March, 2013

Inverse problems for unilevel block α-circulants

William F. Trench, *Trinity University*

Inverse problems for unilevel block α-circulants

William F. Trench

Trinity University, San Antonio, Texas 78212-7200, USA
Mailing address: 659 Hopkinton Road, Hopkinton, NH 03229 USA

Dedicated to Professor Biswa Nath Datta

Abstract

We consider the following inverse problems for the class $\mathcal{C}_\alpha$ of unilevel block α-circulants $C = [C_{s-\alpha r}]_{r,s=0}^{k-1}$, where $k > 1$, C_0, C_1, $\dots$, $C_{k-1} \in \mathbb{C}^{d_1\times d_2}$, $\alpha \in \{1, 2, \dots, k-1\}$, $\gcd(\alpha, k) = 1$, and $\|\cdot\|$ is the Frobenius norm.

Problem 1 Find necessary and sufficient conditions on $Z \in \mathbb{C}^{kd_2\times h}$ and $W \in \mathbb{C}^{kd_1\times h}$ for the existence of $C \in \mathcal{C}_\alpha$ such that $CZ = W$, and find all such C if the conditions are satisfied.

Problem 2 For arbitrary $Z \in \mathbb{C}^{kd_2\times h}$ and $W \in \mathbb{C}^{kd_1\times h}$, find

$$\sigma_\alpha(Z, W) = \min_{C\in\mathcal{C}_\alpha} \|CZ - W\|,$$

characterize the class

$$\mathcal{M}_\alpha(Z, W) = \{C \in \mathcal{C}_\alpha \mid \|CZ - W\| = \sigma_\alpha(Z, W)\},$$

and find C in this class with minimum norm.

Problem 3 If $A \in \mathcal{C}_\alpha$ is given, find

$$\sigma_\alpha(Z, W, A) = \min_{C\in\mathcal{M}_\alpha(Z,W)} \|C - A\|$$

and find $C \in \mathcal{M}_\alpha(Z, W)$ such that $\|C - A\| = \sigma_\alpha(Z, W, A)$.

We also consider slightly modified problems for the case where $\gcd(\alpha, k) > 1$.

MSC: 15A09; 15A15; 15A18; 15A99

Keywords: Block circulant; Discrete Fourier transform; Frobenius norm; Inverse problem; Minimum norm; Moore–Penrose inverse

1 Introduction

We consider inverse problems for the class $\mathcal{C}_\alpha$ of unilevel block α-circulants $C = [C_{s-\alpha r}]_{r,s=0}^{k-1}$, where $k > 1$, $C_0, C_1, \ldots, C_{k-1} \in \mathbb{C}^{d_1 \times d_2}$, $\alpha \in \{1, 2, \ldots, k-1\}$, $\gcd(\alpha, k) = 1$, and all subscripts specifically associated with circulants are to be interpreted modulo k. Throughout the paper $\|\cdot\|$ denotes the Frobenius norm; i.e., if

$$V = [v_{rs}]_{1 \le r \le p, 1 \le s \le q} \in \mathbb{C}^{p \times q}, \quad \text{then} \quad \|V\| = \left(\sum_{r=1}^{p} \sum_{s=1}^{q} |v_{rs}|^2 \right)^{1/2}.$$

Problem 1 Find necessary and sufficient conditions on $Z \in \mathbb{C}^{kd_2 \times h}$ and $W \in \mathbb{C}^{kd_1 \times h}$ for the existence of $C \in \mathcal{C}_\alpha$ such that $CZ = W$, and find all such C if the conditions are satisfied.

Problem 2 For arbitrary $Z \in \mathbb{C}^{kd_2 \times h}$ and $W \in \mathbb{C}^{kd_1 \times h}$, find

$$\sigma_\alpha(Z, W) = \min_{C \in \mathcal{C}_\alpha} \|CZ - W\|,$$

characterize the class

$$\mathcal{M}_\alpha(Z, W) = \left\{ C \in \mathcal{C}_\alpha \,\middle|\, \|CZ - W\| = \sigma_\alpha(Z, W) \right\},$$

and find C in this class with minimum norm.

