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SPECTRAL DECOMPOSITION OF KAC-MURDOCK-SZEGÖ MATRICES

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1 Introduction

Kac, Murdock, and Szegö [1] showed that eigenvalues of

$$K = \left(\rho^{|r-s|} \right)_{r,s=1}^n, \quad 0 < \rho < 1,$$

are

$$\lambda_j = \frac{1 - \rho^2}{1 - 2\rho \cos \theta_j + \rho^2}, \quad 1 \leq j \leq n, \quad (1)$$

where $\theta_1, \theta_2, \dots, \theta_n$ are the zeros of

$$G(\theta) = \sin(n+1)\theta - 2\rho \sin n\theta + \rho^2 \sin(n-1)\theta \quad (2)$$

in $(0, \pi)$. They also showed that

$$\frac{(j-1)\pi}{n+1} < \theta_j < \frac{j\pi}{n+1}, \quad 1 \leq j \leq n. \quad (3)$$

They did not find the associated eigenvectors. We address this and other elementary properties of K , simply for the sake of exposition; nothing that we say is new.

Since K is a symmetric Toeplitz matrix, it has $\lfloor n/2 \rfloor$ linearly independent eigenvectors

$$x = [x_1 \ x_2 \ \dots \ x_n]^T \quad (4)$$

such that **(a)** $x_r = x_{n-r+1}$, $1 \leq r \leq n$, and $\lfloor n/2 \rfloor$ linearly independent eigenvectors such that **(b)** $x_r = -x_{n-r+1}$, $1 \leq r \leq n$. Vectors that satisfy **(a)** are *symmetric* and those that satisfy **(b)** are *skew-symmetric*. In the terminology we introduced in [3], an eigenvalue associated with a symmetric eigenvector is *even* and an eigenvector associated with a skew-symmetric eigenvector is *odd*. In [4] we showed that if

$$t_r = \frac{1}{\pi} \int_0^\pi f(\theta) \cos r\theta \, d\theta, \quad 0 \leq r \leq n-1,$$

and f is nonincreasing, then $T = [t_{r-s}]_{r,s=1}^n$ has distinct eigenvalues $\lambda_1 > \lambda_2 > \dots > \lambda_n$, where λ_{2k-1} is even, $1 \leq k \leq \lfloor n/2 \rfloor$, and λ_{2k} is odd, $1 \leq k \leq \lfloor n/2 \rfloor$. Based on numerical experiments, Laurie [2] conjectured that the components of a λ_j -eigenvector have exactly $j-1$ changes in sign. A matrix with this property is *oscillatory*. Here we show explicitly that K has these properties.

2 Results

Obviously, λ is an eigenvalue of K if and only if $\lambda = 1/\mu$, where μ is an eigenvalue of

$$K^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1 \end{bmatrix}.$$

Denote

$$a = \frac{1+\rho^2}{1-\rho^2} \quad \text{and} \quad b = \frac{\rho}{1-\rho^2}.$$

Then (4) is a μ -eigenvector of K^{-1} – and therefore of K – if and only if

$$-bx_{r-1} + (a-\mu)x_r - bx_{r+1} = 0, \quad 1 \leq r \leq n, \quad (5)$$

where

$$x_0 = \rho x_1 \quad \text{and} \quad x_{n+1} = \rho x_n. \quad (6)$$

The general solution of (5) is

$$x_r = \alpha \zeta^r + \beta \zeta^{-r}, \quad 1 \leq r \leq n, \quad (7)$$

where ζ is a zero of

$$p(z) = -bz^2 + (a-\mu)z - b = -b(z-\zeta)(z-1/\zeta),$$

and the associated eigenvalue of K^{-1} is

$$\mu = a - b(\zeta + 1/\zeta).$$

Since μ is real, $\zeta = e^{i\theta}$; hence $\mu = a - 2b \cos \theta$ is an eigenvalue of K^{-1} and

$$\frac{1}{\mu} = \frac{1}{a - 2b \cos \theta} = \frac{1 - \rho^2}{1 - 2\rho \cos \theta + \rho^2}$$

is an eigenvalue of K .

