Functions Defined by Improper Integrals

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IMPROPER INTEGRALS

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complete instructor’s solution manual is available by email to wtrench@trinity.edu,
subject to verification of the requestor’s faculty status.
1 Foreword

This is a revised version of Section 7.5 of my Advanced Calculus (Harper & Row, 1978). It is a supplement to my textbook Introduction to Real Analysis, which is referenced several times here. You should review Section 3.4 (Improper Integrals) of that book before reading this document.

2 Introduction

In Section 7.2 (pp. 462–484) we considered functions of the form

\[ F(y) = \int_a^b f(x, y) \, dx, \quad c \leq y \leq d. \]

We saw that if \( f \) is continuous on \([a, b] \times [c, d] \), then \( F \) is continuous on \([c, d] \) (Exercise 7.2.3, p. 481) and that we can reverse the order of integration in

\[ \int_c^b F(y) \, dy = \int_c^b \left( \int_a^b f(x, y) \, dx \right) \, dy \]

to evaluate it as

\[ \int_c^b F(y) \, dy = \int_a^b \left( \int_c^b f(x, y) \, dy \right) \, dx \]

(Corollary 7.2.3, p. 466).

Here is another important property of \( F \).

**Theorem 1** If \( f \) and \( f_y \) are continuous on \([a, b] \times [c, d] \), then

\[ F(y) = \int_a^b f(x, y) \, dx, \quad c \leq y \leq d, \quad (1) \]

is continuously differentiable on \([c, d] \) and \( F'(y) \) can be obtained by differentiating (1) under the integral sign with respect to \( y \); that is,

\[ F'(y) = \int_a^b f_y(x, y) \, dx, \quad c \leq y \leq d. \quad (2) \]

Here \( F'(a) \) and \( f_y(x, a) \) are derivatives from the right and \( F'(b) \) and \( f_y(x, b) \) are derivatives from the left.

**Proof** If \( y \) and \( y + \Delta y \) are in \([c, d] \) and \( \Delta y \neq 0 \), then

\[ \frac{F(y + \Delta y) - F(y)}{\Delta y} = \int_a^b \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \, dx. \quad (3) \]

From the mean value theorem (Theorem 2.3.11, p. 83), if \( x \in [a, b] \) and \( y, y + \Delta y \in [c, d] \), there is a \( y(x) \) between \( y \) and \( y + \Delta y \) such that

\[ f(x, y + \Delta y) - f(x, y) = f_y(x, y) \Delta y = f_y(x, y(x)) \Delta y + (f_y(x, y(x)) - f_y(x, y)) \Delta y. \]
From this and (3),
\[
\left| \frac{F(y + \Delta y) - F(y)}{\Delta y} - \int_a^b f_y(x, y) \, dx \right| \leq \int_a^b |f_y(x, y(x)) - f_y(x, y)| \, dx. \tag{4}
\]
Now suppose \( \epsilon > 0 \). Since \( f_y \) is uniformly continuous on the compact set \([a, b] \times [c, d]\) (Corollary 5.2.14, p. 314) and \( y(x) \) is between \( y \) and \( y + \Delta y \), there is a \( \delta > 0 \) such that if \( |\Delta| < \delta \) then
\[
|f_y(x, y) - f_y(x, y(x))| < \epsilon, \quad (x, y) \in [a, b] \times [c, d].
\]
This and (4) imply that
\[
\left| \frac{F(y + \Delta y - F(y)}{\Delta y} - \int_a^b f_y(x, y) \, dx \right| < \epsilon(b-a)
\]
if \( y \) and \( y + \Delta y \) are in \([c, d]\) and \( 0 < |\Delta y| < \delta \). This implies (2). Since the integral in (2) is continuous on \([c, d]\) (Exercise 7.2.3, p. 481, with \( f \) replaced by \( f_y \)), \( F' \) is continuous on \([c, d]\).

**Example 1**
Since
\[
f(x, y) = \cos xy \quad \text{and} \quad f_y(x, y) = -x \sin xy
\]
are continuous for all \((x, y)\), Theorem 1 implies that if
\[
F(y) = \int_0^\pi \cos xy \, dx, \quad -\infty < y < \infty, \tag{5}
\]
then
\[
F'(y) = -\int_0^\pi x \sin xy \, dx, \quad -\infty < y < \infty. \tag{6}
\]
(In applying Theorem 1 for a specific value of \( y \), we take \( R = [0, \pi] \times [-\rho, \rho] \), where \( \rho > |y| \).) This provides a convenient way to evaluate the integral in (6): integrating the right side of (5) with respect to \( x \) yields
\[
F(y) = \left. \frac{\sin xy}{y} \right|_{x=0}^\pi = \frac{\sin \pi y}{y}, \quad y \neq 0.
\]
Differentiating this and using (6) yields
\[
\int_0^\pi x \sin xy \, dx = \frac{\sin \pi y}{y^2} - \frac{\pi \cos \pi y}{y}, \quad y \neq 0.
\]
To verify this, use integration by parts.

We will study the continuity, differentiability, and integrability of
\[
F(y) = \int_a^b f(x, y) \, dx, \quad y \in S,
\]
where $S$ is an interval or a union of intervals, and $F$ is a convergent improper integral for each $y \in S$. If the domain of $f$ is $[a, b) \times S$ where $-\infty < a < b \leq \infty$, we say that $F$ is pointwise convergent on $S$ or simply convergent on $S$, and write

$$
\int_{a}^{b} f(x, y) \, dx = \lim_{r \to b-} \int_{a}^{r} f(x, y) \, dx
$$

(7)

if, for each $y \in S$ and every $\epsilon > 0$, there is an $r = r_{0}(y)$ (which also depends on $\epsilon$) such that

$$
|F(y) - \int_{a}^{r} f(x, y) \, dx| = \left| \int_{r}^{b} f(x, y) \, dx \right| < \epsilon, \quad r_{0}(y) \leq y < b.
$$

(8)

If the domain of $f$ is $(a, b) \times S$ where $-\infty \leq a < b < \infty$, we replace (7) by

$$
\int_{a}^{b} f(x, y) \, dx = \lim_{r \to a+} \int_{r}^{b} f(x, y) \, dx
$$

and (8) by

$$
|F(y) - \int_{r}^{b} f(x, y) \, dx| = \left| \int_{a}^{r} f(x, y) \, dx \right| < \epsilon, \quad a < r \leq r_{0}(y).
$$

In general, pointwise convergence of $F$ for all $y \in S$ does not imply that $F$ is continuous or integrable on $[c, d]$, and the additional assumptions that $f_y$ is continuous and $\int_{a}^{b} f_y(x, y) \, dx$ converges do not imply (2).

**Example 2** The function

$$
f(x, y) = ye^{-|y|x}
$$

is continuous on $[0, \infty) \times (-\infty, \infty)$ and

$$
F(y) = \int_{0}^{\infty} f(x, y) \, dx = \int_{0}^{\infty} ye^{-|y|x} \, dx
$$

converges for all $y$, with

$$
F(y) = \begin{cases} 
-1 & y < 0, \\
0 & y = 0, \\
1 & y > 0;
\end{cases}
$$

therefore, $F$ is discontinuous at $y = 0$.

**Example 3** The function

$$
f(x, y) = y^3 e^{-y^2 x}
$$

is continuous on $[0, \infty) \times (-\infty, \infty)$. Let

$$
F(y) = \int_{0}^{\infty} f(x, y) \, dx = \int_{0}^{\infty} y^3 e^{-y^2 x} \, dx = y, \quad -\infty < y < \infty.
$$
Then
\[ F'(y) = 1, \quad -\infty < y < \infty. \]

However,
\[ \int_0^\infty \frac{\partial}{\partial y} (y^3 e^{-y^2 x}) \, dx = \int_0^\infty (3y^2 - 2y^4 x) e^{-y^2 x} \, dx = \begin{cases} 1, & y \neq 0, \\ 0, & y = 0. \end{cases} \]
so
\[ F'(y) \neq \int_0^\infty \frac{\partial f(x,y)}{\partial y} \, dx \quad \text{if} \quad y = 0. \]

### 3 Preparation

We begin with two useful convergence criteria for improper integrals that do not involve a parameter. Consistent with the definition on p. 152, we say that \( f \) is locally integrable on an interval \( I \) if it is integrable on every finite closed subinterval of \( I \).

