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William F. Trench, Trinity University

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WILLIAM F. TRENCH*

During the powered flight of a space vehicle, such as the Saturn vehicle, the onboard accelerometers provide a source of trajectory information that can be used to obtain a record of position and velocity. However, the data usually is degraded by systematic errors that tend to become large, but whose form is essentially predictable, given a reasonably valid model for the errors in the inertial measurements. This paper describes a postflight technique for employing a relatively small amount of ground-based tracking data to eliminate these systematic errors. The technique has been applied to a simulated two-stage Saturn flight. Bias errors were introduced into the accelerometer data, and noisy ground-based data were simulated. The trajectory estimates were derived from inertial data alone, and from the present method. In all cases considered, a small number of ground-based position measurements yielded position and velocity estimates that were essentially free of bias errors. Even though the analytical basis for the method rests on the assumption of linear time-invariant error model, the method yielded good results in the presence of large, nonlinear, time-varying errors.

1. Introduction

Two independent classes of trajectory data are usually available from the powered flight of a space vehicle: data from ground instrumentation along the tracking range, and data telemetered to the ground. (The Saturn vehicle has its inertial system three mutually perpendicular accelerometers, mounted on a stable platform, from which thrust acceleration data are telemetered.) A postflight trajectory analysis technique, which depends entirely on one or the other of these data sources, is usually inadequate. No single ground instrument can track the vehicle during the entire flight, and abrupt changes in position estimates often occur when the track is handed over from one to another. These discontinuities are caused by differences in the error characteristics of the instruments. Another shortcoming of ground instrumentation is that its accuracy can be severely limited by the geometry of the tracking problem; thus, the term "geometrical dilution of precision" is in common use among trajectory analysts.

Trajectory estimates based on data from the inertial system exhibit prohibitively large bias errors, introduced by platform drift and misalignment, scale factor errors, and nonorthogonality of the accelerometers.

This paper presents a method for combining these two kinds of data to obtain improved postflight estimates of the vehicle's position and velocity during powered flight. It is assumed that errors in the inertial system cause the measured thrust acceleration vector \( A_T(t) \) to be related to the true thrust acceleration \( A_T(t) \) by

\[
A_T(t) = (I + K)A_T(t)
\]

where \( K \) is a constant \( 3 \times 3 \) matrix whose elements are unknown a priori. It is shown below, subject to certain linearizing assumptions, that

\[
X(t) = Y(t) + \sum_{j=1}^{9} q_j P_j(t)
\]

where \( X(t) \) is the true trajectory, and \( Y(t) \) is the trajectory that would be obtained from \( A_T(t) \) alone. Each of the vectors \( [P_j(t)] \) is the solution of a \( 3 \times 3 \) system of linear time-varying differential equations, which can be computed from \( A_T(t) \). The constants \( q_1, \ldots, q_9 \) are unknown a priori, and cannot be determined from the accelerometer data alone. Using standard methods of weighted least squares, they are chosen so that the right side of (2) gives the best fit, of the form (2), to the ground-based measurements. As shown below, \( q_1, \ldots, q_9 \) are used to calculate \( A_T(t) \) from \( A_T(t) \), and an improved estimate of \( X(t) \) is obtained by integrating the equations of motion, using the corrected thrust acceleration data.

2. Analytical Basis for the Method

Let \( X(t) \) be the vehicle's position at time \( t \), relative to an inertial frame. The equation of motion is

\[
\ddot{X} = A_T + G(X)
\]

where \( G(X) \) is the acceleration due to gravity. The vector \( Y(t) \), which is the position estimate based on \( A_T(t) \), is a solution of

\[
\dot{Y} = A_T + G(Y)
\]

The error \( E = X - Y \) satisfies

\[
E = \dot{E} = E(0) = 0
\]

We assume that the error committed in writing

\[
G(X) - G(Y) = HE
\]

where

\[(H)_{ij} = \frac{\partial q_j}{\partial x_i} (i, j = 1, 2, 3)\]

is negligible. Substituting this in (4) yields

\[
\dot{E} - HE = JA_T \quad E(0) = \dot{E}(0) = 0
\]

where

\[ J = -I + (I + K)^{-1} \]

(It is reasonable to assume that \( (I + K)^{-1} \) exists; if it did not, the null space of \( I + K \) would be nontrivial, and there would be a direction in which the inertial system would be unable to detect thrust accelerations.)

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1 Associate Professor of Mathematics.