Problem 3 If $A \in \mathcal{C}_\alpha$ is given, find

$$\sigma_\alpha(Z, W, A) = \min_{C \in \mathcal{M}_\alpha(Z,W)} \|C - A\|,$$

and find $C \in \mathcal{M}_\alpha(Z, W)$ such that $\|C - A\| = \sigma_\alpha(Z, W, A)$.

If $\gcd(\alpha, k) = q > 1$ then the first $p = k/q$ block rows of an α-circulant $C = [C_{s-\alpha r}]_{r,s=0}^{k-1}$ are repeated q times, which obviously restricts the ranges of all such C. Since Problems 1– 3 do not reflect this restriction, it is reasonable to regard them as ill posed in this case. Section 4 is devoted to this question.

These problems and the results contained here are related to previous results and methods developed in [1, 2, 3, 4].

2 Preliminary considerations

Henceforth ζ is a primitive k-th root of unity. If C_0, C_1, $\dots$, $C_{k-1} \in \mathbb{C}^{d_1\times d_2}$, let

$$F_\ell = \sum_{m=0}^{k-1} \zeta^{\ell m} C_m, \quad 0 \le \ell \le k-1; \tag{1}$$

thus $\{F_0, F_1, \dots, F_{k-1}\}$ is the discrete Fourier transform of $\{C_0, C_1, \dots, C_{k-1}\}$. Solving (1) for C_0, C_1, $\dots$, C_{k-1} yields

$$C_m = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{-\ell m} F_\ell, \quad 0 \le m \le k-1. \tag{2}$$

Conversely, solving (2) for F_0, F_1, $\dots$, F_{k-1} yields (1); thus, (1) and (2) are equivalent.

Now let

$$P_\ell = \frac{1}{\sqrt{k}}\begin{bmatrix} I_{d_1} \\ \zeta^\ell I_{d_1} \\ \vdots \\ \zeta^{(k-1)\ell} I_{d_1}\end{bmatrix} \quad \text{and} \quad Q_\ell = \frac{1}{\sqrt{k}}\begin{bmatrix} I_{d_2} \\ \zeta^\ell I_{d_2} \\ \vdots \\ \zeta^{(k-1)\ell} I_{d_2}\end{bmatrix}, \quad 0 \le \ell \le k-1,$$

so

$$P_\ell^* P_\ell = \delta_{\ell m} I_{d_1} \quad \text{and} \quad Q_\ell^* Q_m = \delta_{\ell m} I_{d_2}, \quad 0 \le \ell, m \le k-1. \tag{3}$$

Note that

$$P_{\alpha\ell}^* P_{\alpha m} = \delta_{\ell m} I_{d_1},\ 0 \le \ell, m \le p-1, \ \text{ where}\, p = k/q \ \text{ and}\, q = \gcd(\alpha, k). \tag{4}$$

Lemma 1 *If* $\{C_0, C_1, \dots, C_{k-1}\}$ *and* $\{F_0, F_1, \dots, F_{k-1}\}$ *are related by the equivalent equations* (1) *and* (2), *then*

$$\text{(a)} \quad C = [C_{s-\alpha r}]_{r,s=0}^{k-1} \quad \textit{if and only if} \quad \text{(b)} \quad C = \sum_{\ell=0}^{k-1} P_{\alpha\ell} F_\ell Q_\ell^*. \tag{5}$$

PROOF. Starting from (2) yields

$$
\begin{aligned}
[C_{s-\alpha r}]_{r,s=0}^{k-1} &= \frac{1}{k}\left[\sum_{\ell=0}^{k-1} F_\ell \zeta^{-\ell(s-\alpha r)}\right]_{r,s=0}^{k-1} \\
&= \frac{1}{k}\sum_{\ell=0}^{k-1}\begin{bmatrix} I_{d_1} \\ \zeta^{\ell\alpha} I_{d_1} \\ \vdots \\ \zeta^{(k-1)\ell\alpha} \end{bmatrix} F_\ell \begin{bmatrix} I_{d_2} \\ \zeta^{\ell} I_{d_2} \\ \vdots \\ \zeta^{(k-1)\ell} I_{d_2} \end{bmatrix}^* \\
&= \sum_{\ell=0}^{k-1} P_{\alpha\ell} F_\ell Q_\ell^*,
\end{aligned}
$$

so (5)(a) implies (5)(b). To see that (5)(b) implies (5)(a), start from (1) and work through these equalities in the opposite direction. □