**Theorem 2 (Cauchy Criterion for Convergence of an Improper Integral I)** Suppose \( g \) is locally integrable on \([a,b] \) and denote
\[ G(r) = \int_a^r g(x) \, dx, \quad a \leq r < b. \]
Then the improper integral \( \int_a^b g(x) \, dx \) converges if and only if, for each \( \epsilon > 0 \), there is an \( r_0 \in [a,b] \) such that
\[ |G(r) - G(r_1)| < \epsilon, \quad r_0 \leq r, r_1 < b. \]  \( (9) \)

**Proof** For necessity, suppose \( \int_a^b g(x) \, dx = L. \) By definition, this means that for each \( \epsilon > 0 \) there is an \( r_0 \in [a,b] \) such that
\[ |G(r) - L| < \frac{\epsilon}{2} \quad \text{and} \quad |G(r_1) - L| < \frac{\epsilon}{2}, \quad r_0 \leq r, r_1 < b. \]
Therefore
\[ |G(r) - G(r_1)| = |(G(r) - L) - (G(r_1) - L)| \leq |G(r) - L| + |G(r_1) - L| < \epsilon, \quad r_0 \leq r, r_1 < b. \]
For sufficiency, \( (9) \) implies that
\[ |G(r)| = |G(r_1) + (G(r) - G(r_1))| < |G(r_1)| + |G(r) - G(r_1)| \leq |G(r_1)| + \epsilon, \quad r_0 \leq r \leq r_1 < b. \]
Since \( G \) is also bounded on the compact set \([a,r_0] \) (Theorem 5.2.11, p. 313), \( G \) is bounded on \([a,b] \). Therefore the monotonic functions
\[ \overline{G}(r) = \sup \{G(r_1) \mid r \leq r_1 < b\} \quad \text{and} \quad \underline{G}(r) = \inf \{G(r_1) \mid r \leq r_1 < b\} \]
are well defined on \([a, b]\), and

\[
\lim_{r \to b^-} \overline{G}(r) = \overline{L} \quad \text{and} \quad \lim_{r \to b^-} \underline{G}(r) = \underline{L}
\]

both exist and are finite (Theorem 2.1.11, p. 47). From (9),

\[
|G(r) - G(r_1)| = |(G(r) - G(r_0)) - (G(r_1) - G(r_0))| \\
\leq |G(r) - G(r_0)| + |G(r_1) - G(r_0)| < 2\varepsilon,
\]

so

\[
\overline{G}(r) - \underline{G}(r) \leq 2\varepsilon, \quad r_0 \leq r, r_1 < b.
\]

Since \(\varepsilon\) is an arbitrary positive number, this implies that

\[
\lim_{r \to b^-} (\overline{G}(r) - \underline{G}(r)) = 0,
\]

so \(\overline{L} = \underline{L}\). Let \(L = \overline{L} = \underline{L}\). Since

\[
\underline{G}(r) \leq G(r) \leq \overline{G}(r),
\]

it follows that \(\lim_{r \to b^-} G(r) = L\). □

We leave the proof of the following theorem to you (Exercise 2).

**Theorem 3 (Cauchy Criterion for Convergence of an Improper Integral II)** Suppose \(g\) is locally integrable on \((a, b]\) and denote

\[
G(r) = \int_r^b g(x) \, dx, \quad a \leq r < b.
\]

Then the improper integral \(\int_a^b g(x) \, dx\) converges if and only if, for each \(\varepsilon > 0\), there is an \(r_0 \in (a, b]\) such that

\[
|G(r) - G(r_1)| < \varepsilon, \quad a < r, r_1 \leq r_0.
\]

To see why we associate Theorems 2 and 3 with Cauchy, compare them with Theorem 4.3.5 (p. 204)

### 4 Uniform convergence of improper integrals

Henceforth we deal with functions \(f = f(x, y)\) with domains \(I \times S\), where \(S\) is an interval or a union of intervals and \(I\) is of one of the following forms:

- \([a, b]\) with \(-\infty < a < b \leq \infty\);
- \((a, b]\) with \(-\infty \leq a < b < \infty\);
- \((a, b)\) with \(-\infty \leq a \leq b \leq \infty\).
In all cases it is to be understood that $f$ is locally integrable with respect to $x$ on $I$. When we say that the improper integral $\int_a^b f(x, y) \, dx$ has a stated property “on $S$” we mean that it has the property for every $y \in S$.

**Definition 1** If the improper integral

$$\int_a^b f(x, y) \, dx = \lim_{r \to b} \int_a^r f(x, y) \, dx \quad (10)$$

converges on $S$, it is said to converge uniformly (or be uniformly convergent) on $S$ if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that

$$\left| \int_a^b f(x, y) \, dx - \int_a^r f(x, y) \, dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r < b,$$

or, equivalently,

$$\left| \int_r^b f(x, y) \, dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r < b. \quad (11)$$

The crucial difference between pointwise and uniform convergence is that $r_0(y)$ in (8) may depend upon the particular value of $y$, while the $r_0$ in (11) does not: one choice must work for all $y \in S$. Thus, uniform convergence implies pointwise convergence, but pointwise convergence does not imply uniform convergence.

**Theorem 4 (Cauchy Criterion for Uniform Convergence I)** The improper integral in (10) converges uniformly on $S$ if and only if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that

$$\left| \int_r^{r_1} f(x, y) \, dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r, r_1 < b. \quad (12)$$

**Proof** Suppose $\int_a^b f(x, y) \, dx$ converges uniformly on $S$ and $\epsilon > 0$. From Definition 1, there is an $r_0 \in [a, b)$ such that

$$\left| \int_a^b f(x, y) \, dx \right| < \frac{\epsilon}{2} \quad \text{and} \quad \left| \int_a^y f(x, y) \, dx \right| < \frac{\epsilon}{2}, \quad y \in S, \quad r_0 \leq r, r_1 < b. \quad (13)$$

Since

$$\int_r^{r_1} f(x, y) \, dx = \int_r^b f(x, y) \, dx - \int_{r_1}^b f(x, y) \, dx,$$

(13) and the triangle inequality imply (12).

For the converse, denote

$$F(y) = \int_a^r f(x, y) \, dx.$$
Since (12) implies that
\[ |F(r, y) - F(r_1, y)| < \epsilon, \quad y \in S, \quad r_0 \leq r, r_1 < b, \]  
(14)

Theorem 2 with \( G(r) = F(r, y) \) (\( y \) fixed but arbitrary in \( S \)) implies that \( \int_a^b f(x, y) \, dx \) converges pointwise for \( y \in S \). Therefore, if \( \epsilon > 0 \) then, for each \( y \in S \), there is an \( r_0(y) \in [a, b) \) such that
\[ \left| \int_r^b f(x, y) \, dx \right| < \epsilon, \quad y \in S, \quad r_0(y) \leq r < b. \]  
(15)

For each \( y \in S \), choose \( r_1(y) \geq \max\{r_0(y), r_0\} \). (Recall (14)). Then
\[ \int_r^b f(x, y) \, dx = \int_r^{r_1(y)} f(x, y) \, dx + \int_{r_1(y)}^b f(x, y) \, dx, \]
so (12), (15), and the triangle inequality imply that
\[ \left| \int_r^b f(x, y) \, dx \right| < 2\epsilon, \quad y \in S, \quad r_0 \leq r < b. \]

In practice, we don’t explicitly exhibit \( r_0 \) for each given \( \epsilon \). It suffices to obtain estimates that clearly imply its existence.

**Example 4** For the improper integral of Example 2,
\[ \left| \int_r^\infty f(x, y) \, dx \right| = \int_r^\infty |y| \, dx = e^{-y|x|}, \quad y \neq 0. \]
If \( |y| \geq \rho \), then
\[ \left| \int_r^\infty f(x, y) \, dx \right| \leq e^{-r\rho}, \]
so \( \int_0^\infty f(x, y) \, dx \) converges uniformly on \((\rho, \infty)\) if \( \rho > 0 \); however, it does not converge uniformly on any neighborhood of \( y = 0 \), since, for any \( r > 0 \), \( e^{-r|y|} > \frac{1}{2} \) if \( |y| \) is sufficiently small.

**Definition 2** If the improper integral
\[ \int_a^b f(x, y) \, dx = \lim_{r \to -a^+} \int_r^b f(x, y) \, dx \]
converges on \( S \), it is said to converge uniformly (or be uniformly convergent) on \( S \) if, for each \( \epsilon > 0 \), there is an \( r_0 \in (a, b] \) such that
\[ \left| \int_a^b f(x, y) \, dx - \int_r^b f(x, y) \, dx \right| < \epsilon, \quad y \in S, \quad a < r \leq r_0, \]
or, equivalently,
\[ \left| \int_a^r f(x, y) \, dx \right| < \epsilon, \quad y \in S, \quad a < r \leq r_0. \]
We leave proof of the following theorem to you (Exercise 3).

