Characterization and properties of (R,S)-symmetric, (R,S)-skew symmetric, and (R,S)-conjugate matrices

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CHARACTERIZATION AND PROPERTIES OF (R,S)-SYMMETRIC, (R,S)-SKEW SYMMETRIC, AND (R,S)-CONJUGATE MATRICES

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Abstract. Let $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ be nontrivial involutions; i.e., $R = R^{-1} \ne \pm I_m$ and $S = S^{-1} \ne \pm I_n$. We say that $A \in \mathbb{C}^{m\times n}$ is (R,S)-symmetric ((R,S)-skew symmetric) if $RAS = A$ ($RAS = -A$).

We give an explicit representation of an arbitrary (R,S)-symmetric matrix A in terms of matrices P and Q associated with R and U and V associated with S. If $R = R^*$, then the least squares problem for A can be solved by solving the independent least squares problems for $A_{PU} = P^*AU \in \mathbb{C}^{r\times k}$ and $A_{QV} = Q^*AV \in \mathbb{C}^{s\times \ell}$, where $r+s = m$ and $k+\ell = n$. If, in addition, either $\operatorname{rank}(A) = n$ or $S^* = S$, then $A^\dagger$ can be expressed in terms of $A^\dagger_{PU}$ and $A^\dagger_{QV}$. If $R = R^*$ and $S = S^*$, then a singular value decomposition of A can obtained from singular value decompositions of A_{PU} and A_{QV}. Similar results hold for (R,S)-skew symmetric matrices.

We say that $A \in \mathbb{C}^{m\times n}$ is R-conjugate if $RAS = \overline{R}$, where $R \in \mathbb{R}^{m\times m}$ and $S \in \mathbb{R}^{n\times n}$, $R = R^{-1} \ne \pm I_m$, and $S = S^{-1} \ne \pm I_n$. In this case $\Re(A)$ is (R,S)-symmetric and $\Im(A)$ is (R,S)-skew symmetric, so our results provide explicit representations for (R,S)-conjugate matrices. If $R^T = R$ the least squares problem for the complex matrix A reduces to two least squares problems for a real matrix K. If, in addition, either $\operatorname{rank}(A) = n$ or $S^T = S$, then $A^\dagger$ can be obtained from $K^\dagger$. If both $R^T = R$ and $S^T = S$, a singular value decomposition of A can be obtained from a singular value decomposition of K.

Key words. least squares, Moore–Penrose inverse, optimal solution, (R,S)-conjugate, (R,S)-skew symmetric, (R,S)-symmetric

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1. Introduction. In this paper we expand on a problem initiated by Chen [1], who considered matrices $A \in \mathbb{C}^{m\times n}$ such that

$$RAS = A \quad \text{or} \quad RAS = -A, \tag{1.1}$$

where $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ are involutory Hermitian matrices; i.e., $R = R^*$, $R^2 = I_m$, $S = S^*$, and $S^2 = I_n$. Chen cited applications involving such matrices, developed some of their theoretical properties, and indicated with numerical examples that the least squares problem for a matrix of rank n with either property reduces to two independent least squares problems for matrices of smaller dimensions. He also considered properties of the Moore–Penrose inverses of such matrices but did not obtain explicit expressions for them in terms of Moore–Penrose inverses of lower order matrices.

Here we characterize the matrices $A \in \mathbb{C}^{m\times n}$ satisfying (1.1) without assuming that R and S are Hermitian. We obtain general results on the least squares problem for the case where R is Hermitian, without assuming that S is Hermitian or that $\operatorname{rank}(A) = n$. Under the additional assumption that either S is Hermitian or $\operatorname{rank}(A) = n$, we obtain explicit expressions for $A^\dagger$ in terms of the Moore–Penrose inverses of two related matrices with smaller dimensions. Finally, under the assumption that $R = R^*$ and $S = S^*$, we obtain a singular value decomposition of A in terms of singular value decompositions of these related matrices.

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Under the assumption that $R \in \mathbb{R}^{m\times m}$ and $S \in \mathbb{R}^{n\times n}$, we consider the analogous questions for matrices $A \in \mathbb{C}^{m\times n}$ such that $RAS = \overline{A}$, so that $R\Re(A)S = \Re(A)$ and $R\Im(A)S = -\Im(A)$. We say that such matrices are (R,S)-conjugate.