In connection with Problems 1– 3, we write

$$
Z = \sum_{\ell=0}^{k-1} Q_\ell U_\ell \quad \text{with} \quad U_\ell \in \mathbb{C}^{d_2\times h}, \quad 0 \le \ell \le k-1,
$$

and

$$
W = \sum_{\ell=0}^{k-1} P_\ell V_\ell \quad \text{with} \quad V_\ell \in \mathbb{C}^{d_1\times h}, \quad 0 \le \ell \le k-1. \tag{6}
$$

It is to be understood that Z and W are fixed and U_0, U_1, …, U_{k-1} and V_0, V_1, …, V_{k-1} have these meanings throughout the rest of this paper.

If $\gcd(\alpha, k) = 1$ then $\ell \to \alpha\ell \pmod{k}$ is a permutation of $\{0, 1, \dots, k-1\}$, so we can rewrite (6) as $W = \sum_{\ell=0}^{k-1} P_{\alpha\ell} V_{\alpha\ell}$. (Recall that the subscripts here are to be interpreted modulo k.) Therefore

$$
CZ - W = \sum_{\ell=0}^{k-1} P_{\alpha\ell}(F_\ell U_\ell - V_{\alpha\ell}). \tag{7}
$$

Since $\left[\begin{array}{ccc} P_0 & P_\alpha & \cdots P_{(k-1)\alpha} \end{array}\right]$ is unitary if $\gcd(\alpha, k) = 1$ (see (4)),

$$
\|CZ - W\|^2 = \sum_{\ell=0}^{k-1} \|F_\ell U_\ell - V_{\alpha\ell}\|^2.
$$

Hence Problems 1–3 each reduce to k independent analogous inverse problems for F_0, F_1, …, F_{k-1}.

As usual, $U^\dagger$ denotes the Moore-Penrose inverse of U; i.e., the unique matrix such that

$$UU^\dagger U = U, \quad U^\dagger UU^\dagger = U^\dagger, \quad (UU^\dagger)^* = UU^\dagger, \quad (U^\dagger U)^* = U^\dagger U.$$

We will invoke these properties repeatedly without explicit citation.

The following lemma is from [2]. We include the short proof here for completeness. Parts of the proofs of our main results are also implicit in other lemmas from [2]; however, the self-contained proofs given below require less space than it would take to state the appropriate lemmas from [2] and explain their application here.

Lemma 2 *If $H \in \mathbb{C}^{d_1\times d_2}$ and $U \in \mathbb{C}^{d_2\times p}$, then $HU = 0$ if and only if $H = K(I - UU^\dagger)$ where $K \in \mathbb{C}^{d_1\times d_2}$ is arbitrary.*

Proof. If $H = K(I - UU^\dagger)$ then $HU = 0$. For the converse, suppose $HU = 0$. If $x \in \mathbb{C}^{d_2}$ then $x = v + Uw$, where $v \in \mathbb{C}^{d_2}$, $w \in \mathbb{C}^h$, and $v^*U = 0$. Then $Hx = Hv + HUw = Hv$. Now choose K so that $Kv = Hv$ if $v^*U = 0$. (For example, $K = H$ is acceptable.) Then

$$\begin{aligned} K(I - UU^\dagger)x &= K(I - UU^\dagger)(v + Uw) = K(I - UU^\dagger)v \\ &= Kv - K(v^*UU^\dagger)^* = Kv = Hv = Hx. \end{aligned}$$

Since we have now shown that $Hx = K(I - UU^\dagger)x$ for all $x \in \mathbb{C}^{d_2}$, it follows that $H = K(I - UU^\dagger)$. □

We remind the reader that

$$\|R + S\|^2 = \|R\|^2 + \|S\|^2 \quad \text{if} \quad RS^* = 0. \tag{8}$$

3 Main results for the case where $\gcd(\alpha, k) = 1$

Throughout this section we assume that $\gcd(\alpha, k) = 1$.