**Theorem 5 (Cauchy Criterion for Uniform Convergence II)** The improper integral

\[
\int_a^b f(x, y) \, dx = \lim_{r \to a} \int_r^b f(x, y) \, dx
\]

converges uniformly on \( S \) if and only if, for each \( \varepsilon > 0 \), there is an \( r_0 \in (a, b] \) such that

\[
\left| \int_{r_1}^r f(x, y) \, dx \right| < \varepsilon, \quad y \in S, \quad a < r, r_1 \leq r_0.
\]

We need one more definition, as follows.

**Definition 3** Let \( f = f(x, y) \) be defined on \((a, b) \times S\), where \(-\infty \leq a < b \leq \infty\). Suppose \( f \) is locally integrable on \((a, b)\) for all \( y \in S \) and let \( c \) be an arbitrary point in \((a, b)\). Then \( \int_a^b f(x, y) \, dx \) is said to converge uniformly on \( S \) if \( \int_a^c f(x, y) \, dx \) and \( \int_c^b f(x, y) \, dx \) both converge uniformly on \( S \).

We leave it to you (Exercise 4) to show that this definition is independent of \( c \); that is, if \( \int_a^c f(x, y) \, dx \) and \( \int_c^b f(x, y) \, dx \) both converge uniformly on \( S \) for some \( c \in (a, b) \), then they both converge uniformly on \( S \) for every \( c \in (a, b) \).

We also leave it you (Exercise 5) to show that if \( f \) is bounded on \([a, b] \times [c, d]\) and \( \int_a^b f(x, y) \, dx \) exists as a proper integral for each \( y \in [c, d] \), then it converges uniformly on \([c, d]\) according to all three Definitions 1–3.

**Example 5** Consider the improper integral

\[
F(y) = \int_0^\infty x^{-1/2} e^{-xy} \, dx,
\]

which diverges if \( y \leq 0 \) (verify). Definition 3 applies if \( y > 0 \), so we consider the improper integrals

\[
F_1(y) = \int_0^1 x^{-1/2} e^{-xy} \, dx \quad \text{and} \quad F_2(y) = \int_1^\infty x^{-1/2} e^{-xy} \, dx
\]

separately. Moreover, we could just as well define

\[
F_1(y) = \int_0^c x^{-1/2} e^{-xy} \, dx \quad \text{and} \quad F_2(y) = \int_c^\infty x^{-1/2} e^{-xy} \, dx,
\]

(16)

where \( c \) is any positive number.

Definition 2 applies to \( F_1 \). If \( 0 < r_1 < r \) and \( y \geq 0 \), then

\[
\left| \int_{r_1}^r x^{-1/2} e^{-xy} \, dx \right| < \int_{r_1}^r x^{-1/2} \, dx < 2r^{1/2},
\]

so \( F_1(y) \) converges for uniformly on \([0, \infty)\).
Definition 1 applies to $F_2$. Since
$$\left| \int_r^{r_1} x^{-1/2} e^{-xy} \, dx \right| < r^{-1/2} \int_r^{\infty} e^{-xy} \, dx = \frac{e^{-ry}}{yr^{1/2}},$$
$F_2(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$. It does not converge uniformly on $(0, \rho)$, since the change of variable $u = xy$ yields
$$\int_r^{r_1} x^{-1/2} e^{-xy} \, dx = \int_{ry}^{r_1y} u^{-1/2} e^{-u} \, du,$$
which, for any fixed $r > 0$, can be made arbitrarily large by taking $y$ sufficiently small and $r = 1/y$. Therefore we conclude that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

Note that the constant $c$ in (16) plays no role in this argument.

**Example 6** Suppose we take
$$\int_0^\infty \frac{\sin u}{u} \, du = \frac{\pi}{2}$$
as given (Exercise 31(b)). Substituting $u = xy$ with $y > 0$ yields
$$\int_0^\infty \frac{\sin xy}{x} \, dx = \frac{\pi}{2}, \quad y > 0.$$ 

What about uniform convergence? Since $(\sin xy)/x$ is continuous at $x = 0$, Definition 1 and Theorem 4 apply here. If $0 < r < r_1$ and $y > 0$, then
$$\int_r^{r_1} \frac{\sin xy}{x} \, dx = -\frac{1}{y} \left( \cos xy \bigg|_r^{r_1} + \int_r^{r_1} \frac{\cos xy}{x^2} \, dx \right),$$
so
$$\left| \int_r^{r_1} \frac{\sin xy}{x} \, dx \right| < \frac{3}{ry}.$$ 
Therefore (18) converges uniformly on $[\rho, \infty)$ if $\rho > 0$. On the other hand, from (17), there is a $\delta > 0$ such that
$$\int_{u_0}^{\infty} \frac{\sin u}{u} \, du > \frac{\pi}{4}, \quad 0 \leq u_0 < \delta.$$ 
This and (18) imply that
$$\int_r^{\infty} \frac{\sin xy}{x} \, dx = \int_{yr}^{\infty} \frac{\sin u}{u} \, du > \frac{\pi}{4}$$
for any $r > 0$ if $0 < y < \delta/r$. Hence, (18) does not converge uniformly on any interval $(0, \rho]$ with $\rho > 0$. 

10
5 Absolutely Uniformly Convergent Improper Integrals

Definition 4 (Absolute Uniform Convergence I) The improper integral
\[ \int_{a}^{b} f(x, y) \, dx = \lim_{r \to b-} \int_{a}^{r} f(x, y) \, dx \]
is said to converge absolutely uniformly on \( S \) if the improper integral
\[ \int_{a}^{b} |f(x, y)| \, dx = \lim_{r \to b-} \int_{a}^{r} |f(x, y)| \, dx \]
converges uniformly on \( S \); that is, for each \( \varepsilon > 0 \), there is an \( r_0 \in [a, b) \) such that
\[ \left| \int_{a}^{r_1} |f(x, y)| \, dx - \int_{a}^{r} |f(x, y)| \, dx \right| < \varepsilon, \quad y \in S, \quad r_0 < r < b. \]

To see that this definition makes sense, recall that if \( f \) is locally integrable on \([a, b]\) for all \( y \) in \( S \), then so is \( |f| \) (Theorem 3.4.9, p. 161). Theorem 4 with \( f \) replaced by \( |f| \) implies that \( \int_{a}^{b} f(x, y) \, dx \) converges absolutely uniformly on \( S \) if and only if, for each \( \varepsilon > 0 \), there is an \( r_0 \in [a, b) \) such that
\[ \int_{r}^{r_1} |f(x, y)| \, dx < \varepsilon, \quad y \in S, \quad r_0 < r < b. \]

Since
\[ \left| \int_{r}^{r_1} f(x, y) \, dx \right| \leq \int_{r}^{r_1} |f(x, y)| \, dx, \]
Theorem 4 implies that if \( \int_{a}^{b} f(x, y) \, dx \) converges absolutely uniformly on \( S \) then it converges uniformly on \( S \).

Theorem 6 (Weierstrass’s Test for Absolute Uniform Convergence I) Suppose \( M = M(x) \) is nonnegative on \([a, b]\), \( \int_{a}^{b} M(x) \, dx < \infty \), and
\[ |f(x, y)| \leq M(x), \quad y \in S, \quad a \leq x < b. \quad (19) \]
Then \( \int_{a}^{b} f(x, y) \, dx \) converges absolutely uniformly on \( S \).

Proof Denote \( \int_{a}^{b} M(x) \, dx = L < \infty \). By definition, for each \( \varepsilon > 0 \) there is an \( r_0 \in [a, b) \) such that
\[ L - \varepsilon < \int_{a}^{r} M(x) \, dx \leq L, \quad r_0 < r < b. \]
Therefore, if \( r_0 < r \leq r_1 \), then
\[ 0 \leq \int_{r}^{r_1} M(x) \, dx = \left( \int_{a}^{r_1} M(x) \, dx - L \right) - \left( \int_{a}^{r} M(x) \, dx - L \right) < \varepsilon \]
This and (19) imply that
\[
\int_{r}^{r_1} |f(x, y)| \, dx \leq \int_{r}^{r_1} M(x) \, dx < \epsilon, \quad y \in S, \quad a \leq r < r_1 < b.
\]
Now Theorem 4 implies the stated conclusion.