We gave related results for square matrices with $R = S$ in [5] and studied other approximation problems for (R,S)-symmetric and (R,S)-skew symmetric matrices in [6].

2. Preliminary considerations. Let $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ be nontrivial involutions; thus $R = R^{-1} \neq \pm I_m$ and $S = S^{-1} \neq \pm I_n$. Then the minimal and characteristic polynomials of R are

$$m_R(x) = (x-1)(x+1) \quad \text{and} \quad c_R(x) = (x-1)^r(x+1)^s,$$

where $1 \le r, s \le m$ and $r + s = m$. Therefore there are matrices $P \in \mathbb{C}^{m\times r}$ and $Q \in \mathbb{C}^{m\times s}$ such that

$$P^*P = I_r, \quad Q^*Q = I_s, \tag{2.1}$$

$$RP = P, \quad \text{and} \quad RQ = -Q. \tag{2.2}$$

Thus, the columns of P (Q) form an orthonormal basis for the eigenspace of R associated with the eigenvalue $\lambda = 1$ ($\lambda = -1$). Although P and Q are not unique, suitable P and Q can be obtained by applying the Gram–Schmidt procedure to the columns of $I + R$ and $I - R$, respectively. If R is a signed permutation matrix, this requires little computation. For example, if J is the flip matrix with ones on the secondary diagonal and zeros elsewhere and $R = J_{2k}$, we can take

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} I_k \\ J_k \end{bmatrix} \quad \text{and} \quad Q = \frac{1}{\sqrt{2}}\begin{bmatrix} I_k \\ -J_k \end{bmatrix},$$

while if $R = J_{2k+1}$, we can take

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} I_k & 0_{k\times 1} \\ 0_{1\times k} & \sqrt{2} \\ J_k & 0_{k\times 1} \end{bmatrix} \quad \text{and} \quad Q = \frac{1}{\sqrt{2}}\begin{bmatrix} I_k \\ 0_{1\times k} \\ -J_k \end{bmatrix}.$$

If we define

$$\widehat{P} = \frac{P^*(I+R)}{2} \quad \text{and} \quad \widehat{Q} = \frac{Q^*(I-R)}{2}, \tag{2.3}$$

then

$$\widehat{P}P = I_r, \quad \widehat{P}Q = 0 \quad \widehat{Q}P = 0, \quad \text{and} \quad \widehat{Q}Q = I_s,$$

so

$$[P\ Q]^{-1} = \begin{bmatrix} \widehat{P} \\ \widehat{Q} \end{bmatrix}. \tag{2.4}$$

Similarly, there are integers k and ℓ such that $k + \ell = n$ and matrices $U \in \mathbb{C}^{n\times k}$ and $V \in \mathbb{C}^{n\times \ell}$ such that

$$U^*U = I_k, \quad V^*V = I_\ell,$$

$$SU = U, \quad \text{and} \quad SV = -V. \tag{2.5}$$

Moreover, if we define

$$\widehat{U} = \frac{U^*(I+S)}{2} \quad \text{and} \quad \widehat{V} = \frac{V^*(I-S)}{2}, \tag{2.6}$$

then

$$\widehat{U}U = I_k, \quad \widehat{U}V = 0, \quad \widehat{V}U = 0, \quad \text{and} \quad \widehat{V}V = I_\ell, \tag{2.7}$$

so

$$[U\ V]^{-1} = \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}. \tag{2.8}$$

It is straightforward to verify that if $R = R^*$, then $[P\ Q]$ and $[P\ iQ]$ are both unitary. Similarly, if $S = S^*$, then $[U\ V]$ and $[U\ iV]$ are both unitary. We will use this observation in several places without restating it.

