An elementary view of Weyl's theory of equal distribution

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Abstract

Suppose that \(-\infty < a < b < \infty\), \(a \leq u_{1n} \leq u_{2n} \leq \cdots \leq u_{nn} \leq b\), and \(a \leq v_{1n} \leq v_{2n} \leq \cdots \leq v_{nn} \leq b\) for \(n \geq 1\). We simplify and strengthen Weyl’s definition of equal distribution of \(\{(u_{in})_{i=1}^n\}_{n=1}^\infty\) and \(\{(v_{in})_{i=1}^n\}_{n=1}^\infty\) by showing that the following statements are equivalent:

(i) \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n (F(u_{in}) - F(v_{in})) = 0\) for all \(F \in C[a, b]\),

(ii) \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n |u_{in} - v_{in}| = 0\),

(iii) \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n |F(u_{in}) - F(v_{in})| = 0\) for all \(F \in C[a, b]\).

We relate this to Weyl’s definition of uniform distribution and Szegő’s distribution formula for the eigenvalues of a family of Toeplitz matrices \(\{t_{r-s}\}_{r,s=1}^\infty\), where \(t_r = \frac{1}{2\pi} \int_{-\pi}^\pi e^{irx} g(x) \, dx\) and \(g\) is real-valued and continuous on \([-\pi, \pi]\).

1 Introduction

We consider four definitions of “distribution” that can be traced back to H. Weyl. We assume throughout that the doubly-indexed sequences

\[ \mathbf{U} = \{(u_{in})_{i=1}^n\}_{n=1}^\infty \quad \text{and} \quad \mathbf{V} = \{(v_{in})_{i=1}^n\}_{n=1}^\infty \]

are contained in a finite interval \([a, b]\). As usual, \(C[a, b]\) is the family of real-valued continuous functions on \([a, b]\). To avoid annoying repetition, every occurrence of “distributed” is to be interpreted as “distributed in \([a, b]\).”

We have presented part of this discussion in [4] and [5]. However, [4] is interesting mainly to linear algebraists and operator theorists, and [5] is not widely circulated. Moreover, the arguments given here are simpler and we think that the conclusions will be interesting to a wider audience.

Our first definition is stated and attributed to H. Weyl in [1, p. 62].
Definition 1 U and V are equally distributed if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (F(u_i) - F(v_i)) = 0 \text{ for all } F \in C[a, b].
\] (1)

Definition 2 V is uniformly distributed if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F(v_i) = \frac{1}{b-a} \int_{a}^{b} F(x) \, dx \text{ for all } F \in C[a, b].
\] (2)

Definition 3 A sequence \(\{x_i\}_{i=1}^{\infty} \subset [a, b]\) is uniformly distributed if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F(x_i) = \frac{1}{b-a} \int_{a}^{b} F(x) \, dx \text{ for all } F \in C[a, b].
\] (3)

Put another way, \(\{x_i\}_{i=1}^{\infty}\) is uniformly distributed if \(\{\hat{x}_i\}_{i=1}^{\infty}\) is uniformly distributed as in Definition 2.

Definition 4 If a and b are respectively the minimum and maximum values of a continuous function \(g\) on a closed interval \([c, d]\), then U is distributed like the values of \(g\) if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F(u_i) = \frac{1}{d-c} \int_{c}^{d} F(g(x)) \, dx \text{ for all } F \in C[a, b].
\] (4)

In the setting of linear algebra and operator theory, the members of U and V could be the eigenvalues of two families \(\{A_n\}_{n=1}^{\infty}\) and \(\{B_n\}_{n=1}^{\infty}\) of Hermitian matrices, and the problem is to find conditions on \(\{A_n - B_n\}_{n=1}^{\infty}\) which imply that U and V are equally distributed.

It is well known that (2) is equivalent to
\[
\lim_{n \to \infty} \frac{C_n(\mathcal{I})}{n} = \frac{\ell(\mathcal{I})}{b-a}
\]
for every subinterval \(\mathcal{I}\) of \([a, b]\), where \(\ell(\mathcal{I})\) is the length of \(\mathcal{I}\) and \(C_n(\mathcal{I})\) is the cardinality of \(\{v_{i_n}\}_{i=1}^{n} \cap \mathcal{I}\).