Example 7 Suppose \( g = g(x, y) \) is locally integrable on \([0, \infty)\) for all \( y \in S \) and, for some \( a_0 \geq 0 \), there are constants \( K \) and \( p_0 \) such that
\[
|g(x, y)| \leq Ke^{p_0x}, \quad y \in S, \quad x \geq a_0.
\]
If \( p > p_0 \) and \( r \geq a_0 \), then
\[
\int_{r}^{\infty} e^{-px} |g(x, y)| \, dx = \int_{r}^{\infty} e^{-(p-p_0)x} e^{-p_0x} |g(x, y)| \, dx \\
\leq K \int_{r}^{\infty} e^{-(p-p_0)x} \, dx = \frac{Ke^{-(p-p_0)x}}{p-p_0},
\]
so \( \int_{r}^{\infty} e^{-px} g(x, y) \, dx \) converges absolutely on \( S \). For example, since
\[
|x^\alpha \sin xy| < e^{p_0x} \quad \text{and} \quad |x^\alpha \cos xy| < e^{p_0x}
\]
for \( x \) sufficiently large if \( p_0 > 0 \), Theorem 4 implies that \( \int_{0}^{\infty} e^{-px} x^\alpha \sin xy \, dx \) and \( \int_{0}^{\infty} e^{-px} x^\alpha \cos xy \, dx \) converge absolutely uniformly on \((-\infty, \infty)\) if \( p > 0 \) and \( \alpha \geq 0 \). As a matter of fact, \( \int_{0}^{\infty} e^{-px} x^\alpha \sin xy \, dx \) converges absolutely on \((-\infty, \infty)\) if \( p > 0 \) and \( \alpha > -1 \). (Why?)

Definition 5 (Absolute Uniform Convergence II) The improper integral
\[
\int_{a}^{b} f(x, y) \, dx = \lim_{r \to a^+} \int_{r}^{b} f(x, y) \, dx
\]
is said to converge absolutely uniformly on \( S \) if the improper integral
\[
\int_{a}^{b} |f(x, y)| \, dx = \lim_{r \to a^+} \int_{r}^{b} |f(x, y)| \, dx
\]
converges uniformly on \( S \); that is, if, for each \( \epsilon > 0 \), there is an \( r_0 \in (a, b] \) such that
\[
\left| \int_{a}^{b} |f(x, y)| \, dx - \int_{r}^{b} |f(x, y)| \, dx \right| < \epsilon, \quad y \in S, \quad a < r < r_0 \leq b.
\]
We leave it to you (Exercise 7) to prove the following theorem.

Theorem 7 (Weierstrass’s Test for Absolute Uniform Convergence II) Suppose \( M = M(x) \) is nonnegative on \((a, b]\), \( \int_{a}^{b} M(x) \, dx < \infty \), and
\[
|f(x, y)| \leq M(x), \quad y \in S, \quad x \in (a, b].
\]
Then \( \int_{a}^{b} f(x, y) \, dx \) converges absolutely uniformly on \( S \).
Example 8 If \( g = g(x, y) \) is locally integrable on \( (0, 1] \) for all \( y \in S \) and 
\[
|g(x, y)| \leq Ax^{-\beta}, \quad 0 < x \leq x_0,
\]
for each \( y \in S \), then 
\[
\int_0^1 x^\alpha g(x, y) \, dx
\]
converges absolutely uniformly on \( S \) if \( \alpha > \beta - 1 \). To see this, note that if \( 0 < r < r_1 \leq x_0 \), then 
\[
\int_{r_1}^r x^\alpha |g(x, y)| \, dx \leq A \int_{r_1}^r x^{\alpha-\beta} \, dx = \frac{Ax^{\alpha-\beta+1}}{\alpha - \beta + 1} \bigg|_{r_1}^r < \frac{Ar_1^{\alpha-\beta+1}}{\alpha - \beta + 1}.
\]
Applying this with \( \beta = 0 \) shows that 
\[
F(y) = \int_0^1 x^\alpha \cos xy \, dx
\]
converges absolutely uniformly on \((-\infty, \infty)\) if \( \alpha > -1 \) and 
\[
G(y) = \int_0^1 x^\alpha \sin xy \, dx
\]
converges absolutely uniformly on \((-\infty, \infty)\) if \( \alpha > -2 \).

By recalling Theorem 4.4.15 (p. 246), you can see why we associate Theorems 6 and 7 with Weierstrass.

6 Dirichlet’s Tests

Weierstrass’s test is useful and important, but it has a basic shortcoming: it applies only to absolutely uniformly convergent improper integrals. The next theorem applies in some cases where \( \int_a^b f(x, y) \, dx \) converges uniformly on \( S \), but \( \int_a^b |f(x, y)| \, dx \) does not.

Theorem 8 (Dirichlet’s Test for Uniform Convergence I) If \( g, g_x, \) and \( h \) are continuous on \([a, b] \times S\), then 
\[
\int_a^b g(x, y)h(x, y) \, dx
\]
converges uniformly on \( S \) if the following conditions are satisfied:

(a) \( \lim_{x \to b^-} \left\{ \sup_{y \in S} |g(x, y)| \right\} = 0; \)

(b) There is a constant \( M \) such that 
\[
\sup_{y \in S} \left| \int_a^x h(u, y) \, du \right| < M, \quad a \leq x < b;
\]
(c) $\int_a^b |g_x(x, y)| \, dx$ converges uniformly on $S$.

**Proof**  If

$$H(x, y) = \int_a^x h(u, y) \, du,$$

then integration by parts yields

$$
\int_r^{r_1} g(x, y) h(x, y) \, dx = \int_r^{r_1} g(x, y) H_x(x, y) \, dx \\
= g(r_1, y) H(r_1, y) - g(r, y) H(r, y) - \int_r^{r_1} g_x(x, y) H(x, y) \, dx.
$$

(21)

Since assumption (b) and (20) imply that $|H(x, y)| \leq M$, $(x, y) \in (a, b) \times S$, Eqn. (21) implies that

$$
\left| \int_r^{r_1} g(x, y) h(x, y) \, dx \right| < M \left( 2 \sup_{x \geq r} |g(x, y)| + \int_r^{r_1} |g_x(x, y)| \, dx \right)
$$

(22)

on $[r, r_1] \times S$.

Now suppose $\epsilon > 0$. From assumption (a), there is an $r_0 \in [a, b]$ such that $|g(x, y)| < \epsilon$ on $S$ if $r_0 \leq x < b$. From assumption (c) and Theorem 6, there is an $s_0 \in [a, b]$ such that

$$\int_r^{r_1} |g_x(x, y)| \, dx < \epsilon, \quad y \in S, \quad s_0 < r < r_1 < b.$$

Therefore (22) implies that

$$
\left| \int_r^{r_1} g(x, y) h(x, y) \, dx \right| < 3M\epsilon, \quad y \in S, \quad \max(r_0, s_0) < r < r_1 < b.
$$

Now Theorem 4 implies the stated conclusion.

The statement of this theorem is complicated, but applying it isn’t; just look for a factorization $f = gh$, where $h$ has a bounded antiderivative on $[a, b)$ and $g$ is “small” near $b$. Then integrate by parts and hope that something nice happens. A similar comment applies to Theorem 9, which follows.

**Example 9** Let

$$I(y) = \int_0^{\infty} \frac{\cos xy}{x + y} \, dx, \quad y > 0.$$

The obvious inequality

$$\frac{\cos xy}{x + y} \leq \frac{1}{x + y}$$

is useless here, since

$$\int_0^{\infty} \frac{dx}{x + y} = \infty.$$
However, integration by parts yields
\[\int_r^{r_1} \frac{\cos xy}{x+y} \, dx = \sin xy \left|_r^{r_1}\right. + \int_r^{r_1} \frac{\sin xy}{y(x+y)^2} \, dx\]
\[= \frac{\sin r_1 y}{y(r_1 + y)} - \frac{\sin r y}{y(r + y)} + \int_r^{r_1} \frac{\sin xy}{y(x+y)^2} \, dx.\]

Therefore, if \(0 < r < r_1\), then
\[\left|\int_r^{r_1} \frac{\cos xy}{x+y} \, dx\right| < \frac{1}{y} \left(\frac{2}{r+y} + \int_r^{\infty} \frac{1}{(x+y)^2}\right) \leq \frac{3}{y(r+y)^2} \leq \frac{3}{\rho(r+\rho)}\]
if \(y \geq \rho > 0\). Now Theorem 4 implies that \(I(y)\) converges uniformly on \([\rho, \infty)\) if \(\rho > 0\).

We leave the proof of the following theorem to you (Exercise 10).