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Denote the elements of \( J \) by

\[
J = \begin{pmatrix}
q_1 & q_2 & q_3 \\
q_1 & q_2 & q_3 \\
q_1 & q_2 & q_3 \\
\end{pmatrix}
\]

From (5), \( E(t) \) can be written in the form

\[
E(t) = \sum_{j=1}^{9} q_j P_j(t)
\]

where

\[
P_j = H P_j + F_j \quad P_j(0) = 0
\]

If we denote the components of \( A_M \) by \( (\bar{X}_M, \bar{\eta}_M, \bar{\xi}_M) \), the forcing functions are given by

\[
F_1 = \begin{pmatrix}
\bar{X}_M \\
0 \\
0
\end{pmatrix} \quad F_2 = \begin{pmatrix}
\bar{\eta}_M \\
0 \\
0
\end{pmatrix} \quad F_3 = \begin{pmatrix}
\bar{\xi}_M \\
0 \\
0
\end{pmatrix}
\]

\( F_1, F_2, F_3, F_4, F_5 \) are similar, with \( \bar{X}_M \) replaced by \( \bar{\eta}_M \) and \( \bar{\xi}_M \), respectively. We will refer to \( P_1, \ldots, P_9 \) as error functions, and to \( q_1, \ldots, q_9 \) as error coefficients.

Consistent with the linearization assumption made in the derivation of (6), the derivatives appearing in \( H \) can be evaluated at points on the uncorrected trajectory (that is, the trajectory obtained from \( A_M \)). Hence, the error functions can be calculated from \( A_M \), and if the error coefficients were known, the true trajectory could be obtained from

\[
X(t) = Y(t) + \sum_{j=1}^{9} q_j P_j(t)
\]

(6)

As an alternative, one can compute \( A_T = (I + J) A_M \) and obtain the corrected trajectory from (3). This was the method used in the calculations described subsequently.

3. Estimation of the Error Coefficients: Theoretical Considerations

If estimates of \( X(t) \) are available from other data sources, such as ground instrumentation, the error coefficients can be obtained by weighted least squares estimation, where the data are fitted to a linear combination of \( P_1(t), \ldots, P_9(t) \). Let \( \hat{X}(t_1), \ldots, \hat{X}(t_n) \) be estimates of \( X(t) \) at times \( t_1, \ldots, t_n \), derived from ground instruments. Assume that the errors in \( \hat{X}(t_k) \) have zero mean and known covariance matrix \( C \).

The \( i,j \)th element of \( C \) is given by

\[
c_{ik} = [\hat{X}(t_k) - x_i(t_k)][\hat{X}(t_k) - x_j(t_k)]
\]

(\( i, j = 1, 2, 3 \))

(The bar denotes expected value.) The joint covariance matrix of the errors in \( \hat{X}(t_1), \ldots, \hat{X}(t_n) \) is

\[
C = \begin{pmatrix}
C_1 & 0 & \cdots & 0 \\
0 & C_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C_n
\end{pmatrix}
\]

where the zeros are \( 3 \times 3 \) matrices with zero elements, so that \( C \) is of order \( 3K \).

Define the \( 3K \)-dimensional vectors

\[
U = \begin{pmatrix}
X(t_1) \\
\vdots \\
X(t_n)
\end{pmatrix} \quad \hat{U} = \begin{pmatrix}
\hat{X}(t_1) \\
\vdots \\
\hat{X}(t_n)
\end{pmatrix} \quad V = \begin{pmatrix}
Y(t_1) \\
\vdots \\
Y(t_n)
\end{pmatrix}
\]

the nine-dimensional vector

\[
Q = \begin{pmatrix}
q_1 \\
\vdots \\
q_9
\end{pmatrix}
\]

and the \( 3K \) by \( 9 \) matrix

\[
P = \begin{pmatrix}
P_1(t_1) \cdots P_9(t_1) \\
\vdots \cdots \cdots \cdots \vdots \\
P_1(t_n) \cdots P_9(t_n)
\end{pmatrix}
\]

From (6), \( U = V + PQ \). For every nine-dimensional vector \( Q' \), we can define a vector, \( U' = V + PQ' \), and a positive number,

\[
\sigma^2 = (U' - \hat{U})^T C^{-1} (U' - \hat{U})
\]