Definition 3 is a special case of Definition 2; nevertheless, a special case of Definition 3 is probably the most famous of all the definitions that we are considering. If \(x\) is an arbitrary real, let \([x]\) denote the greatest integer not greater than \(x\), and let \(\hat{x} = x - [x]\), so \(0 \leq \hat{x} < 1\). According to another definition of Weyl, \(\{x_i\}_{i=1}^{\infty}\) is equidistributed modulo 1 or uniformly distributed modulo 1 if \(\{\hat{x}_i\}_{i=1}^{\infty}\) is uniformly distributed in \([0, 1]\) as in Definition 3, with \(a = 0\) and \(b = 1\).

The most famous example of Definition 4 is related to a special case of Szegö’s distribution theorem [1, p. 64]. Suppose \(g\) is real-valued and continuous on \([-\pi, \pi]\). Let
\[
t_r = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-irx} g(x) \, dx, \quad r = 0, \pm 1, \pm 2, \ldots,
\]
and

\[ T_n = [t_{r-s}]_{r,s=1}^n, \quad n = 1, 2, 3, \ldots \]

These are Toeplitz matrices. Since \( g \) is real-valued, \( t_{-\ell} = \overline{t_\ell} \), so \( T_n \) is Hermitian and therefore has real eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \); in fact, they are all in \([a, b]\), where \( a \) and \( b \) are respectively the minimum and maximum values of \( g \) on \([-\pi, \pi]\).

Szegő showed that \( \{\{\lambda_{in}\}_{i=1}^n\}_{n=1}^\infty \) is distributed like the values of \( g \); i.e.,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n F(\lambda_{in}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(g(x)) \, dx \quad \text{for all } F \in C[a, b]
\]

if \( g \) is essentially bounded and Lebesgue integrable on \([-\pi, \pi]\). Moreover, there are many results on this question under still weaker assumptions on \( g \). We consider only the case where \( g \) is continuous.

Although we have stated four definitions to provide a historical perspective, Definitions 2–4 are special cases of Definition 1. In connection with Definitions 2 and 3, let

\[ w_{in} = a + \frac{i}{n}(b - a) \quad \text{for } 1 \leq i \leq n \quad \text{and } n = 2, 3, \ldots . \quad (5) \]

From first year calculus, we know that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n F(w_{in}) = \frac{1}{b - a} \int_a^b F(x) \, dx \quad \text{for all } F \in C[a, b].
\]

From this and (1), \( U \) is uniformly distributed if and only if \( U \) and \( \{\{w_{in}\}_{i=1}^n\}_{n=1}^\infty \) are equally distributed. Similarly, from (3), \( \{x_i\}_{i=1}^\infty \) is uniformly distributed if and only if \( \{\{x_i\}_{i=1}^n\}_{n=1}^\infty \) and \( \{\{w_{in}\}_{i=1}^n\}_{n=1}^\infty \) are equally distributed.

As for Definition 4, let

\[ y_{in} = c + \frac{i}{n}(d - c) \quad \text{for } 1 \leq i \leq n \quad \text{and } n = 2, 3, \ldots . \quad (6) \]

Since \( g \) is continuous on \([c, d]\), it follows that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n F(g(y_{in})) = \frac{1}{d - c} \int_c^d F(g(x)) \, dx \quad \text{for all } F \in C[a, b].
\]

From this and (4), \( U \) is distributed like the values of \( F \) if and only if \( U \) and \( \{\{g(y_{in})\}_{i=1}^n\}_{n=1}^\infty \) are equally distributed.

2 The Main Theorem and Corollaries

Henceforth we assume – without loss of generality – that

\[ a \leq u_{1n} \leq u_{2n} \leq \cdots \leq u_{nn} \leq b \quad \text{and } a \leq v_{1n} \leq v_{2n} \leq \cdots \leq v_{nn} \leq b. \]

Here is our main result. We will prove it in Section 4.
Theorem 1 The following assertions are equivalent:

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (F(u_i) - F(v_i)) = 0 \quad \text{for all } F \in C[a, b];
\]

(7)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |u_i - v_i| = 0;
\]

(8)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |F(u_i) - F(v_i)| = 0 \quad \text{for all } F \in C[a, b].
\]

(9)

This theorem and the discussion of Definitions 2–4 in Section 1 yield the following corollaries.