Theorem 1 *If $Z = \sum_{\ell=0}^{k-1} Q_\ell U_\ell$ and E is an α-circulant, then $EZ = 0$ if and only if*

$$E = [E_{s-\alpha r}]_{r,s=0}^{k-1} \text{ where } E_m = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{-\ell m} K_\ell(I - U_\ell U_\ell^\dagger), \quad 0 \le m \le k-1, \tag{9}$$

and $K_0, K_1, \ldots, K_{k-1} \in \mathbb{C}^{d_1\times d_2}$ are arbitrary.

PROOF. From Lemma 1, if $E = [E_{s-\alpha r}]_{r,s=0}^{k-1}$ then there are $H_0, H_1, \dots, H_{k-1} \in \mathbb{C}^{d_1 \times d_2}$ such that

$$E = \sum_{\ell=0}^{k-1} P_{\alpha\ell} H_\ell Q_\ell^* \quad \text{and} \quad E_m = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{-\ell m} H_\ell, \quad 0 \le m \le k-1.$$

Therefore (3) implies that

$$EZ = \left(\sum_{\ell=0}^{k-1} P_{\alpha\ell} H_\ell Q_\ell^*\right)\left(\sum_{\ell=0}^{k-1} Q_\ell U_\ell\right) = \sum_{\ell=0}^{k-1} P_{\alpha\ell} H_\ell U_\ell,$$

so (4) with $q = 1$ and (8) imply that

$$\|EZ\|^2 = \sum_{\ell=0}^{k-1} \|H_\ell U_\ell\|^2;$$

hence, $EZ = 0$ if and only if $H_\ell U_\ell = 0$, $0 \le \ell \le k-1$. Now Lemma 2 implies that $H_\ell = K_\ell(I - U_\ell U_\ell^\dagger)$, $0 \le \ell \le k-1$, so Lemma 1 (with $C = E$ and $F_\ell = H_\ell$, $0 \le \ell \le k-1$) implies the conclusion. □

The following theorem solves Problem 2.

Theorem 2 *Let $\mathcal{E}_\alpha$ be the set of all α-circulants of the form* (9), *and let*

$$C^{(\alpha)} = [C_{s-\alpha r}^{(\alpha)}]_{r,s=0}^{k-1}, \quad \textit{where} \quad C_m^{(\alpha)} = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{-\ell m} V_{\alpha\ell} U_\ell^\dagger, \quad 0 \le m \le k-1. \tag{10}$$

Then

$$\sigma_\alpha(Z, W) =_{\text{def}} \min_{C \in \mathcal{C}_\alpha} \|CZ - W\| = \left(\sum_{\ell=0}^{k-1} \|V_{\alpha\ell}(I - U_\ell^\dagger U_\ell)\|^2\right)^{1/2}, \tag{11}$$

and this minimum is attained if and only if

$$C = C^{(\alpha)} + E, \quad \textit{where} \quad E \in \mathcal{E}_\alpha. \tag{12}$$

Moreover, $C^{(\alpha)}$ is the unique circulant of this form with minimum norm

$$\|C^{(\alpha)}\| = \left(\sum_{\ell=0}^{k-1} \|V_{\alpha\ell} U_\ell^\dagger\|^2\right)^{1/2},$$

PROOF. Since (4) with $q=1$ implies that $\begin{bmatrix} P_0 & P_\alpha & \cdots & P_{(k-1)\alpha} \end{bmatrix}$ is unitary, (7) and (8) imply that

$$\|CZ-W\|^2 = \sum_{\ell=0}^{k-1} \|F_\ell U_\ell - V_{\alpha\ell}\|^2. \qquad (13)$$

Now write

$$F_\ell U_\ell - V_{\alpha\ell} = (F_\ell - V_{\alpha\ell}U_\ell^\dagger)U_\ell - V_{\alpha\ell}(I - U_\ell^\dagger U_\ell).$$

Since

$$(F_\ell - V_{\alpha\ell}U_\ell^\dagger)U_\ell[V_{\alpha\ell}(I-U_\ell^\dagger U_\ell)]^* = (F_\ell - V_{\alpha\ell}U_\ell^\dagger)U_\ell(I-U_\ell^\dagger U_\ell)V_{\alpha\ell}^* = 0,$$