**Theorem 9 (Dirichlet’s Test for Uniform Convergence II)** If \(g, g_x,\) and \(h\) are continuous on \((a, b) \times S\), then
\[\int_a^b g(x, y)h(x, y) \, dx\]
converges uniformly on \(S\) if the following conditions are satisfied:

(a) \(\lim_{x \to a^+} \left\{\sup_{y \in S} |g(x, y)|\right\} = 0;\)

(b) There is a constant \(M\) such that
\[\sup_{y \in S} \left|\int_x^b h(u, y) \, du\right| \leq M, \quad a < x \leq b;\]

(c) \(\int_a^b |g_x(x, y)| \, dx\) converges uniformly on \(S\).

By recalling Theorems 3.4.10 (p. 163), 4.3.20 (p. 217), and 4.4.16 (p. 248), you can see why we associate Theorems 8 and 9 with Dirichlet.

### 7 Consequences of uniform convergence

**Theorem 10** If \(f = f(x, y)\) is continuous on either \([a, b] \times [c, d]\) or \((a, b) \times [c, d]\) and
\[F(y) = \int_a^b f(x, y) \, dx\]
converges uniformly on \([c, d]\), then \(F(y)\) is continuous on \([c, d]\). Moreover,
\[\int_c^d \left(\int_a^b f(x, y) \, dx\right) \, dy = \int_a^b \left(\int_c^d f(x, y) \, dy\right) \, dx.\]
Proof We will assume that \( f \) is continuous on \( [a, b] \times [c, d] \). You can consider the other case (Exercise 4).

We will first show that \( F \) in (23) is continuous on \( [c, d] \). Since \( F \) converges uniformly on \( [c, d] \), Definition 1 (specifically, (11)) implies that if \( \epsilon > 0 \), there is an \( r \in [a, b] \) such that

\[
\left| \int_{a}^{b} f(x, y) \, dx \right| < \epsilon, \quad c \leq y \leq d.
\]

Therefore, if \( c \leq y, y_0 \leq d \), then

\[
|F(y) - F(y_0)| = \left| \int_{a}^{b} f(x, y) \, dx - \int_{a}^{b} f(x, y_0) \, dx \right|
\leq \left| \int_{a}^{r} [f(x, y) - f(x, y_0)] \, dx \right| + \left| \int_{r}^{b} f(x, y) \, dx \right|
+ \left| \int_{r}^{b} f(x, y_0) \, dx \right|,
\]

so

\[
|F(y) - F(y_0)| \leq \int_{a}^{r} |f(x, y) - f(x, y_0)| \, dx + 2\epsilon. \tag{25}
\]

Since \( f \) is uniformly continuous on the compact set \([a, r] \times [c, d]\) (Corollary 5.2.14, p. 314), there is a \( \delta > 0 \) such that

\[
|f(x, y) - f(x, y_0)| < \epsilon
\]
if \( (x, y) \) and \( (x, y_0) \) are in \([a, r] \times [c, d]\) and \( |y - y_0| < \delta \). This and (25) imply that

\[
|F(y) - F(y_0)| < (r - a)\epsilon + 2\epsilon < (b - a + 2)\epsilon
\]
if \( y \) and \( y_0 \) are in \([c, d]\) and \( |y - y_0| < \delta \). Therefore \( F \) is continuous on \([c, d]\), so the integral on left side of (24) exists. Denote

\[
I = \int_{c}^{d} \left( \int_{a}^{b} f(x, y) \, dx \right) \, dy. \tag{26}
\]

We will show that the improper integral on the right side of (24) converges to \( I \). To this end, denote

\[
I(r) = \int_{a}^{r} \left( \int_{c}^{d} f(x, y) \, dy \right) \, dx.
\]

Since we can reverse the order of integration of the continuous function \( f \) over the rectangle \([a, r] \times [c, d]\) (Corollary 7.2.2, p. 466),

\[
I(r) = \int_{c}^{d} \left( \int_{a}^{r} f(x, y) \, dx \right) \, dy.
\]
From this and (26),
\[
I - I(r) = \int_c^d \left( \int_r^d f(x, y) \, dx \right) \, dy.
\]
Now suppose \( \epsilon > 0 \). Since \( \int_a^b f(x, y) \, dx \) converges uniformly on \([c, d]\), there is an \( r_0 \in (a, b) \) such that
\[
\left| \int_r^b f(x, y) \, dx \right| < \epsilon, \quad r < r_0 < b,
\]
so \( |I - I(r)| < (d - c)\epsilon \) if \( r_0 < r < b \). Hence,
\[
\lim_{r \to b^-} \int_a^r \left( \int_c^d f(x, y) \, dy \right) \, dx = \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy,
\]
which completes the proof of (24).

**Example 10** It is straightforward to verify that
\[
\int_0^\infty e^{-xy} \, dx = \frac{1}{y}, \quad y > 0,
\]
and the convergence is uniform on \([\rho, \infty)\) if \( \rho > 0 \). Therefore Theorem 10 implies that if \( 0 < y_1 < y_2 \), then
\[
\int_{y_1}^{y_2} \frac{dy}{y} = \int_{y_1}^{y_2} \left( \int_0^\infty e^{-xy} \, dx \right) \, dy = \int_0^\infty \left( \int_{y_1}^{y_2} e^{-xy} \, dy \right) \, dx
\]
\[
= \int_0^\infty \frac{e^{-yx_1} - e^{-yx_2}}{x} \, dx.
\]
Since
\[
\int_{y_1}^{y_2} \frac{dy}{y} = \log \frac{y_2}{y_1}, \quad y_2 \geq y_1 > 0,
\]
it follows that
\[
\int_0^\infty \frac{e^{-yx_1} - e^{-yx_2}}{x} \, dx = \log \frac{y_2}{y_1}, \quad y_2 \geq y_1 > 0.
\]

**Example 11** From Example 6,
\[
\int_0^\infty \frac{\sin xy}{x} \, dx = \frac{\pi}{2}, \quad y > 0,
\]
and the convergence is uniform on \([\rho, \infty)\) if \( \rho > 0 \). Therefore, Theorem 10 implies that if \( 0 < y_1 < y_2 \), then
\[
\frac{\pi}{2} (y_2 - y_1) = \int_{y_1}^{y_2} \left( \int_0^\infty \frac{\sin xy}{x} \, dx \right) \, dy = \int_0^\infty \left( \int_{y_1}^{y_2} \frac{\sin xy}{x} \, dy \right) \, dx
\]
\[
= \int_0^\infty \frac{\cos xy_1 - \cos xy_2}{x^2} \, dx.
\]
(27)
The last integral converges uniformly on \((-\infty, \infty)\) (Exercise 10(h)), and is therefore continuous with respect to \(y_1\) on \((-\infty, \infty)\), by Theorem 10; in particular, we can let \(y_1 \to 0^+\) in (27) and replace \(y_2\) by \(y\) to obtain
\[
\int_0^\infty \frac{1 - \cos xy}{x^2} \, dx = \frac{\pi y}{2}, \quad y \geq 0.
\]

The next theorem is analogous to Theorem 4.4.20 (p. 252).

**Theorem 11** Let \(f\) and \(f_y\) be continuous on either \([a, b] \times [c, d]\) or \((a, b) \times [c, d]\). Suppose that the improper integral
\[
F(y) = \int_a^b f(x, y) \, dx
\]
converges for some \(y_0 \in [c, d]\) and
\[
G(y) = \int_a^b f_y(x, y) \, dx
\]
converges uniformly on \([c, d]\). Then \(F\) converges uniformly on \([c, d]\) and is given explicitly by
\[
F(y) = F(y_0) + \int_{y_0}^y G(t) \, dt, \quad c \leq y \leq d.
\]
Moreover, \(F\) is continuously differentiable on \([c, d]\); specifically,
\[
F'(y) = G(y), \quad c \leq y \leq d,
\]
where \(F'(c)\) and \(f_y(x, c)\) are derivatives from the right, and \(F'(d)\) and \(f_y(x, d)\) are derivatives from the left.

**Proof** We will assume that \(f\) and \(f_y\) are continuous on \([a, b] \times [c, d]\). You can consider the other case (Exercise 15).