Corollary 1 \(U\) and \(V\) are equally distributed if and only if (8) is true.

Corollary 2 \(V\) is uniformly distributed if and only if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |v_i - w_i| = 0,
\]

(10)

with \(\{\{w_i\}_{i=1}^{n}\}_{n=1}^{\infty}\) as in (5). Moreover, each of the following statements is equivalent to (10):

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (F(v_i) - F(w_i)) = 0 \quad \text{for all } F \in C[a, b],
\]

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |F(v_i) - F(w_i)| = 0 \quad \text{for all } F \in C[a, b].
\]

Corollary 3 For each \(n \geq 2\), let \(\sigma_n\) be a permutation of \(\{1, 2, \ldots, n\}\) such that \(x_{\sigma_n(1)} \leq x_{\sigma_n(2)} \leq \cdots \leq x_{\sigma_n(n)}\).

Then \(\{x_i\}_{i=1}^{\infty}\) is uniformly distributed if and only if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |x_{\sigma_n(i)} - w_i| = 0.
\]

(11)

Moreover, each of the following statements is equivalent to (11):

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (F(x_i) - F(w_i)) = 0 \quad \text{for all } F \in C[a, b],
\]

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |F(x_{\sigma_n(i)}) - F(w_i)| = 0 \quad \text{for all } F \in C[a, b].
\]
Corollary 4. Let \( \{y_{in}\}_{i=1}^{n} \) be as in (6) and, for each \( n \geq 2 \), let \( \rho_{n} \) be a permutation of \( \{1, 2, \ldots, n\} \) such that
\[
g(y_{\rho_{n}(1), n}) \leq g(y_{\rho_{n}(2), n}) \leq \cdots \leq g(y_{\rho_{n}(n), n}).
\]
Then \( U \) is distributed like the values of \( g \) if and only
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |u_{in} - g(y_{\rho_{n}(i), n})| = 0.
\]
Moreover, each of the following statements is equivalent to (12):
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (F(u_{in}) - F(g(y_{in}))) = 0 \quad \text{for all } F \in C[a, b],
\]
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |F(u_{in}) - F(g(y_{\rho_{n}(i), n}))| = 0 \quad \text{for all } F \in C[a, b].
\]

We hope that the following suggestion will be taken as constructive rather than offensive: if (7), (8), and (9) are equivalent, then regarding (7) as the definition of equal distribution is putting the cart before the horse. Therefore – with some trepidation – we suggest that (8) should be the definition. Analogous suggestions apply to (10), (11), and (12) in connection with Definitions 2–4.

For examples that support this suggestion, suppose
\[
U_{\ell} = \left\{ \{u_{in}^{(\ell)}\}_{i=1}^{n} \right\}_{n=1}^{\infty} \subset [a, b], \quad V_{\ell} = \left\{ \{v_{in}^{(\ell)}\}_{i=1}^{n} \right\}_{n=1}^{\infty} \subset [a, b] \quad \text{for } 1 \leq \ell \leq k,
\]
where \( \lambda_{in} \geq 0 \), \( 1 \leq i \leq n \), and \( \lambda_{1n} + \lambda_{2n} + \cdots + \lambda_{nn} = 1 \).

Further, let \( U = \left\{ \{u_{in}\}_{i=1}^{n} \right\}_{n=1}^{\infty} \) and \( V = \left\{ \{v_{in}\}_{i=1}^{n} \right\}_{n=1}^{\infty}, \) where
\[
u_{in} = \lambda_{1n}u_{in}^{(1)} + \lambda_{2n}u_{in}^{(2)} + \cdots + \lambda_{nn}u_{in}^{(k)} \quad \text{and} \quad v_{in} = \lambda_{1n}v_{in}^{(1)} + \lambda_{2n}v_{in}^{(2)} + \cdots + \lambda_{nn}v_{in}^{(k)}.
\]

Then Corollary 1 obviously implies that \( U \) and \( V \) are equally distributed if \( U_{\ell} \) and \( V_{\ell} \) are equally distributed for \( i = 1, 2, \ldots, k \), and Corollary 2 obviously implies that \( V \) is uniformly distributed if \( V_{1}, V_{2}, \ldots, V_{k} \) are uniformly distributed. These conclusions are not obvious from Definitions 1 and 2.