(8) implies that (13) can be rewritten as

$$\|CZ-W\|^2 = \sum_{\ell=0}^{k-1} \|(F_\ell - V_{\alpha\ell}U_\ell^\dagger)U_\ell\|^2 + \sum_{\ell=0}^{k-1} \|V_{\alpha\ell}(I-U_\ell^\dagger U_\ell)\|^2.$$

This implies (11), and that the minimum is attained if and only

$$(F_\ell - V_{\alpha\ell}U_\ell^\dagger)U_\ell = 0, \quad 0 \le \ell \le k-1.$$

From Lemma 2, this is equivalent to

$$F_\ell = V_{\alpha\ell}U_\ell^\dagger + K_\ell(I - U_\ell U_\ell^\dagger), \quad 0 \le \ell \le k-1,$$

which is equivalent to (12), by Lemma 1. Moreover, since

$$V_{\alpha\ell}U_\ell^\dagger[K_\ell(I-U_\ell U_\ell^\dagger)]^* = V_{\alpha\ell}U_\ell^\dagger(I-U_\ell U_\ell^\dagger)K_\ell^* = 0, \quad 0 \le \ell \le k-1,$$

(8) implies that

$$\|F_\ell\|^2 = \|V_{\alpha\ell}U_\ell^\dagger\|^2 + \|K_\ell(I-U_\ell U_\ell^\dagger)\|^2.$$

This implies the last sentence of Theorem 2. □

This implies the following theorem, which solves Problem 1.

Theorem 3 *There is an α-circulant C such that $CZ=W$ if and only if*

$$V_{\alpha\ell}(I-U_\ell^\dagger U_\ell) = 0, \quad 0 \le \ell \le k-1,$$

in which case $CZ=W$ if and only if C is as in (12).

Corollary 1 *If* $Z = \sum_{\ell=0}^{k-1} Q_\ell U_\ell$ *then* $\sigma_\alpha(Z,W) = 0$ *for all* $W \in \mathbb{C}^{kd_1 \times h}$ *if and only if* $\operatorname{rank}(U_\ell) = h$, $0 \le \ell \le k-1$.

The following theorem solves Problem 3.

Theorem 4 *Let*

$$A = [A_{s-\alpha r}]_{r,s=0}^{k-1} = \sum_{\ell=0}^{k-1} P_{\alpha\ell} G_\ell Q_\ell^* \ \text{ with } G_\ell = \sum_{\ell=0}^{k-1} \zeta^{\ell m} A_m,\ 0 \le m \le k-1, \quad (14)$$

be a given member of $\mathcal{C}_\alpha$. *Then*

$$\sigma_\alpha(Z,W,A) =_{\text{def}} \min_{C \in \mathcal{M}_\alpha(Z,W)} \|C - A\| = \|(V_{\alpha\ell} - G_\ell U_\ell) U_\ell^\dagger\|^2, \qquad (15)$$

which is attained if and only if $C = C^{(\alpha)} + [\widehat{E}_{s-\alpha r}]_{r,s=0}^{k-1}$, *where*

$$\widehat{E}_m = \frac{1}{k} \sum_{\ell=0}^{k-1} \zeta^{-\ell m} G_\ell (I - U_\ell U_\ell^\dagger), \quad 0 \le m \le k-1. \qquad (16)$$

Proof. From Theorem 2, $C \in \mathcal{M}_\alpha$ if and only if $C = C^{(\alpha)} + E$, where E is as in (9). For any such C, (9), (10), and (14) imply that

$$C - A = C^{(\alpha)} + E - A = \sum_{\ell=0}^{k-1} P_{\alpha\ell} H_\ell Q_\ell^*,$$

where

$$\begin{aligned} H_\ell &= V_{\alpha\ell} U_\ell^\dagger + K_\ell (I - U_\ell U_\ell^\dagger) - G_\ell \\ &= (V_{\alpha\ell} U_\ell^\dagger + K_\ell - G_\ell)(I - U_\ell U_\ell^\dagger) + (V_{\alpha\ell} U_\ell^\dagger - G_\ell) U_\ell U_\ell^\dagger. \end{aligned}$$

Since $(I - U_\ell U_\ell^\dagger)(U_\ell U_\ell^\dagger)^* = 0$, (8) implies that

$$\|H_\ell\|^2 = \|V_{\alpha\ell} U_\ell^\dagger + K_\ell - G_\ell\|^2 + \|(V_{\alpha\ell} - G_\ell U_\ell) U_\ell^\dagger\|^2.$$