Let
\[
F_r(y) = \int_a^r f(x, y) \, dx, \quad a \leq r < b, \quad c \leq y \leq d.
\]
Since \(f\) and \(f_y\) are continuous on \([a, r] \times [c, d]\), Theorem 1 implies that
\[
F'_r(y) = \int_a^r f_y(x, y) \, dx, \quad c \leq y \leq d.
\]
Then
\[
F_r(y) = F_r(y_0) + \int_{y_0}^y \left( \int_a^r f_y(x, t) \, dx \right) \, dt
\]
\[
= F(y_0) + \int_{y_0}^y G(t) \, dt
\]
\[
+(F_r(y_0) - F(y_0)) - \int_{y_0}^y \left( \int_r^b f_y(x, t) \, dx \right) \, dt, \quad c \leq y \leq d.
\]
Therefore,
\[
\left| F_r(y) - F(y_0) - \int_{y_0}^y G(t) \, dt \right| \leq \left| F_r(y_0) - F(y_0) \right| + \left| \int_{y_0}^y \int_r^b f_y(x, t) \, dx \right| \, dt.
\] (29)

Now suppose \( \epsilon > 0 \). Since we have assumed that \( \lim_{r \to b^-} F_r(y_0) = F(y_0) \) exists, there is an \( r_0 \) in \((a, b)\) such that
\[
|F_r(y_0) - F(y_0)| < \epsilon, \quad r_0 < r < b.
\]

Since we have assumed that \( G(y) \) converges for \( y \in [c, d] \), there is an \( r_1 \in [a, b) \) such that
\[
\left| \int_r^b f_y(x, t) \, dx \right| < \epsilon, \quad t \in [c, d], \quad r_1 < r < b.
\]

Therefore, (29) yields
\[
\left| F_r(y) - F(y_0) - \int_{y_0}^y G(t) \, dt \right| < \epsilon(1 + |y - y_0|) \leq \epsilon(1 + d - c)
\]
if \( \max(r_0, r_1) \leq r < b \) and \( t \in [c, d] \). Therefore \( F(y) \) converges uniformly on \([c, d]\) and
\[
F(y) = F(y_0) + \int_{y_0}^y G(t) \, dt, \quad c \leq y \leq d.
\]

Since \( G \) is continuous on \([c, d]\) by Theorem 10, (28) follows from differentiating this (Theorem 3.3.11, p. 141).

**Example 12** Let
\[
I(y) = \int_0^\infty e^{-yx^2} \, dx, \quad y > 0.
\]

Since
\[
\int_0^r e^{-yx^2} \, dx = \frac{1}{\sqrt{y}} \int_0^r e^{-t^2} \, dt,
\]

it follows that
\[
I(y) = \frac{1}{\sqrt{y}} \int_0^\infty e^{-t^2} \, dt,
\]
and the convergence is uniform on \([\rho, \infty)\) if \( \rho > 0 \) (Exercise 8(i)). To evaluate the last integral, denote \( J(\rho) = \int_0^\rho e^{-t^2} \, dt \); then
\[
J^2(\rho) = \left( \int_0^\rho e^{-u^2} \, du \right) \left( \int_0^\rho e^{-v^2} \, dv \right) = \int_0^\rho \int_0^\rho e^{-(u^2+v^2)} \, du \, dv.
\]

Transforming to polar coordinates \( r = r \cos \theta, v = r \sin \theta \) yields
\[
J^2(\rho) = \int_0^{\pi/2} \int_0^\rho r e^{-r^2} \, dr \, d\theta = \frac{\pi(1 - e^{-\rho^2})}{4}, \quad \text{so} \quad J(\rho) = \frac{\sqrt{\pi(1 - e^{-\rho^2})}}{2}.
\]

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Therefore
\[
\int_0^\infty e^{-t^2} \, dt = \lim_{\rho \to \infty} J(\rho) = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \int_0^\infty e^{-y^2} \, dy = \frac{1}{\sqrt{y}} \quad y > 0.
\]
Differentiating this \( n \) times with respect to \( y \) yields
\[
\int_0^\infty x^{2n} e^{-yx^2} \, dx = \frac{1 \cdot 3 \cdots (2n - 1) \sqrt{\pi}}{2^n y^{n+1/2}} \quad y > 0, \quad n = 1, 2, 3, \ldots,
\]
where Theorem 11 justifies the differentiation for every \( n \), since all these integrals converge uniformly on \([\rho, \infty)\) if \( \rho > 0 \) (Exercise 8(i)).

Some advice for applying this theorem: Be sure to check first that \( F(y_0) = \int_a^b f(x, y_0) \, dx \) converges for at least one value of \( y \). If so, differentiate \( \int_a^b f(x, y) \, dx \) formally to obtain \( \int_a^b f_y(x, y) \, dx \). Then \( F'(y) = \int_a^b f_y(x, y) \, dx \) if \( y \) is in some interval on which this improper integral converges uniformly.

## 8 Applications to Laplace transforms

The Laplace transform of a function \( f \) locally integrable on \([0, \infty)\) is
\[
F(s) = \int_0^\infty e^{-sx} f(x) \, dx
\]
for all \( s \) such that integral converges. Laplace transforms are widely applied in mathematics, particularly in solving differential equations.

We leave it to you to prove the following theorem (Exercise 26).

**Theorem 12** Suppose \( f \) is locally integrable on \([0, \infty)\) and \(| f(x) | \leq Me^{s_0x}\) for sufficiently large \( x \). Then the Laplace transform of \( F \) converges uniformly on \([s_1, \infty)\) if \( s_1 > s_0 \).

**Theorem 13** If \( f \) is continuous on \([0, \infty)\) and \( H(x) = \int_0^\infty e^{-s_0t} f(u) \, du \) is bounded on \([0, \infty)\), then the Laplace transform of \( f \) converges uniformly on \([s_1, \infty)\) if \( s_1 > s_0 \).

**Proof** If \( 0 \leq r \leq r_1 \),
\[
\int_r^{r_1} e^{-sx} f(x) \, dx = \int_r^{r_1} e^{-(s-s_0)x} e^{-s_0x} f(x) \, dt = \int_r^{r_1} e^{-(s-s_0)t} H'(x) \, dt.
\]
Integration by parts yields
\[
\int_r^{r_1} e^{-sx} f(x) \, dx = e^{-(s-s_0)x} H(x) \bigg|_r^{r_1} + (s-s_0) \int_r^{r_1} e^{-(s-s_0)x} H(x) \, dx.
\]
Therefore, if $|H(x)| \leq M$, then
\[
\left| \int_{r}^{r_1} e^{-sx} f(x) \, dx \right| \leq M \left| e^{-(s-s_0)x_{r_1}} + e^{-(s-s_0)x_{r}} + (s-s_0) \int_{r}^{r_1} e^{-(s-s_0)x} \, dx \right|
\leq 3M e^{-(s-s_0)r} \leq 3M e^{-(s_1-s_0)r}, \quad s \geq s_1.
\]

Now Theorem 4 implies that $F(s)$ converges uniformly on $[s_1, \infty)$.

The following theorem draws a considerably stronger conclusion from the same assumptions.

**Theorem 14** If $f$ is continuous on $[0, \infty)$ and
\[
H(x) = \int_{0}^{x} e^{-s_0 u} f(u) \, du
\]
is bounded on $[0, \infty)$, then the Laplace transform of $f$ is infinitely differentiable on $(s_0, \infty)$, with
\[
F^{(n)}(s) = (-1)^n \int_{0}^{s} e^{-sx} x^n f(x) \, dx; \quad (30)
\]
that is, the $n$-th derivative of the Laplace transform of $f(x)$ is the Laplace transform of $(-1)^n x^n f(x)$.

**Proof** First we will show that the integrals
\[
I_n(s) = \int_{0}^{s} e^{-sx} x^n f(x) \, dx, \quad n = 0, 1, 2, \ldots
\]
al converge uniformly on $[s_1, \infty)$ if $s_1 > s_0$. If $0 < r < r_1$, then
\[
\int_{r}^{r_1} e^{-sx} x^n f(x) \, dx = \int_{r}^{r_1} e^{-(s-s_0)x} e^{-s_0x} x^n f(x) \, dx = \int_{r}^{r_1} e^{-(s-s_0)x} x^n H'(x) \, dx.
\]
Integrating by parts yields
\[
\int_{r}^{r_1} e^{-sx} x^n f(x) \, dx = r_1^n e^{-(s-s_0)r_1} H(r) - r^n e^{-(s-s_0)r} H(r) - \int_{r}^{r_1} H(x) \left( e^{-(s-s_0)x} x^n \right)' \, dx,
\]
where ' indicates differentiation with respect to $x$. Therefore, if $|H(x)| \leq M \leq \infty$ on $[0, \infty)$, then
\[
\left| \int_{r}^{r_1} e^{-sx} x^n f(x) \, dx \right| \leq M \left( e^{-(s-s_0)r} r_1^n + e^{-(s-s_0)r} r^n + \int_{r}^{\infty} |(e^{-(s-s_0)x} x^n)'| \, dx \right).
\]
Therefore, since $e^{-(s-s_0)r} r^n$ decreases monotonically on $(n, \infty)$ if $s > s_0$ (check!),
\[
\left| \int_{r}^{r_1} e^{-sx} x^n f(x) \, dx \right| < 3M e^{-(s-s_0)r} r^n, \quad n < r < r_1,
\]
so Theorem 4 implies that $I_n(s)$ converges uniformly $[s_1, \infty)$ if $s_1 > s_0$. Now Theorem 11 implies that $F_{n+1} = -F_{n}'$, and an easy induction proof yields (30) (Exercise 25).
Example 13 Here we apply Theorem 12 with \( f(x) = \cos ax \) \((a \neq 0)\) and \( s_0 = 0 \).