From (2.4) and (2.8), any $A \in \mathbb{C}^{m\times n}$ can be written conformably in block form as

$$A = [P\ Q] \begin{bmatrix} A_{PU} & A_{PV} \\ A_{QU} & A_{QV} \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}. \tag{2.9}$$

We say that $A \in \mathbb{C}^{m\times n}$ is (R, S)-symmetric if $RAS = A$, or (R, S)-skew symmetric if $RAS = -A$. From (2.2), (2.5), and (2.6),

$$RAS = [P\ Q] \begin{bmatrix} A_{PU} & -A_{PV} \\ -A_{QU} & A_{QV} \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}. \tag{2.10}$$

Henceforth, $z \in \mathbb{C}^n$, $x \in \mathbb{C}^k$, $y \in \mathbb{C}^\ell$, $w \in \mathbb{C}^m$, $\phi \in \mathbb{C}^r$, and $\psi \in \mathbb{C}^s$. We say that w is R-symmetric (R-skew symmetric) if $Rw = w$ ($Rw = -w$). An arbitrary w can be written uniquely as $w = P\phi + Q\psi$ with $\phi = \widehat{P}w$ and $\psi = \widehat{Q}w$. From (2.2), $P\phi$ is R-symmetric and $Q\psi$ is R-skew symmetric. Similarly, we say that z is S-symmetric (S-skew symmetric) if $Sz = z$ ($Sz = -z$). An arbitrary z can be written uniquely as $z = Ux + Vy$ with $x = \widehat{U}z$ and $y = \widehat{V}z$. From (2.5), Ux is S-symmetric and Vy is S-skew symmetric.

3. Two useful lemmas. Suppose that $B \in \mathbb{C}^{m\times n}$ and consider the least squares problem for B: If $w \in \mathbb{C}^m$, find $z \in \mathbb{C}^n$ such that

$$\|Bz - w\| = \min_{\zeta\in\mathbb{C}^n} \|B\zeta - w\|, \tag{3.1}$$

where $\|\cdot\|$ is the 2-norm. This problem has a unique solution if and only if $\operatorname{rank}(B) = n$. In this case, $z = (B^*B)^{-1}B^*w$. In any case, the optimal solution of (3.1) is the unique n-vector z_0 of minimum norm that satisfies (3.1); thus, $z_0 = B^\dagger w$ where $B^\dagger$ is the Moore–Penrose inverse of B. The general solution of (3.1) is $z = z_0 + q$ with q in the null space of B, and

$$\|Bz - w\| = \|(BB^\dagger - I)w\|$$

for all such z.

The proof of the next lemma is motivated in part by a theorem of Meyer and Painter [3].

LEMMA 3.1. *Suppose that*

$$B = CKF, \tag{3.2}$$

where $C \in \mathbb{C}^{m\times m}$ *is unitary and* $F \in \mathbb{C}^{n\times n}$ *is invertible. Then the general solution of* (3.1) *is*

$$z = F^{-1}K^{\dagger}C^{*}w + (I - F^{-1}K^{\dagger}KF)h, \quad h \in \mathbb{C}^{n}, \tag{3.3}$$

and

$$\|Bz - w\| = \|(KK^{\dagger} - I)C^{*}w\| \tag{3.4}$$

for all such z. *If either* $\operatorname{rank}(B) = n$ *or* F *is unitary, then*

$$B^{\dagger} = F^{-1}K^{\dagger}C^{*},$$

so the optimal solution of (3.1) *is*

$$z_0 = F^{-1}K^{\dagger}C^{*}w. \tag{3.5}$$

Moreover, z_0 *is the unique solution of* (3.1) *if* $\operatorname{rank}(B) = n$.

Proof. Recall [4] that $Z = W^{\dagger}$ and $W = Z^{\dagger}$ if and only if Z and W satisfy the Penrose conditions

$$WZW = W, \quad ZWZ = Z, \quad (ZW)^{*} = ZW, \quad \text{and} \quad (WZ)^{*} = WZ. \tag{3.6}$$

Let

$$B^{L} = F^{-1}K^{\dagger}C^{*}. \tag{3.7}$$

By letting $W = K$ and $Z = K^{\dagger}$ in (3.6), it is straightforward to verify that

$$B^{L}BB^{L} = B^{L},\ BB^{L}B = B,\ (BB^{L})^{*} = BB^{L},\ \text{and}\ \ B^{L}B = F^{-1}K^{\dagger}KF. \tag{3.8}$$

Any $\zeta \in \mathbb{C}^{n\times n}$ can be written as $\zeta = B^{L}w + q$, so

$$B\zeta - w = (BB^{L} - I)w + Bq.$$

From the second and third equalities in (3.8),

$$[(BB^{L} - I)w]^{*}Bq = 0,$$

so

$$\|B\zeta - w\|^{2} = \|(BB^{L} - I)w\|^{2} + \|Bq\|^{2},$$

which is a minimum if and only if $Bq = 0$.

