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Orbifold Quantum Cohomology¹

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This is a research announcement on a theory of Gromov-Witten invariants and quantum cohomology of symplectic or projective orbifolds. Our project started in the summer of 98 where our original motivation was to study the quantum cohomology under singular flops in complex dimension three. In this setting, we allow our three-fold to have terminal singularities which can be deformed into a symplectic orbifold. We spent the second half of 98 and most of spring of 99 to develop the foundation of Gromov-Witten invariants over orbifolds, including the key conceptual ingredient — the notion of good map. In the April of 99, we were lucky to meet R. Dijkgraaf who explained to us the orbifold string theory and the role of twisted sectors. The twisted sector provides the precise topological framework for our orbifold quantum cohomology. Our theory of orbifold quantum cohomology was virtually completed in the summer of 99. Here, we give an overlook of the foundation of orbifold quantum cohomology while we leave its applications in other fields such as birational geometry for future research. The first part of our work has already appeared in [CR1]. The second part [CR2] will appear shortly. We would like to thank R. Dijkgraaf for bringing orbifold string theory to our attention at the critical moment of our project. We are also benefited from many discussions with E. Zaslow. Finally, the second author would like to thank E. Witten for many stimulating discussions about orbifold string theory.

1 Orbifold String Theory

In 1985, Dixon, Harvey, Vafa and Witten considered string compactification on Calabi-Yau orbifolds (arising as global quotients X/G by a finite group G) for the purpose of symmetry breaking [DHVW]. Later on, orbifold string theory became an important part of string theory. For example, orbifolds provide some of the simplest nontrivial models in string theory. Until very recently, only physical argument for mirror symmetry had been given for orbifold models [GP]. In fact, orbifolds are such a popular topic in string theory that a search on hep-th yields more than 200 papers whose title contains orbifold. The reason that orbifold string theory is interesting mathematically is that it contains information which we do not have in the smooth case. Roughly speaking, to have a consistent string theory, string Hilbert space has to contain factors called twisted sectors. Twisted sectors can be viewed as the contribution from singularities. All other quantities such as correlation functions have to contain the contribution from the twisted sectors. So far, the twisted sectors are best understood in the context of conformal field theory. It is our intention to initiate a program to investigate the new geometry and topology of orbifolds caused by the inclusion of twisted sectors. We should mention that the orbifold string theory construction has only been carried out for global quotients. However, it is well-known that most of Calabi-Yau orbifolds are not global quotients. It seems to also be important to be able to construct orbifold string theory over general Calabi-Yau

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orbifolds. Our orbifold quantum cohomology theory works over arbitrary symplectic or projective orbifolds. We hope that our construction will shed some light on the construction of orbifold string theory for general Calabi-Yau orbifolds.

An orbifold, by definition, is a singular space. One can try to desingularize a Calabi-Yau orbifold by the means of resolution or deformation. To preserve the Calabi-Yau condition, we have to restrict ourselves to the so-called crepant resolutions. It is natural to ask for the relation between orbifold string theory and string theory of its desingularization. In fact, this link provides some of the most interesting mathematics from orbifold string theory. In physics, orbifold string theory and ordinary string theory of its crepant resolution appear to be two members in a family of theories. This strongly suggests that there must be a relation between them. The strongest predication is that they are the same. Indeed, this is what physicists hope for. For quantum cohomology, this translates into the following orbifold string theory prediction:

(1.1) *The quantum cohomology of a crepant resolution should be “isomorphic” to the “quantum cohomology” of the orbifold.*

Here, the “quantum cohomology” of the orbifold should be understood as orbifold quantum cohomology. The goal of our project is to establish a mathematical theory for orbifold quantum cohomology.

Actually, the above prediction is false in general. A counterexample is a $K3$ -surface with ADE-singularities. But this is not the end of the story. Recall that the most general form of mirror symmetry fails for rigid Calabi-Yau 3-folds. But this did not stop research from unearthing the layer and layer of mathematical treasures from mirror symmetry. In fact, it is entirely possible that weaker forms or the current form (1.1) for a more restrictive class of orbifolds are still true. The authors believe that this link to crepant resolutions will greatly enrich this subject. Therefore, it is useful to keep this strongest form of prediction in mind for the direction of future research.

The weakest form of the orbifold string theory prediction is to replace quantum cohomology by orbifold Euler number. Here, it has a good chance to hold. The orbifold Euler number is defined as the sum of Euler numbers over all sectors. It has a natural interpretation as the Euler characteristic of orbifold K-theory [AR]. A weak orbifold string theory prediction is that Euler number of a crepant resolution is the same as the orbifold Euler number of itself. On the mathematical side, a similar phenomenon was independently discovered earlier by John McKay, which is now known as *McKay correspondence* [McK, Re]. A version of McKay correspondence is stated as follows:

Let $G \subset SL(n, \mathbf{C})$ be a finite group, and $\pi : Y \rightarrow X = \mathbf{C}^n/G$ be a crepant resolution, then there exist “natural” bijections between conjugacy classes of G and a basis of $H_(Y; \mathbf{Z})$.*

Based on these ideas, Batyrev-Dais [BD] proposed the so-called *strong McKay correspondence* and defined *string-theoretic Hodge numbers*.

The classical part of our orbifold quantum cohomology is a new cohomology of orbifolds which we call orbifold cohomology (see section 2). In the case of Gorenstein orbifolds, Batyrev-Dais’s string-theoretic Hodge number is just the Hodge number of our orbifold cohomology. The next level of the orbifold string theory prediction is to identify the orbifold cohomology group with the ordinary cohomology group of a crepant resolution. This is best described through the orbifold K-theory [AR]. It is unlikely that one can identify the cohomology ring structures because of quantum corrections. The third level would be the last level concerning quantum cohomology, which is the most challenging one. At this moment, it is not clear how to formulate the prediction without the risk of finding a simple counterexample.