3 Required Lemmas

We need the following lemmas, in which \( V_{a}^{b}(\phi) \) is the total variation of a function \( \phi \) on \([a, b]\).

Lemma 1 (Helly’s First Theorem) Let \( \{\phi_{m}\}_{m=1}^{\infty} \) be an infinite sequence of functions on \([a, b]\) and suppose that
\[
|\phi_{m}(x)| \leq K < \infty \quad \text{for } a \leq x \leq b \quad \text{and} \quad V_{a}^{b}(\phi_{m}) \leq K, \quad m \geq 1.
\]
Then there is a subsequence of \( \{\phi_{m}\}_{m=1}^{\infty} \) that converges at every point of \([a, b]\) to a function of bounded variation on \([a, b]\).
Lemma 2 (Helly’s Second Theorem) Let \( \{ \phi_m \}_{m=1}^{\infty} \) be an infinite sequence of functions on \([a, b]\) such that \( V_a^b(\phi_m) \leq K < \infty, m \geq 1 \), and
\[
\lim_{m \to \infty} \phi_m(x) = \phi(x) \text{ for } a \leq x \leq b.
\]
Then \( V_a^b(\phi) \leq K \) and
\[
\lim_{m \to \infty} \int_a^b F(x) \, d\phi_m(x) = \int_a^b F(x) \, d\phi(x) \text{ for all } F \in C[a, b].
\]

Lemma 3 Suppose \( \phi(a) = \phi(b) = 0 \), \( \phi \) is of bounded variation on \([a, b]\), and
\[
\int_a^b F(x) \, d\phi(x) = 0, \text{ for all } F \in C[a, b].
\]
Then \( \phi(x) = 0 \) at all points of continuity of \( \phi \). Thus, \( \phi(x) \neq 0 \) for at most countably many values of \( x \).

For proofs of Lemmas 1–3, see [2, p. 222], [2, p. 233], and [3, p. 111].

Lemma 4 Suppose \( x_1 \leq x_2 \leq \cdots \leq x_n \) and \( y_1 \leq y_2 \leq \cdots \leq y_n \). Let \( \{ \ell_1, \ell_2, \ldots, \ell_n \} \) be a permutation of \( \{1, 2, \ldots, n\} \) and define
\[
S(\ell_1, \ell_2, \ldots, \ell_n) = \sum_{i=1}^{n} |x_i - y_{\ell_i}|. \tag{13}
\]

Then
\[
S(\ell_1, \ell_2, \ldots, \ell_n) \geq S(1, 2, \ldots, n) = \sum_{i=1}^{n} |x_i - y_i|. \tag{14}
\]

Proof The proof is by induction. Let \( P_n \) be the stated proposition. \( P_1 \) is trivial. Suppose that \( n > 1 \) and \( P_{n-1} \) is true. If \( \ell_n = n \) then \( P_{n-1} \) implies \( P_n \).
If \( \ell_n = s < n \) then choose \( r \) so that \( \ell_r = n \), and define
\[
\ell'_i = \begin{cases} 
\ell_i & \text{if } i \neq r \text{ and } i \neq n, \\
s & \text{if } i = r, \\
n & \text{if } i = n.
\end{cases}
\]

Then
\[
S(\ell_1, \ell_2, \ldots, \ell_n) - S(\ell'_1, \ell'_2, \ldots, \ell'_n) = \sigma(x_n) - \sigma(x_r), \tag{15}
\]
where
\[
\sigma(x) = |x - y_s| - |x - y_n| = \begin{cases} 
y_s - y_n, & x < y_s, \\
2x - y_s - y_n, & y_s \leq x \leq y_n, \\
y_n - y_s, & x > y_n.
\end{cases}
\]
Since \( \sigma \) nondecreasing, (15) implies that
\[
S(\ell_1, \ell_2, \ldots, \ell_n) \geq S'(\ell_1', \ell_2', \ldots, \ell_n').
\]

Since \( \ell_n' = n \), \( P_{n-1} \) implies that
\[
S(\ell_1', \ell_2', \ldots, \ell_n') \geq S(1, 2, \ldots, n).
\]

Therefore (15) implies (14), which completes the induction.