The quantity \( \sigma^2 \) is a function of \( (q_1', \ldots, q_9') \), and it can be written directly in terms of the vector \( Q' \) as follows:

\[
\sigma^2(Q') = (V - \hat{U} + PQ')^T C^{-1} (V - \hat{U} + PQ')
\]

It can be shown that \( \sigma^2 \) has a unique minimum value, attained when \( Q' = \hat{Q} \), a solution of

\[
\Delta \hat{Q} = P^T C^{-1} (U - V) = L
\]

(7)

where

\[
\lambda = P^T C^{-1} P
\]

If \( \lambda \) is nonsingular, \( \hat{Q} \) is unique; \( \hat{Q} = \lambda^{-1} L \). If \( \lambda \) is singular, there are infinitely many vectors \( \hat{Q} \) for which the same minimum value of \( \sigma^2 \) is attained. Since this case is of considerable interest, we will examine it further.

The singularity of \( \lambda \) is equivalent to the existence of a nontrivial vector

\[
Z = \begin{pmatrix}
z_1 \\
\vdots \\
z_k
\end{pmatrix}
\]

such that \( \lambda Z = 0 \). From the positive definiteness of \( C \), it follows that \( \lambda Z = 0 \) if and only if \( PZ = 0 \) and from the definition of \( P \), this is equivalent to

\[
\sum_{i=1}^{9} \xi_i P_j(t_i) = 0 \quad (k = 1, \ldots, K)
\]

(8)

From this, we can conclude that either

\[
\sum_{j=1}^{9} \xi_j P_j(t) = 0
\]

identically in \( t \), or \( \lambda \) can be rendered nonsingular by introducing additional ground-based data. We will assume that if \( \lambda \) is singular, it is because of (8). It can be shown that (8) is satisfied if and only if \( A_M \) lies in the same inertial plane for all \( t \); that is,

\[
n_1 \bar{X}_M + n_2 \bar{\eta}_M + n_3 \bar{\xi}_M = 0
\]

where \( n_1, n_2, \) and \( n_3 \) are the (constant) components of a normal to the plane of \( A_M \). It follows from (1) that \( A_T \) lies in a fixed inertial plane if and only if \( A_M \) does. We will denote these planes by \( \Gamma_T \) and \( \Gamma_M \).

It is reasonable to expect that \( \Gamma_T \) will be known in advance; in fact, \( \Gamma_T \) is often defined by

\[
\xi_T(t) = 0
\]
because the $\xi$ axis of the inertial system is erected normal to the plane of the nominal trajectory. From this, it can also be seen intuitively that $\mathbf{\ddot{Q}}$ cannot be obtained uniquely in this situation, because, if it could, then the matrix $K$ could be obtained uniquely. This would be absurd, since the coefficients $k_a$, $k_b$, and $k_c$, which cause errors arising from accelerations along the $\xi$ axis, cannot be determined if there are no such accelerations. However, even though $\mathbf{\ddot{Q}}$ is not unique, this does not preclude finding the best correction for the trajectory, since it can be shown that if $\mathbf{\ddot{Q}}$ and $\mathbf{\ddot{Q}}$ are both minimizing vectors, then

$$\sum_{i=1}^{q} (\mathbf{\ddot{Q}}_i - \mathbf{\ddot{Q}}_i) P_i(t) = 0$$

4. Estimation of the Error Coefficients: Practical Considerations

It is unlikely that the vector $A_x$ will be strictly confined to a plane, since guidance errors will cause some motion of the vehicle out of the nominal plane of the trajectory. Thus, it is not likely that $A_x$ will be exactly singular. Rather, the question of interest is how "nearly singular," or "ill conditioned" $A_x$ is. An equivalent question is how "close" the vector $A_x$ stays to a plane. We give a procedure that simultaneously leads to a precise definition of the latter situation, and also provides the approximate reduction of the system (7) to a lower order, if necessary.

Let the times at which thrust accelerations are measured be of the form $l_1 = 0, 1, 2, \ldots, N_1$, and define the $3 \times 3$ matrix $\Psi = (\Psi_{ij})$ by

$$\Psi_{ij} = \frac{1}{N_1} \sum_{n=0}^{N_1} A_{i}(nh)A_{j}(nh) \quad (i, j = 1, 2, 3)$$

where we have adopted the notations $A_{1M} = \xi M$, $A_{31} = \eta M$, $A_{13} = \zeta M$.

It can be shown that $\Psi$ is symmetric and positive semi-definite, and it is singular if and only if $A_x$ is singular. In this case,

$$n^T \Psi_n = 0$$

where $n$ is the unit normal to $\Gamma_x$.