This verifies (15) and implies that the minimum is attained if and only if

$$K_\ell = G_\ell - V_{\alpha\ell} U_\ell^\dagger, \quad 0 \le \ell \le k-1.$$

Then

$$K_\ell (I - U_\ell U_\ell^\dagger) = (G_\ell - V_{\alpha\ell} U_\ell^\dagger)(I - U_\ell U_\ell^\dagger) = G_\ell (I - U_\ell U_\ell^\dagger),$$

which implies (16). □

4 The case where $\gcd(\alpha, k) > 1$

Throughout this section,

$$\gcd(\alpha, k) = q \quad \text{and} \quad p = k/q. \tag{17}$$

In this case every integer in $\{0, 1, \ldots, k-1\}$ can be written uniquely as $\ell+\nu p$ with $0 \le \ell \le p-1$ and $0 \le \nu \le q-1$. Therefore the second equality in (5) can be rewritten as

$$C = \sum_{\ell=0}^{p-1} \sum_{\nu=0}^{q-1} P_{\alpha(\ell+\nu p)} F_{\ell+\nu p} Q^*_{\ell+\nu p} = \sum_{\ell=0}^{p-1} P_{\alpha\ell} \sum_{\nu=0}^{q-1} F_{\ell+\nu p} Q^*_{\ell+\nu p}, \tag{18}$$

since $\alpha p = (\alpha/q)k \equiv 0 \pmod{k}$. Now define

$$\mathbf{F}_\ell = \left[\begin{array}{cccc} F_\ell & F_{\ell+p} & \cdots & F_{\ell+(q-1)p} \end{array}\right] \in \mathbb{C}^{d_1 \times q d_2}, \quad 0 \le \ell \le p-1, \tag{19}$$

and

$$\mathbf{Q}_\ell = \left[\begin{array}{cccc} Q_\ell & Q_{\ell+p} & \cdots & Q_{\ell+(q-1)p} \end{array}\right] \in \mathbb{C}^{d_1 \times q d_2}, \quad 0 \le \ell \le p-1,$$

so (18) becomes

$$C = \sum_{\ell=0}^{p-1} P_{\alpha\ell} \mathbf{F}_\ell \mathbf{Q}^*_\ell. \tag{20}$$

To be more specific: (17) implies that $[C_{s-\alpha r}]_{r,s=0}^{k-1}$ can be written as (20). On the other hand, if C is presented in the form (20), then $C = [C_{s-\alpha r}]_{r,s=0}^{k-1}$, where C_0, C_1, …, C_{k-1} can be computed from (2) after determining F_0, F_1, …, F_{k-1} by partitioning $\mathbf{F}_0$, $\mathbf{F}_1$, …, $\mathbf{F}_{p-1}$ as in (19). For brevity, we omit this step in the theorems stated in this section.

Since $\left[\begin{array}{cccc} \mathbf{Q}_0 & \mathbf{Q}_1 & \cdots & \mathbf{Q}_{p-1} \end{array}\right]$ is unitary, we can write an arbitrary $Z \in \mathbb{C}^{kd_2 \times h}$ as

$$Z = \sum_{\ell=0}^{p-1} \mathbf{Q}_\ell \mathbf{U}_\ell \quad \text{with} \quad \mathbf{U}_\ell \in \mathbb{C}^{qd_2 \times h}, \quad 0 \le \ell \le p-1. \tag{21}$$

This and (20) imply that

$$CZ = \sum_{\ell=0}^{p-1} P_{\alpha\ell} \mathbf{F}_\ell \mathbf{U}_\ell,$$

so every W in the range of any $C = [C_{s-\alpha r}]_{r,s=0}^{k-1}$ is necessarily of the form

$$W = \sum_{\ell=0}^{p-1} P_{\alpha\ell} V_{\alpha\ell} \quad \text{with} \quad V_{\alpha\ell} \in \mathbb{C}^{d_1 \times p}. \tag{22}$$

Therefore, if (17) holds, we can repair Problems 1-3 by simply replacing "$W \in \mathbb{C}^{kd_1 \times h}$" in all of its occurences by "W of the form (22)." Then

$$CZ - W = \sum_{\ell=0}^{p-1} P_{\alpha\ell}(\mathbf{F}_\ell \mathbf{U}_\ell - V_{\alpha\ell}).$$

Now (4) and (8) imply that

$$\|CZ - W\|^2 = \sum_{\ell=0}^{p-1} \|\mathbf{F}_\ell \mathbf{U}_\ell - V_{\alpha\ell}\|^2,$$

and the proofs of the following four theorems are analogous to the proofs of Theorems 1– 4.