Since
\[
\int_0^x \cos au \, du = \frac{\sin ax}{a}
\]
is bounded on \((0, \infty)\), Theorem 12 implies that
\[
F(s) = \int_0^\infty e^{-sx} \cos ax \, dx
\]
converges and
\[
F^{(n)}(s) = (-1)^n \int_0^\infty e^{-sx} x^n \cos ax \, dx, \quad s > 0. \tag{31}
\]
(Note that this is also true if \( a = 0 \).) Elementary integration yields
\[
F(s) = \frac{s}{s^2 + a^2}.
\]
Hence, from (31),
\[
\int_0^\infty e^{-sx} x^n \cos ax = (-1)^n \frac{d^n}{ds^n} \frac{s}{s^2 + a^2}, \quad n = 0, 1, \ldots.
\]
9 Exercises

1. Suppose \( g \) and \( h \) are differentiable on \([a, b]\), with
   \[ a \leq g(y) \leq b \quad \text{and} \quad a \leq h(y) \leq b, \quad c \leq y \leq d. \]
   Let \( f \) and \( f_y \) be continuous on \([a, b] \times [c, d]\). Derive Liebniz’s rule:
   \[ \frac{d}{dy} \int_{g(y)}^{h(y)} f(x, y) \, dx = f(h(y), y)h'(y) - f(g(y), y)g'(y) + \int_{g(y)}^{h(y)} f_y(x, y) \, dx. \]
   (Hint: Define \( H(y, u, v) = \int_u^v f(x, y) \, dx \) and use the chain rule.)

2. Adapt the proof of Theorem 2 to prove Theorem 3.

3. Adapt the proof of Theorem 4 to prove Theorem 5.

4. Show that Definition 3 is independent of \( c \); that is, if \( \int_a^c f(x, y) \, dx \) and \( \int_c^b f(x, y) \, dx \)
   both converge uniformly on \( S \) for some \( c \in (a, b) \), then they both converge uniformly on \( S \)
   and every \( c \in (a, b) \).

5. (a) Show that if \( f \) is bounded on \([a, b] \times [c, d]\) and \( \int_a^b f(x, y) \, dx \) exists as a proper integral
   for each \( y \in [c, d] \), then it converges uniformly on \([c, d]\) according to all of Definition 1–3.
   (b) Give an example to show that the boundedness of \( f \) is essential in (a).

6. Working directly from Definition 1, discuss uniform convergence of the following integrals:
   (a) \( \int_0^\infty \frac{dx}{1 + y^2 x^2} \)
   (b) \( \int_0^\infty e^{-xy} x^2 \, dx \)
   (c) \( \int_0^\infty x^{2n} e^{-yx^2} \, dx \)
   (d) \( \int_0^\infty \sin xy^2 \, dx \)
   (e) \( \int_0^\infty (3y^2 - 2xy)e^{-y^2 x} \, dx \)
   (f) \( \int_0^\infty (2xy - y^2 x^2)e^{-xy} \, dx \)

7. Adapt the proof of Theorem 6 to prove Theorem 7.

8. Use Weierstrass’s test to show that the integral converges uniformly on \( S \):
   (a) \( \int_0^\infty e^{-xy} \sin x \, dx, \quad S = [0, \infty), \quad \rho > 0 \)
   (b) \( \int_0^\infty \frac{\sin x}{xy} \, dx, \quad S = [c, d], \quad 1 < c < d < 2 \)
9. (a) Show that

\[ \Gamma(y) = \int_0^\infty x^{y-1} e^{-x} \, dx \]

converges if \( y > 0 \), and uniformly on \([c, d]\) if \( 0 < c < d < \infty \).

(b) Use integration by parts to show that

\[ \Gamma(y) = \frac{\Gamma(y+1)}{y}, \quad y \geq 0, \]

and then show by induction that

\[ \Gamma(y) = \frac{\Gamma(y+n)}{y(y+1) \cdots (y+n-1)}, \quad y > 0, \quad n = 1, 2, 3, \ldots. \]

How can this be used to define \( \Gamma(y) \) in a natural way for all \( y \neq 0, -1, -2, \ldots \)? (This function is called the gamma function.)

(c) Show that \( \Gamma(n + 1) = n! \) if \( n \) is a positive integer.

(d) Show that

\[ \int_0^\infty e^{-x} x^\alpha \, dx = \Gamma(\alpha + 1), \quad \alpha > -1, \quad s > 0. \]

10. Show that Theorem 8 remains valid with assumption (c) replaced by the assumption that \( |g_s(x, y)| \) is monotonic with respect to \( x \) for all \( y \in S \).

11. Adapt the proof of Theorem 8 to prove Theorem 9.

12. Use Dirichlet’s test to show that the following integrals converge uniformly on \( S = [\rho, \infty) \) if \( \rho > 0 \):

\[ \int_1^\infty e^{-p x} \frac{\sin(xy)}{x} \, dx, \quad p > 0, \quad S = (-\infty, \infty) \]

\[ \int_0^1 \frac{e^{xy}}{(1 - x)^\rho} \, dx, \quad S = (-\infty, b), \quad b < 1 \]

\[ \int_{-\infty}^\infty \frac{\cos(xy)}{1 + x^2 y^2} \, dx, \quad S = (-\infty, -\rho] \cup [\rho, \infty), \quad \rho > 0. \]

\[ \int_1^\infty e^{-x/y} \, dx, \quad S = [\rho, \infty), \quad \rho > 0 \]

\[ \int_{-\infty}^\infty e^{xy} e^{-x^2} \, dx, \quad S = [-\rho, \rho], \quad \rho > 0 \]

\[ \int_0^\infty \frac{\cos(xy) - \cos(ax)}{x^2} \, dx, \quad S = (-\infty, \infty) \]

\[ \int_0^{\infty} x^{2n} e^{-y x^2} \, dx, \quad S = [\rho, \infty), \quad \rho > 0, \quad n = 0, 1, 2, \ldots \]
13. Suppose \( g, g_x \) and \( h \) are continuous on \( [a, b] \times S \), and denote \( H(x, y) = \int_a^x h(u, y) \, du, \ a \leq x < b \). Suppose also that

\[
\lim_{x \to b^-} \left\{ \sup_{y \in S} |g(x, y)H(x, y)| \right\} = 0 \quad \text{and} \quad \int_a^b g_x(x, y)H(x, y) \, dx
\]

converges uniformly on \( S \). Show that \( \int_a^b g(x, y)h(x, y) \, dx \) converges uniformly on \( S \).

14. Prove Theorem 10 for the case where \( f = f(x, y) \) is continuous on \( (a, b] \times [c, d] \).

15. Prove Theorem 11 for the case where \( f = f(x, y) \) is continuous on \( (a, b] \times [c, d] \).

16. Show that

\[
C(y) = \int_{-\infty}^{\infty} f(x) \cos xy \, dx \quad \text{and} \quad S(y) = \int_{-\infty}^{\infty} f(x) \sin xy \, dx
\]

are continuous on \( (-\infty, \infty) \) if

\[
\int_{-\infty}^{\infty} |f(x)| \, dx < \infty.
\]

17. Suppose \( f \) is continuously differentiable on \( [a, \infty) \), \( \lim_{x \to \infty} f(x) = 0 \), and

\[
\int_a^{\infty} |f'(x)| \, dx < \infty.
\]

Show that the functions

\[
C(y) = \int_a^{\infty} f(x) \cos xy \, dx \quad \text{and} \quad S(y) = \int_a^{\infty} f(x) \sin xy \, dx
\]

are continuous for all \( y \neq 0 \). Give an example showing that they need not be continuous at \( y = 0 \).