The second equality in (3.8) implies that $\operatorname{rank}(B^{L}B) = \operatorname{rank}(B)$, so $\operatorname{rank}(I - B^{L}B)$ equals the dimension of the null space of B. Now the second equality in (3.8) implies that $Bq = 0$ if and only if $q = (I - B^L B)h$, $h \in \mathbb{C}^{n\times n}$. Hence, the general solution of (3.1) is

$$z = B^L w + (I - B^L B)h, \quad h \in \mathbb{C}^{n\times n}.$$

Substituting (3.2) and (3.7) into this yields (3.3). From (3.2) and (3.3),

$$Bz - w = C(KK^\dagger - I)C^* w,$$

since C is unitary. This implies (3.4).

If $\text{rank}(B) = n$, then $\text{rank}(K) = n$, so $K^\dagger K = I$ and the fourth equality in (3.8) reduces to $B^L B = I$. If F is unitary, the fourth equality in (3.8) implies that $(B^L B)^* = B^L B$. In either case, (3.8) implies that $B^L = B^\dagger$, so (3.5) is the optimal solution of (3.1). If $\text{rank}(B) = n$, then (3.3) reduces to (3.5). ∎

The following lemma is obvious.

Lemma 3.2. *Suppose that $B \in \mathbb{C}^{m\times n}$ and $B = CKF$, where $C \in \mathbb{C}^{m\times m}$ and $F \in \mathbb{C}^{n\times n}$ are unitary and $K = ZDW^*$ is a singular value decomposition of K. Then $B = (CZ)D(FW)^*$ is a singular value decomposition of B.*

4. Characterization and properties of (R,S)-symmetric matrices. The following theorem characterizes (R,S)-symmetric matrices.

Theorem 4.1. *A is (R,S)-symmetric if and only if*

$$A = [P\ Q]\begin{bmatrix} A_{PU} & 0 \\ 0 & A_{QV} \end{bmatrix}\begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}, \tag{4.1}$$

where

$$A_{PU} = P^* A U \quad \textit{and} \quad A_{QV} = Q^* A V. \tag{4.2}$$

Proof. From (2.9) and (2.10), $RAS = A$ if and only if (4.1) holds. If (4.1) holds, then (2.8) implies that

$$A[U\ V] = [P\ Q]\begin{bmatrix} A_{PU} & 0 \\ 0 & A_{QV} \end{bmatrix},$$

so $AU = PA_{PU}$ and $AV = QA_{QV}$. Therefore (2.1) implies (4.2).

The verification of the converse is straightforward. ∎

The following theorem reduces the least squares problem

$$\|Az - w\| = \min_{\zeta\in\mathbb{C}^n} \|A\zeta - w\| \tag{4.3}$$

to the independent $r \times k$ and $s \times \ell$ least squares problems

$$\|A_{PU}x - \phi\| = \min_{\xi\in\mathbb{C}^k} \|A_{PU}\xi - \phi\|$$

and

$$\|A_{QV}y - \psi\| = \min_{\eta\in\mathbb{C}^\ell} \|A_{QV}\eta - \psi\|.$$

THEOREM 4.2. *Suppose that A is (R,S)-symmetric, $R=R^*$, and $w=P\phi+Q\psi$. Then the general solution of* (4.3) *is*

$$z=U[A_{PU}^\dagger\phi+(I_k-A_{PU}^\dagger A_{PU})\xi]+V[A_{QV}^\dagger\psi+(I_\ell-A_{QV}^\dagger A_{QV})\eta],\ \xi\in\mathbb{C}^k,\ \eta\in\mathbb{C}^\ell,$$

and

$$\|Az-w\|^2=\|(A_{PU}A_{PU}^\dagger-I_r)\phi\|^2+\|(A_{QV}A_{QV}^\dagger-I_s)\psi\|^2$$

for all such z. If either $\operatorname{rank}(A)=n$ *or $S=S^*$, then*

$$A^\dagger=[U\ V]\begin{bmatrix} A_{PU}^\dagger & 0\\ 0 & A_{QV}^\dagger\end{bmatrix}\begin{bmatrix} P^*\\ Q^*\end{bmatrix}$$

and $z_0=UA_{PU}^\dagger\phi+VA_{QV}^\dagger\psi$ is the optimal solution of (4.3). *Moreover, z_0 is the unique solution of* (4.3) *if* $\operatorname{rank}(A)=n$.