Theorem 5 *If E is an α-circulant and Z is as in* (21), *then $EZ = 0$ if and only if*

$$E = \sum_{\ell=0}^{p-1} P_{\alpha\ell} \mathbf{K}_\ell (I - \mathbf{U}_\ell \mathbf{U}_\ell^\dagger) \mathbf{Q}_\ell^* \quad \textit{with} \quad \mathbf{K}_\ell \in \mathbb{C}^{d_1 \times qd_2}, \quad 0 \le \ell \le p-1. \tag{23}$$

Theorem 6 *Let $\mathcal{E}_\alpha$ be the set of all α-circulants of the form* (23), *and let*

$$C^{(\alpha)} = \sum_{\ell=0}^{p-1} P_{\alpha\ell} V_{\alpha\ell} \mathbf{U}_\ell^\dagger \mathbf{Q}_\ell^*.$$

Then

$$\sigma_\alpha(Z, W) =_{\text{def}} \min_{C \in \mathcal{C}_\alpha} \|CZ - W\| = \left(\sum_{\ell=0}^{p-1} \|V_{\alpha\ell}(I - \mathbf{U}_\ell^\dagger \mathbf{U}_\ell)\|^2 \right)^{1/2},$$

and this minimum is attained if and only if

$$C = C^{(\alpha)} + E \quad \textit{where} \quad E \in \mathcal{E}_\alpha. \tag{24}$$

Moreover, $C^{(\alpha)}$ is the unique circulant of this form with minimum norm, which is

$$\|C^{(\alpha)}\| = \left(\sum_{\ell=0}^{p-1} \|V_{\alpha\ell} \mathbf{U}_\ell^\dagger\|^2. \right)^{1/2},$$

Theorem 7 *There is an α-circulant C such that $CZ=W$ if and only if*

$$V_{\alpha\ell}(I-\mathbf{U}_\ell^\dagger\mathbf{U}_\ell)=0, \quad 0\le\ell\le p-1,$$

in which case $CZ=W$ if and only if C is as in (24).

Theorem 8 *Let $A=\sum_{\ell=0}^{p-1}P_{\alpha\ell}\mathbf{G}_\ell\mathbf{Q}_\ell^*$ be a given member of $\mathcal{C}_\alpha$. Then*

$$\sigma_\alpha(Z,W,A)=_{\rm def}\min_{C\in\mathcal{M}_\alpha(Z,W)}\|C-A\|=\left(\sum_{\ell=0}^{p-1}\|(V_{\alpha\ell}-\mathbf{G}_\ell\mathbf{U}_\ell)\mathbf{U}_\ell^\dagger\|^2\right)^{1/2},$$

which is attained if and only if

$$C=C^{(\alpha)}+\widehat{E} \quad \textit{where} \quad \widehat{E}=\sum_{\ell=0}^{p-1}P_{\alpha\ell}\mathbf{G}_\ell(I-\mathbf{U}_\ell\mathbf{U}_\ell^\dagger)\mathbf{Q}_\ell^*.$$

References

[1] Jiao-fen Li and Xi-yan Hu, Procrustes problems and associated approximation problems for matrices with k-involutory symmetries, Linear Algebra Appl. 434 (2011), 820-829.

[2] W. F. Trench, Minimization problems for (R,S)-symmetric and (R,S)-skew symmetric matrices, Linear Algebra Appl. 389 (2004) 23–31.

[3] W. F. Trench, Properties of unilevel block circulants, Linear Algebra Appl. 430 (2009) 2012-2025.

[4] W. F. Trench, Characterization and properties of matrices with k-involutory symmetries II, Linear Algebra Appl. 432, (2010) 2282-2797.