \text{ for all } G\in C[a,b].$$

For reasons that will soon become clear, Lemma 1 leads us to assume that

$$(f_\ell(x)-f_\ell(y))(f_k(x)-f_k(y))\ge 0 \ \text{ for } \ a\le x,y\le b \ \text{ and } \ 1\le k,\ell\le m. \tag{7}$$

Now we can prove (6). From Szegö's theorem,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}G(\lambda_{rn}^{(\ell)})=\frac{1}{2\pi}\int_{-\pi}^{\pi}G(f_\ell(x))\,dx \ \text{ for all } \ G\in C[a,b] \tag{8}$$

and $1 \le \ell \le k$, and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}G(\lambda_{rn})=\frac{1}{2\pi}\int_{-\pi}^{\pi}G(f(x))\,dx \ \text{ for all } \ G\in C[a,b]. \tag{9}$$

Let

$$x_{rn}=\left(\frac{2(r-1)}{n}-1\right)\pi \ \text{ for } \ 1\le r\le n.$$

From the definition of the Riemann integral,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} G(f_\ell(x_{rn})) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(f_\ell(x))\,dx \quad \text{for all } G \in C[a,b], \tag{10}$$

where $1 \le \ell \le k$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} G(f(x_{rn})) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(f(x))\,dx \quad \text{for all } G \in C[a,b]. \tag{11}$$

From (7), for each $n \ge 2$ there is a *single* permutation σ_n of $\{1, 2, \dots, n\}$ such that

$$f_\ell(x_{\sigma_n(1),n}) \le f_\ell(x_{\sigma_n(2),n}) \le \cdots \le f_\ell(x_{\sigma_n(n),n}) \text{ for } 1 \le \ell \le m. \tag{12}$$

This, (1), and (4) imply that

$$f(x_{\sigma_n(1),n}) \le f(x_{\sigma_n(2),n}) \le \cdots \le f(x_{\sigma_n(n),n}). \tag{13}$$

Rearranging the terms on the left sides of (10) and (11) yields

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} G\left(f_\ell\left(x_{\sigma_n(r),n}\right)\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} G(f_\ell(x))\,dx \quad \text{for all } G \in C[a,b], \tag{14}$$

where $1 \le \ell \le k$, and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} G\left(f\left(x_{\sigma_n(r),n}\right)\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} G(f(x))\,dx \quad \text{for all } G \in C[a,b]. \tag{15}$$

From (8) and (14),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(G\left(\lambda_{rn}^{(\ell)}\right) - G\left(f_\ell\left(x_{\sigma_n(r),n}\right)\right)\right) = 0 \quad \text{for all } G \in C[a,b],$$

where $1 \le \ell \le k$, so (2), (12), and Lemma 1 imply that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left|\lambda_{rn}^{(\ell)} - f_\ell\left(x_{\sigma_n(r),n}\right)\right| = 0 \quad \text{for} \quad 1 \le \ell \le k. \tag{16}$$

From (9) and (15),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} (G(\lambda_{rn}) - G(f(x_{\sigma_n(r),n}))) = 0 \quad \text{for all } G \in C[a,b],$$

so (3), (13), and Lemma 1 imply that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{m} |\lambda_{rn} - f(x_{\sigma_n(r),n})| = 0. \tag{17}$$

From (1), (4), and (5),

$$\mu_{rn} - f(x_{\sigma_n(r),n})) = \sum_{\ell=1}^{k} w_\ell(\lambda_{rn}^{(\ell)} - f_\ell(x_{\sigma_n(r),n})),$$

so (16) implies that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \left|\mu_{rn} - f\left(x_{\sigma_n(r),n}\right)\right| = 0.$$

This and (17) imply (6), which completes the proof.

Incidentally, (6) and Lemma 1 imply that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} |G(\lambda_{rn} - G(\mu_{rn})| = 0 \text{ for all } G \in C[a,b].$$

It is important to note that this paper is about a very special case of Szegö's distribution theorem, which – in its original formulation – is about families of hermitian Toeplitz matrices $T_n = [t_{r-s}]_{r,s=1}^{n}$, where $\{t_r\}_{r=-\infty}^{\infty}$ are the Fourier coefficients of a real-valued (not necessarily bounded) Lebesgue integrable function on $[-\pi, \pi]$. This theorem and its numerous extensions comprise part of the core of modern operator theory. Although the arguments in this paper use the properties of the Riemann integral in an essential way, it seems reasonable to hope that more knowledgeable investigators will be motivated to extend our result to a more general setting.

References

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