Proof. Starting from Theorem 4.1, we apply Lemma 3.1 with

$$C=[P\ Q],\quad K=\begin{bmatrix} A_{PU} & 0\\ 0 & A_{QV}\end{bmatrix},\quad F=\begin{bmatrix}\widehat{U}\\ \widehat{V}\end{bmatrix},$$

$$z=Ux+Vy,\quad w=P\phi+Q\psi\quad\text{and}\quad h=U\xi+V\eta.$$

It is straightforward to verify that

$$K^\dagger=\begin{bmatrix} A_{PU}^\dagger & 0\\ 0 & A_{QV}^\dagger\end{bmatrix},$$

and the other details follow easily, if we recall that since $R=R^*$, $\widehat{P}=P^*$ and $\widehat{Q}=Q^*$. ∎

Theorem 4.1 and Lemma 3.2 imply the following theorem. (Recall that $\widehat{U}=U^*$ and $\widehat{V}=V^*$ if $S=S^*$.)

THEOREM 4.3. *Suppose that $R=R^*$, $S=S^*$, and A is (R,S)-symmetric. Let*

$$A_{PU}=\Phi D_{PU}X^*\quad\textit{and}\quad A_{QV}=\Psi D_{QV}Y^*$$

be singular value decompositions of A_{PU} and A_{QV}. Then

$$A=[P\Phi\ Q\Psi]\begin{bmatrix} D_{PU} & 0\\ 0 & D_{QV}\end{bmatrix}[UX\ VY]^*$$

is a singular value decomposition of A. Thus, the singular values of A_{PU} are singular values of A with associated R-symmetric left singular vectors and S-symmetric right singular vectors, and the singular values of A_{QV} are singular values of A with associated R-skew symmetric left singular vectors and S-skew symmetric right singular vectors.

5. Characterization and properties of (R,S)-skew symmetric matrices. The following theorem characterizes (R,S)-skew symmetric matrices.

THEOREM 5.1. *A is (R,S)-skew symmetric if and only if*

$$A = [P\ Q] \begin{bmatrix} 0 & A_{PV} \\ A_{QU} & 0 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}, \tag{5.1}$$

where

$$A_{PV} = P^*AV \quad \text{and} \quad A_{QU} = Q^*AU. \tag{5.2}$$

Proof. From (2.9) and (2.10), $RAS = -A$ if and only if (5.1) holds. If (5.1) holds, then (2.8) implies that

$$A[U\ V] = [P\ Q] \begin{bmatrix} 0 & A_{PV} \\ A_{QU} & 0 \end{bmatrix},$$

so $AU = QA_{QU}$ and $AV = PA_{PV}$. Therefore (2.1) implies (5.2).

The verification of the converse is straightforward. □

Theorem 5.1 and Lemma 3.1 imply the following theorem, which reduces (4.3) to the independent $s \times k$ and $r \times \ell$ least squares problems

$$\|A_{QU}x - \psi\| = \min_{\xi \in \mathbb{C}^k} \|A_{QU}\xi - \psi\|$$

and

$$\|A_{PV}y - \phi\| = \min_{\eta \in \mathbb{C}^\ell} \|A_{PV}\eta - \phi\|.$$

The proof is similar to the proof of Theorem 4.2, noting that in this case

$$K = \begin{bmatrix} 0 & A_{PV} \\ A_{QU} & 0 \end{bmatrix} \quad \text{and} \quad K^\dagger = \begin{bmatrix} 0 & A_{QU}^\dagger \\ A_{PV}^\dagger & 0 \end{bmatrix}.$$

THEOREM 5.2. *Suppose that A is (R,S)-skew symmetric, $R^* = R$, and $w = P\phi + Q\psi$. Then the general solution of* (4.3) *is*

$$z = U[A_{QU}^\dagger \psi + (I_k - A_{QU}^\dagger A_{QU})\xi] + V[A_{PV}^\dagger \phi + (I_\ell - A_{PV}^\dagger A_{PV})\eta],\ \xi \in \mathbb{C}^k,\ \eta \in \mathbb{C}^\ell,$$

and

$$\|Az - w\|^2 = \|(A_{QU}A_{QU}^\dagger - I_s)\psi\|^2 + \|(A_{PV}A_{PV}^\dagger - I_r)\phi\|^2$$