4 Proof of Theorem 1

Obviously, (9) implies (7). To see that (8) implies (9), suppose that \( F \in C[a, b] \) and \( \epsilon > 0 \). By the Weierstrass approximation theorem, there is a polynomial \( P \) such that
\[
|F(x) - P(x)| < \epsilon/2 \text{ for } a \leq x \leq b.
\]

By the triangle inequality,
\[
|F(u_{in}) - F(v_{in})| \leq |F(u_{in}) - P(u_{in})| + |P(u_{in}) - P(v_{in})| + |P(v_{in}) - F(v_{in})|
\]
\[
< |P(u_{in}) - P(v_{in})| + \epsilon.
\]

Let \( M = \max_{a \leq x \leq b} |P'(x)| \). By the mean value theorem,
\[
|P(u_{in}) - P(v_{in})| \leq M|u_{in} - v_{in}|.
\]

This and (16) imply that
\[
\frac{1}{n} \sum_{i=1}^{n} |F(u_{in}) - F(v_{in})| < \epsilon + \frac{M}{n} \sum_{i=1}^{n} |u_{in} - v_{in}|.
\]

From this and (8),
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |F(u_{in}) - F(v_{in})| \leq \epsilon.
\]

Since \( \epsilon \) is arbitrary, this implies (9).

To complete the proof, we must show that (7) implies (8). The proof is by contradiction. If (8) is false, there is an \( \epsilon_0 > 0 \) and an increasing sequence \( \{\ell_k\}_{k=1}^{\infty} \) of positive integers such that
\[
\frac{1}{\ell_k} \sum_{i=1}^{\ell_k} |u_{i\ell_k} - v_{i\ell_k}| \geq \epsilon_0, \quad k \geq 1.
\]

However, we will show that if (7) holds, then any increasing infinite sequence \( \{\ell_k\}_{k=1}^{\infty} \) of positive integers has a a subsequence \( \{n_k\}_{k=1}^{\infty} \) such that
\[
\lim_{k \to \infty} \frac{1}{n_k} \sum_{i=1}^{n_k} |u_{in_k} - v_{in_k}| = 0,
\]
which contradicts (17).

If \( S \) is a set, let \( \text{card}(S) \) be the cardinality of \( S \). For \( a \leq x \leq b \), let

\[
\nu_n(x; U) = \text{card}\{i \mid u_{in} < x\} \quad \text{and} \quad \nu_n(x; V) = \text{card}\{i \mid v_{in} < x\}.
\]

Define

\[
\rho_n(x; U) = \begin{cases} 
\nu_n(x; U)/n, & a \leq x < b, \\
1, & x = b, 
\end{cases}
\]

and

\[
\rho_n(x; V) = \begin{cases} 
\nu_n(x; V)/n, & a \leq x < b, \\
1, & x = b, 
\end{cases}
\]

Then

\[
\frac{1}{n} \sum_{i=1}^{n} F(u_{in}) = \int_{a}^{b} F(x) \, d\rho_n(x; U) \quad \text{for all } F \in C[a, b]
\]

and

\[
\frac{1}{n} \sum_{i=1}^{n} F(v_{in}) = \int_{a}^{b} F(x) \, d\rho_n(x; V) \quad \text{for all } F \in C[a, b]
\]

[2, p. 231]. If

\[
\phi_n = \rho_n(\cdot; U) - \rho_n(\cdot; V),
\]

then (7), (21), and (22) imply that

\[
\lim_{n \to \infty} \int_{a}^{b} F(x) \, d\phi_n(x) = 0 \quad \text{for all } F \in C[a, b].
\]

Since

\[
|\phi_n(x)| \leq 1, \quad a \leq x \leq b, \quad \text{and} \quad V_a^{b}(\phi_n) \leq 2, \quad n \geq 1,
\]

Lemma 1 implies that every sequence \( \{\ell_k\}_{k=1}^{\infty} \) of positive integers has a subsequence \( \{n_k\}_{k=1}^{\infty} \) such that

\[
\lim_{k \to \infty} \phi_{n_k}(x) = \phi(x) \quad \text{for } a \leq x \leq b,
\]

where \( \phi \) is of bounded variation on \([a, b]\). From (23) and Lemma 2,

\[
\int_{a}^{b} F(x) \, d\phi(x) = 0 \quad \text{for all } F \in C[a, b].
\]

This and Lemma 3 imply that \( \phi(x) = 0 \) for all but countably many values of \( x \).