18. Evaluate \( F(y) \) and use Theorem 11 to evaluate \( I \):

\[
\begin{align*}
(a) \quad F(y) &= \int_0^{\infty} \frac{dx}{1 + y^2x^2}, \ y \neq 0; \quad I &= \int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} bx}{x} \, dx,
\end{align*}
\]

\( a, b > 0 \).
(b) \( F(y) = \int_0^{\infty} x^y \, dx, \ y > -1; \quad I = \int_0^{\infty} \frac{x^a - x^b}{\log x} \, dx, \ a, b > -1 \)

(c) \( F(y) = \int_0^{\infty} e^{-xy} \cos x \, dx, \ y > 0 \)
\[
I = \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \cos x \, dx, \ a, b > 0
\]

(d) \( F(y) = \int_0^{\infty} e^{-xy} \sin x \, dx, \ y > 0 \)
\[
I = \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \sin x \, dx, \ a, b > 0
\]

(e) \( F(y) = \int_0^{\infty} e^{-xy} \sin xy \, dx; \ I = \int_0^{\infty} e^{-y} \frac{1 - \cos ax}{x} \, dx \)

(f) \( F(y) = \int_0^{\infty} e^{-xy} \cos xy \, dx; \ I = \int_0^{\infty} e^{-y} \frac{\sin ax}{x} \, dx \)

19. Use Theorem 11 to evaluate:

(a) \( \int_0^{1} (\log x)^n x^y \, dx, \quad y > -1, \quad n = 0, 1, 2, \ldots \)

(b) \( \int_0^{\infty} \frac{dx}{(x^2 + y)^{n+1}} \, dx, \quad y > 0, \quad n = 0, 1, 2, \ldots \)

(c) \( \int_0^{\infty} x^{2n+1} e^{-xy^2} \, dx, \quad y > 0, \quad n = 0, 1, 2, \ldots \)

(d) \( \int_0^{\infty} x^y \, dx, \quad 0 < y < 1. \)

20. (a) Use Theorem 11 and integration by parts to show that
\[
F(y) = \int_0^{\infty} e^{-x^2} \cos 2xy \, dx
\]
satisfies
\[
F' + 2yF = 0.
\]

(b) Use part (a) to show that
\[
F(y) = \frac{\sqrt{\pi}}{2} e^{-y^2}.
\]

21. Show that
\[
\int_0^{\infty} e^{-x^2} \sin 2xy \, dx = e^{-y^2} \int_0^{y} e^{u^2} \, du.
\]
(Hint: See Exercise 20.)

22. State a condition implying that
\[
C(y) = \int_a^{\infty} f(x) \cos xy \, dx \quad \text{and} \quad S(y) = \int_a^{\infty} f(x) \sin xy \, dx
\]
are \( n \) times differentiable on for all \( y \neq 0 \). (Your condition should imply the hypotheses of Exercise 16.)

23. Suppose \( f \) is continuously differentiable on \([a, \infty)\),
\[
\int_a^\infty |(x^k f(x))'| \, dx < \infty, \quad 0 \leq k \leq n,
\]
and \( \lim_{x \to \infty} x^n f(x) = 0 \). Show that if
\[
C(y) = \int_a^\infty f(x) \cos xy \, dx \quad \text{and} \quad S(y) = \int_a^\infty f(x) \sin xy \, dx,
\]
then
\[
C^{(k)}(y) = \int_a^\infty x^k f(x) \cos xy \, dx \quad \text{and} \quad S^{(k)}(y) = \int_a^\infty x^k f(x) \sin xy \, dx,
\]
\( 0 \leq k \leq n \).

24. Differentiating
\[
F(y) = \int_1^\infty \cos \frac{y}{x} \, dx
\]
under the integral sign yields
\[
-\int_1^\infty \frac{1}{x} \sin \frac{y}{x} \, dx,
\]
which converges uniformly on any finite interval. (Why?) Does this imply that \( F \) is differentiable for all \( y \)?

25. Show that Theorem 11 and induction imply Eq. (30).


27. Show that if \( F(s) = \int_0^\infty e^{-sx} f(x) \, dx \) converges for \( s = s_0 \), then it converges uniformly on \([s_0, \infty)\). (What’s the difference between this and Theorem 13?)

28. Prove: If \( f \) is continuous on \([0, \infty)\) and \( \int_0^\infty e^{-s_0 x} f(x) \, dx \) converges, then
\[
\lim_{s \to s_0^+} \int_0^\infty e^{-sx} f(x) \, dx = \int_0^\infty e^{-s_0 x} f(x) \, dx.
\]
(Hint: See the proof of Theorem 4.5.12, p. 273.)

29. Under the assumptions of the proof of Theorem 4.5.12, p. 273., show that
\[
\lim_{s \to s_0^+} \int_r^\infty e^{-sx} f(x) \, dx = \int_r^\infty e^{-s_0 x} f(x) \, dx, \quad r > 0.
\]
30. Suppose $f$ is continuous on $[0, \infty)$ and

$$F(s) = \int_0^\infty e^{-sx} f(x) \, dx$$

converges for $s = s_0$. Show that $\lim_{s \to \infty} F(s) = 0$. (Hint: Integrate by parts.)

31. (a) Starting from the result of Exercise 18(d), let $b \to \infty$ and invoke Exercise 30 to evaluate

$$\int_0^\infty e^{-ax} \frac{\sin x}{x} \, dx, \quad a > 0.$$ 

(b) Use (a) and Exercise 28 to show that

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}.$$ 

32. (a) Suppose $f$ is continuously differentiable on $[0, \infty)$ and

$$|f(x)| \leq Me^{s_0x}, \quad 0 \leq x \leq \infty.$$ 

Show that

$$G(s) = \int_0^\infty e^{-sx} f'(x) \, dx$$

converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$. (Hint: Integrate by parts.)

(b) Show from part (a) that

$$G(s) = \int_0^\infty e^{-sx} xe^{x^2} \sin e^{x^2} \, dx$$

converges uniformly on $[\rho, \infty)$ if $\rho > 0$. (Notice that this does not follow from Theorem 6 or 8.)

33. Suppose $f$ is continuous on $[0, \infty)$,

$$\lim_{x \to 0+} \frac{f(x)}{x}$$

exists, and

$$F(s) = \int_0^\infty e^{-sx} f(x) \, dx$$

converges for $s = s_0$. Show that

$$\int_{s_0}^\infty F(u) \, du = \int_0^\infty e^{-s_0x} \frac{f(x)}{x} \, dx.$$
10 Answers to selected exercises

5. (b) If \( f(x, y) = 1/y \) for \( y \neq 0 \) and \( f(x, 0) = 1 \), then \( \int_a^b f(x, y) \, dx \) does not converge uniformly on \([0, d]\) for any \( d > 0 \).

6. (a), (d), and (e) converge uniformly on \((\infty, \rho] \cup [\rho, \infty)\) if \( \rho > 0 \); (b), (c), and (f) converge uniformly on \([\rho, \infty)\) if \( \rho > 0 \).

17. Let \( C(y) = \int_1^\infty \frac{\cos xy}{x} \, dx \) and \( S(y) = \int_1^\infty \frac{\sin xy}{x} \, dx \). Then \( C(0) = \infty \) and \( S(0) = 0 \), while \( S(y) = \pi/2 \) if \( y \neq 0 \).

18. (a) \( F(y) = \frac{\pi}{2|y|} \); \( I = \frac{\pi}{2} \log \frac{a}{b} \) \quad (b) \( F(y) = \frac{1}{y+1} \); \( I = \log \frac{a+1}{b+1} \)

(c) \( F(y) = \frac{y}{y^2+1} \); \( I = \frac{1}{2} \frac{b^2+1}{a^2+1} \)

(d) \( F(y) = \frac{1}{y^2+1} \); \( I = \tan^{-1} b - \tan^{-1} a \)

(e) \( F(y) = \frac{y}{y^2+1} \); \( I = \frac{1}{2} \log(1+a^2) \)

(f) \( F(y) = \frac{1}{y^2+1} \); \( I = \tan^{-1} a \)

19. (a) \( (-1)^n n! (y+1)^{-n-1} \) \quad (b) \( \pi^{2n-1} \left( \frac{2n}{n} \right) y^{-n-1/2} \)

(c) \( \frac{n!}{2^ny^{n+1}} (\log y)^{-2} \) \quad (d) \( \frac{1}{(\log x)^2} \)

22. \( \int_\infty^{-\infty} |x^n f(x)| \, dx < \infty \)

24. No; the integral defining \( F \) diverges for all \( y \).

31. (a) \( \frac{\pi}{2} - \tan^{-1} a \)