for all such z. If either $\operatorname{rank}(A) = n$ *or $S = S^*$, then*

$$A^\dagger = [U\ V] \begin{bmatrix} 0 & A_{QU}^\dagger \\ A_{PV}^\dagger & 0 \end{bmatrix} \begin{bmatrix} P^* \\ Q^* \end{bmatrix}$$

and $z_0 = UA_{QU}^\dagger \psi + VA_{PV}^\dagger \phi$ is the optimal solution of (4.3). *Moreover, z_0 is the unique solution of* (4.3) *if* $\operatorname{rank}(A) = n$.

Theorem 5.1 and Lemma 3.2 imply the following theorem.

Theorem 5.3. *Suppose that $R = R^*$, $S = S^*$, and A is (R,S)-skew symmetric. Let*

$$A_{PV} = \Phi D_{PV} Y^* \quad \textit{and} \quad A_{QU} = \Psi D_{QU} X^*$$

be singular value decompositions of A_{PV} and A_{QU}. Then

$$A = [P\Phi \ Q\Psi] \begin{bmatrix} D_{PV} & 0 \\ 0 & D_{QU} \end{bmatrix} [VY \ UX]^*$$

is a singular value decomposition of A. Thus, the singular values of A_{PV} are singular values of A with R-symmetric left singular vectors and S-skew symmetric right singular vectors, and the singular values of A_{QU} are singular values of A with R-skew symmetric left singular vectors and S-symmetric right singular vectors.

6. Characterization and properties of (R,S)-conjugate matrices. In this section we impose the following standing assumption.

Assumption A. *$R \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{n\times n}$ $R^{-1} = R \neq \pm I_m$, $S^{-1} = S \neq \pm I_n$, $P \in \mathbb{R}^{m\times r}$, $Q \in \mathbb{R}^{m\times s}$, $U \in \mathbb{R}^{n\times k}$, and $V \in \mathbb{R}^{n\times \ell}$. Also, $A = B + iC$ with B, $C \in \mathbb{R}^{m\times n}$.*

Under this assumption, (2.3) reduces to

$$\widehat{P} = \frac{P^T(I+R)}{2} \quad \text{and} \quad \widehat{Q} = \frac{Q^T(I-R)}{2},$$

and (2.6) reduces to

$$\widehat{U} = \frac{U^T(I+S)}{2}, \quad \text{and} \quad \widehat{V} = \frac{V^T(I-S)}{2}.$$

Moreover, if $R = R^T$, then $\widehat{P} = P^T$, $\widehat{Q} = Q^T$, and $[P \ iQ]$ is unitary. Similarly, if $S = S^T$, then $\widehat{U} = U^T$, $\widehat{V} = V^T$, and $[U \ iV]$ is unitary.

We say that A is (R,S)-conjugate if $RAS = \overline{A}$. The following theorem characterizes the class of (R,S)-conjugate matrices.

Theorem 6.1. *$A = B + iC$ is (R,S)-conjugate if and only if*

$$A = [P \ iQ] \begin{bmatrix} B_{PU} & -C_{PV} \\ C_{QU} & B_{QV} \end{bmatrix} \begin{bmatrix} \widehat{U} \\ -i\widehat{V} \end{bmatrix}, \tag{6.1}$$

where

$$B_{PU} = P^T BU, \quad B_{QV} = Q^T BV, \quad C_{PV} = P^T CV, \quad C_{QU} = Q^T CU. \tag{6.2}$$

Proof. If $RAS = \overline{A}$, then $RBS = B$ and $RCS = -C$. Therefore Theorem 4.1 implies that

$$B = [P \ Q] \begin{bmatrix} B_{PU} & 0 \\ 0 & B_{QV} \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}$$

with B_{PU} and B_{QV} as in (6.2) and Theorem 5.1 implies that

$$C = [P \ Q] \begin{bmatrix} 0 & C_{PV} \\ C_{QU} & 0 \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}$$

with C_{PV} and C_{QU} as in (6.2). Therefore

$$A = B + iC = [P\ Q] \begin{bmatrix} B_{PU} & iC_{PV} \\ iC_{QU} & B_{QV} \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix},$$

which is equivalent to (6.1).