Since \( \lim_{k \to \infty} \phi_{n_k}(x) = 0 \) for all but countably many values of \( x \), (19) and (20) imply that

\[
\lim_{k \to \infty} \frac{\nu_{n_k}(x, U) - \nu_{n_k}(x, V)}{n_k} = 0
\]

for all but countably many values of \( x \). Therefore, given \( \epsilon > 0 \), we can choose \( x_0, x_1, \ldots, x_m \) so that

\[
a = x_0 < x_1 < \cdots < x_m = b,
\]
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\[ x_j - x_{j-1} < \epsilon \quad \text{for} \quad 1 \leq j \leq m, \quad (24) \]

and

\[ \lim_{k \to \infty} \frac{\nu_{n_k}(x_j; U) - \nu_{n_k}(x_j; V)}{n_k} = 0. \quad (25) \]

Let

\[ I_j = [x_{j-1}, x_j) \quad \text{for} \quad 1 \leq j \leq m-1, \quad \text{and} \quad I_m = [x_{m-1}, x_m], \]

and denote

\[ U_{jk} = \text{card} \{ i \mid u_{i_{nk}} \in I_j \}, \quad V_{jk} = \text{card} \{ i \mid v_{i_{nk}} \in I_j \}. \]

Since

\[ U_{jk} = \begin{cases} \nu_{n_k}(x_1; U), & j = 1, \\ \nu_{n_k}(x_j; U) - \nu_{n_k}(x_{j-1}; U), & 2 \leq j \leq m-1, \\ n_k - \nu_{n_k}(x_{m-1}; U), & j = m, \end{cases} \]

and

\[ V_{jk} = \begin{cases} \nu_{n_k}(x_1; V), & j = 1, \\ \nu_{n_k}(x_j; V) - \nu_{n_k}(x_{j-1}; V), & 2 \leq j \leq m-1, \\ n_k - \nu_{n_k}(x_{m-1}; V), & j = m, \end{cases} \]

(25) implies that

\[ \lim_{k \to \infty} \frac{U_{jk} - V_{jk}}{n_k} = 0 \quad \text{for} \quad 1 \leq j \leq m. \quad (26) \]

Since

\[ \min(U_{jk}, V_{jk}) = \frac{U_{jk} + V_{jk} - |U_{jk} - V_{jk}|}{2}, \]

and

\[ \sum_{j=1}^{m} U_{jk} = \sum_{j=1}^{m} V_{jk} = n_k, \]

it follows that

\[ \sum_{j=1}^{m} \min(U_{jk}, V_{jk}) = n_k - r_k, \quad (27) \]

where

\[ r_k = \frac{1}{2} \sum_{j=1}^{m} |U_{jk} - V_{jk}|. \]

From (26),

\[ \lim_{k \to \infty} \frac{r_k}{n_k} = 0. \quad (28) \]

From (24) and (27), there is a permutation \( \tau_k \) of \{1, \ldots, n_k\} such that

\[ |u_{i_{nk}} - v_{\tau_k(i_{nk})}| < \epsilon \]
for at least $n_k - r_k$ values of $i$; hence

$$\sum_{i=1}^{n_k} |u_{in_k} - v_{rk(i),n_k}| < n_k \epsilon + r_k(b - a).$$

Now Lemma 4 implies that

$$\sum_{i=1}^{n_k} |u_{in_k} - v_{in_k}| < n_k \epsilon + r_k|b - a|.$$

Hence, from (28),

$$\limsup_{k \to \infty} \frac{1}{n_k} \sum_{i=1}^{n_k} |u_{in_k} - v_{in_k}| \leq \epsilon.$$

Since $\epsilon$ is arbitrary, this implies (18), which completes the proof.

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abled me to complete the proof of Theorem 1 in my earlier papers [4, 5].

Lemma 4 and its proof are similar to a well known result [6, p. 108] applicable
in the case where (13) is replaced by

$$S(\ell_1, \ell_2, \ldots, \ell_n) = \sum_{i=1}^{n} (x_i - y_{\ell_i})^2.$$

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References


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asymptotic equal distribution of two families of finite sets*, Cubo A Mathe-

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