The eigenvalues of $\Psi$ are nonnegative. Denote them by $n_1 \geq n_2 \geq \cdots \geq n_3 \geq 0$. Let $l_i$, $m_i$, and $n_i$ be the principal axes of $\Psi$ and $n_1$, $n_2$, and $n_3$ are the mean square values of the projection of $A_{xnm}$ on $l_i$, $m_i$, and $n_i$, respectively.

$\Psi$ is singular if and only if $n_3$, its smallest eigenvalue, is equal to zero. In this case, the vector $n$ is the normal to the plane $\Gamma_x$. However, even if $\Psi$ is not exactly singular, a convenient measure of the "closeness" to singularity is afforded by

$$r = (n_1^2 + n_2^2)/n_3$$

which is the ratio of the mean square value of the component $A_x$ in the plane of $l_i$ and $m_i$ to the mean square value of the component perpendicular to it. If $r = \infty$, $\Psi$, and consequently $A_x$ are singular. Clearly, the bigger the value of $r$, the more nearly singular $\Psi$ becomes. If $r = \infty$, or is very large, define the matrix $\mathbf{R}$.

5. Results with Simulated Data

To test the method presented in this paper, two computer programs were written. One was a simulation program, that accepts thrust acceleration data as input, and produces the following outputs:

a) A reference trajectory, obtained by numerical integration of the equations of motion with error-free input data. This is used as a standard in later computations.

b) A simulated time history of the output of the accelerometers. This is obtained by means of a program that introduces systematic errors into the thrust acceleration data in accordance with an error model that accounts for scale factor errors, cross coupling between accelerometer axes, non-orthogonality of the axes, misalignment, and drift. The last two types of errors do not match the assumed time-invariant error model of Eq. (1).

c) A simulated range data tape, containing estimates of the position of the vehicle as a function of time, as would be provided by ground based data. This tape is obtained by computing the local coordinates of the target, corresponding to the reference trajectory, and adding noise and bias errors. The resulting estimates of the local coordinates then are transformed into the central coordinate system, and stored on a tape that represents the input that would be provided by ground instrumentation. In addition, this tape contains a covariance matrix for each time of observation.

The output tapes described by b) and c) are used as input to a second program that combines the data from the two tapes, corrects for the systematic errors in the tape b) data, and computes a corrected trajectory.

When the errors in the inertial data were of the form (1), the corrected trajectory was found to lie extremely close to the reference trajectory. However, the more interesting result is that, if errors in the guidance system are simulated with a time-varying error model in the simulation program, the method still provides excellent corrections to the trajectory, despite the fact that it is based on a constant error model.

The fact that good corrections to the trajectory can be obtained with the constant error model (1), even in the case when the actual errors are considerably more complicated, shows that the operational program is a useful tool for obtaining improved estimates of the trajectory under practical conditions.

Several simulated applications of the method were computed, using a thrust acceleration profile for a two-stage, 600-see powered flight of a Saturn vehicle. Tracking range data were introduced every 20 sec. We will summarize some typical runs, where we used three different error models for the inertial system, described as follows:
Case 3: Constant scale factor error on each axis of 1%, nonorthogonality of axes = 0.1°. Platform drift about all three axes at rate of 0.0015° sec/m.

Case 1 is a time-varying error model. Case 2 is a constant error model of the form (1). Case 3 is again time varying, with very large errors. This case was computed primarily to test the linearizing assumptions.

The tracking range that was simulated can be described as follows:

Ground sensors: One radar and one Azusa in the launch area, and one radar and one Azusa down range.

Errors:

Tracking range 1:

\[
\sigma_K = 20 \text{ m} \quad \sigma_A = \sigma_E = 0.3 \text{ m}
\]

Tracking range 2:

\[
\sigma_K = \Delta R = 100 \text{ m} \quad \sigma_A = \Delta \bar{A} = \sigma_E = \Delta \bar{E} = 3 \text{ m}
\]

In tracking range 2, \(\Delta A\), \(\Delta E\), \(\Delta l\), and \(\Delta m\) are constant bias errors.

The results are given in terms of a tangent plane coordinate system, with origin at the launch point. The \(z\) coordinate is positive down range, \(y\) is positive along the vertical at the launch point, and \(x\) completes the right-hand system. The results are summarized in Table 1.

### Reference