For the converse, if A satisfies (6.1) where the center matrix is in $\mathbb{R}^{m\times n}$, then $RAS = \overline{A}$. Moreover, $A = B + iC$ with

$$B = PB_{PU}\widehat{U} + QB_{QV}\widehat{V} \quad \text{and} \quad C = QC_{QU}\widehat{U} + PC_{PV}\widehat{V}.$$

Now we invoke (2.1) (with $^* = ^T$) and (2.7) to verify (6.2). ◻

Theorem 6.1 with $m = n$ and $R = S$ is related to a a result of Ikramov [2]. See also [5, Theorem 19].

Henceforth

$$K = \begin{bmatrix} B_{PU} & -C_{PV} \\ C_{QU} & B_{QV} \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

An arbitrary z can be written uniquely as $z = Ux + iVy$ with $x = \widehat{U}z$ and $y = -i\widehat{V}z$. An arbitrary w can be written uniquely as $w = P\phi + iQ\psi$ with $\phi = \widehat{P}w$ and $\psi = -i\widehat{Q}w$. For our present purposes it is useful to write x, y, ϕ, and ψ in terms of their real and imaginary parts; thus,

$$x = x_1 + ix_2, \quad x_1, x_2 \in \mathbb{R}^k, \quad y = y_1 + iy_2, \quad y_1, y_2 \in \mathbb{R}^\ell,$$

$$\phi = \phi_1 + i\phi_2, \quad \phi_1, \phi_2 \in \mathbb{R}^r, \quad \psi = \psi_1 + i\psi_2, \quad \psi_1, \psi_2 \in \mathbb{R}^s.$$

Theorem 6.1 and Lemma 3.1 imply the following theorem, which reduces (4.3) to two independent least squares problems for the real matrix K:

$$\left\| K \begin{bmatrix} x_j \\ y_j \end{bmatrix} - \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} \right\| = \min_{\xi_j\in\mathbb{R}^k, \eta_j\in\mathbb{R}^\ell} \left\| K \begin{bmatrix} \xi_j \\ \eta_j \end{bmatrix} - \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} \right\|, \quad j = 1, 2.$$

THEOREM 6.2. *Suppose that A is (R,S)-conjugate, $R^T = R$, and $w = P\phi + Q\psi$. Then the general solution of* (4.3) *is*

$$z = [U\ iV]\left(K^\dagger \begin{bmatrix} \psi \\ \phi \end{bmatrix} + (I - K^\dagger K) \begin{bmatrix} \xi \\ \eta \end{bmatrix} \right), \quad \xi \in \mathbb{C}^k, \quad \eta \in \mathbb{C}^\ell,$$

and

$$\|Az - w\|^2 = \left\| (KK^\dagger - I) \begin{bmatrix} \psi_1 \\ \phi_1 \end{bmatrix} \right\|^2 + \left\| (KK^\dagger - I) \begin{bmatrix} \psi_2 \\ \phi_2 \end{bmatrix} \right\|^2$$

for all such z. If either $\operatorname{rank}(A) = n$ *or $S = S^T$, then*

$$A^\dagger = [U\ iV]K^\dagger \begin{bmatrix} P^T \\ -iQ^T \end{bmatrix}$$

and

$$z_0 = [U\ iV]K^\dagger \begin{bmatrix} \psi \\ \phi \end{bmatrix}$$

is the optimal solution of (4.3). *Moreover,* z_0 *is the unique solution of* (4.3) *if* $\operatorname{rank}(A) = n$.

Finally, Lemma 3.2 and Theorem 6.1 imply the following theorem.

THEOREM 6.3. *Suppose* $R^T = R$, $S^T = S$, *and* A *is* (R, S)*-conjugate. Let* $K = WDZ^T$ *be a singular value decomposition of* K. *Then*

$$A = [P \;\; iQ]WD([U \;\; iV]Z)^*$$

is a singular value decomposition of A. *Therefore the left singular vectors of* A *can be written as* $w_j = P\phi_j + iQ\psi_j$, *with* $\phi_j \in \mathbb{R}^r$ *and* $\psi_j \in \mathbb{R}^s$, $1 \le j \le m$, *and the right singular vectors of* A *can be written as* $z_j = Ux_j + iVy_j$ *with* $x_j \in \mathbb{R}^k$ *and* $y_j \in \mathbb{R}^\ell$, $1 \le j \le n$.

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