From (6) and (7),

$$\begin{bmatrix} 1 - \rho\zeta & 1 - \rho/\zeta \\ \zeta^{n+1} - \rho\zeta^n & \zeta^{-n-1} - \rho\zeta^{-n} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (8)$$

The determinant of this matrix is

$$D(\zeta) = \zeta^{-n-1}(1 - \rho\zeta)^2 - \zeta^{n+1}(1 - \rho/\zeta)^2 = G(\theta).$$

(Recall (2).) If $D(\zeta) = 0$, then

$$\alpha = 1 - \rho/\zeta, \quad \beta = -(1 - \rho\zeta)$$

is a solution of (8), and (7) becomes (4) with

$$x_r = c(1 - \rho/\zeta)\zeta^r - (1 - \rho\zeta)\zeta^{-r}, \quad 1 \leq r \leq n, \quad (9)$$

with $c \neq 0$ is an eigenvector K . With the appropriate choice of c ,

$$x_r = \sin r\theta - \rho \sin(r-1)\theta = (1 - \rho \cos \theta) \sin r\theta + \rho \sin \theta \cos r\theta, \quad 1 \leq r \leq n,$$

which we can rewrite as

$$x_r = A \sin(r\theta + \alpha), \quad 1 \leq r \leq n, \quad (10)$$

where

$$A = \sqrt{1 - 2\rho \cos \theta + \rho^2}, \quad \cos \alpha = \frac{1 - \rho \cos \theta}{A}, \quad \text{and} \quad \sin \alpha = \frac{\rho \sin \theta}{A}.$$

To separate the even and odd spectra of K , we note that

$$\begin{aligned} D(\zeta) &= \left[\zeta^{-(n+1)/2}(1 - \rho\zeta) + \zeta^{(n+1)/2}(1 - \rho/\zeta) \right] \\ &\times \left[\zeta^{-(n+1)/2}(1 - \rho\zeta) - \zeta^{(n+1)/2}(1 - \rho/\zeta) \right], \end{aligned}$$

If ζ is a zero of the first factor, (9) becomes

$$\begin{aligned} x_r &= c(1 - \rho/\zeta)(\zeta^r + \zeta^{n-r+1}) \\ &= c(1 - \rho/\zeta)\zeta^{(n+1)/2} \left(\zeta^{-(n-2r+1)/2} + \zeta^{(n-2r+1)/2} \right), \quad 1 \leq r \leq n, \end{aligned}$$

so the associated eigenvector is symmetric. With $\zeta = e^{i\theta}$, the first factor is multiple of

$$\begin{aligned} C(\theta) &= \cos \frac{(n+1)\theta}{2} - \rho \cos \frac{(n-1)\theta}{2} \\ &= (1 - \rho) \cos \frac{\theta}{2} \cos \frac{n\theta}{2} - (1 + \rho) \sin \frac{\theta}{2} \sin \frac{n\theta}{2}, \end{aligned}$$

and a suitable choice of c yields

$$x_r = \cos \frac{(n-2r+1)\theta}{2}, \quad 1 \leq r \leq n,$$

If ζ is a zero of the second factor, (9) becomes

$$\begin{aligned} x_r &= c(1 - \rho/\zeta)(\zeta^r + \zeta^{n-r+1}) \\ &= c(1 - \rho/\zeta)\zeta^{(n+1)/2} \left(\zeta^{-(n-2r+1)/2} - \zeta^{(n-2r+1)/2} \right), \quad 1 \leq r \leq n. \end{aligned}$$

so the eigenvector is skew-symmetric. With $\zeta = e^{i\theta}$, the second factor is a multiple of

$$\begin{aligned} S(\theta) &= \sin \frac{(n+1)\theta}{2} - \rho \sin \frac{(n-1)\theta}{2} \\ &= (1-\rho) \cos \frac{\theta}{2} \sin \frac{n\theta}{2} + (1+\rho) \sin \frac{\theta}{2} \cos \frac{n\theta}{2}, \end{aligned}$$

. and an appropriate choice of c yields

$$x_r = \sin \frac{(n-2r+1)\theta}{2}, \quad 1 \leq r \leq n,$$

It is easy to verify that C changes sign in

$$\left(\frac{2(j-1)\pi}{n}, \frac{2j\pi}{n} \right), \quad 1 \leq j \leq \lceil n/2 \rceil,$$

and S changes sign in

$$\left(\frac{(2j-1)\pi}{n}, \frac{2j\pi}{n} \right), \quad 1 \leq j \leq \lfloor n/2 \rfloor.$$

Since the right side of (1) is a decreasing function of j , it follows that the even and odd eigenvalues of K are interlaced and the largest is even. Moreover,

$$\frac{(j-1)\pi}{n} < \theta_j < \frac{j\pi}{n}, \quad 1 \leq j \leq n.$$

This and (3) yield the slightly improved estimates

$$\frac{(j-1)\pi}{n} < \theta_j < \frac{j\pi}{n+1}, \quad 1 \leq j \leq n.$$

Either of the last two inequalities implies that $r\theta_j$ increases by at least $(j-1)\pi$ but less than $j\pi$ as $r = 1, 2, \dots, n$. This and (10) imply that K is oscillatory.

References

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