In the following sections, we shall outline the construction of an *orbifold Gromov-Witten theory*, which obeys almost all of the axioms of ordinary Gromov-Witten theory, as it should according to

physics.

2 Orbifold Cohomology

The ordinary quantum cohomology ring appears as a (quantum) deformation of the ordinary cohomology ring with the cup product. In the orbifold Gromov-Witten theory, the role of ordinary cohomologies is played by the so called *orbifold cohomologies*, which we shall describe in this section. One of our main results is the construction of an *orbifold cup product* on the total orbifold cohomology group, which makes it into a ring with unit. We will call the resulting ring *the orbifold cohomology ring*. The orbifold quantum cohomology ring is just the corresponding quantum deformation of the orbifold cohomology ring. Details are in [CR1].

Let X be a closed, almost complex orbifold of dimension n , with almost complex structure J . For any $p \in X$, let (V_p, G_p, π_p) be a uniformizing system, which can be taken so that V_p is a small ball in \mathbf{C}^n centered at the origin and G_p acts on V_p as a finite subgroup of $U(n)$. We consider the set

$$(2.1) \quad \tilde{X} := \{(p, (g)_{G_p}) | p \in X, g \in G_p\}.$$

Here $(g)_{G_p}$ represents the conjugacy class of g in G_p . There is a locally constant function $\iota : \tilde{X} \rightarrow \mathbf{Q}$ defined as follows: write g as a diagonal matrix

$$\text{diag}(e^{2\pi i m_{1,g}/m_g}, \dots, e^{2\pi i m_{n,g}/m_g}),$$

where m_g is the order of g in G_p , and $0 \leq m_{i,g} < m_g$ for $i = 1, \dots, n$. We define

$$(2.2) \quad \iota(p, (g)_{G_p}) = \sum_{i=1}^n \frac{m_{i,g}}{m_g}.$$

Let $I : \tilde{X} \rightarrow \tilde{X}$ be the involution defined by $I(p, (g)_{G_p}) = (p, (g^{-1})_{G_p})$.

Lemma 2.1: *There is an equivalence relation among the $(g)_{G_p}$, and if we let $T = \{(g)\}$ be the set of equivalence classes and define $X_{(g)} = \{(p, (g)_{G_p}) \in \tilde{X} | (g)_{G_p} \in (g)\}$, then each $X_{(g)}$ is naturally a closed, connected, almost complex orbifold, and \tilde{X} is decomposed as a disjoint union $\sqcup_{(g) \in T} X_{(g)}$. Furthermore, if we denote the value of the locally constant function $\iota : \tilde{X} \rightarrow \mathbf{Q}$ by $\iota_{(g)}$, and let (g^{-1}) denote the image of (g) under the involution I , and (1) denote the equivalence class of the trivial element $(1)_{G_p}$, the following conditions are satisfied:*

1. $\iota : \tilde{X} \rightarrow \mathbf{Q}$ is integer-valued iff each G_p is contained in $SL(n, \mathbf{C})$.

2.

$$(2.3) \quad \iota_{(g)} + \iota_{(g^{-1})} = \dim_{\mathbf{C}} X - \dim_{\mathbf{C}} X_{(g)}.$$

3. $\iota_{(g)} \geq 0$ for all $(g) \in T$, and $\iota_{(g)} = 0$ iff $(g) = (1)$.

Note that for Calabi-Yau orbifolds, each $\iota_{(g)}$ is integer-valued. When $X = Y/G$ is a global quotient, \tilde{X} can be identified with $\sqcup_{(g), g \in G} Y^g / C(g)$.

The orbifold cohomologies are just direct sums of ordinary cohomologies of $X_{(g)}$ with degrees shifted by $2\iota_{(g)}$. More precisely,

Definition 2.2: Let X be a closed almost complex orbifold with $\dim_{\mathbb{C}} X = n$. For any rational number $d \in [0, 2n]$, the orbifold cohomology group of degree d is defined to be the direct sum

$$(2.4) \quad H_{orb}^d(X; \mathbb{Q}) = \oplus_{(g) \in T} H^{d-2\iota(g)}(X_{(g)}; \mathbb{Q}).$$

We will call $\iota_{(g)}$ *degree shifting numbers*, which have been referred as *fermion shift numbers* in physics [Z]. The orbifold $X_{(g)}$ or its cohomology will be called a *twisted sector* if $(g) \neq (1)$, and called the *nontwisted sector* if $(g) = (1)$. The construction of \tilde{X} (cf. (2.1)) first appeared in [Ka].

The following theorem is proved in [CR1], whose construction is based on genus-zero, degree zero orbifold Gromov-Witten invariants.

Theorem 2.3: Let (X, J) be a closed almost complex orbifold of dimension n . Then

1. There is a non-degenerate pairing $\langle, \rangle_{orb} : H_{orb}^d(X; \mathbb{Q}) \times H_{orb}^{2n-d}(X; \mathbb{Q}) \rightarrow \mathbb{Q}$ extending the ordinary Poincaré pairing on the nontwisted sectors $H^*(X; \mathbb{Q})$.
2. There is a cup product $\cup_{orb} : H_{orb}^p(X; \mathbb{Q}) \times H_{orb}^q(X; \mathbb{Q}) \rightarrow H_{orb}^{p+q}(X; \mathbb{Q})$ for any $0 \leq p, q \leq 2n$ such that $p + q \leq 2n$, which has the following properties:
 - The total orbifold cohomology group $H_{orb}^*(X; \mathbb{Q}) = \oplus_{0 \leq d \leq 2n} H_{orb}^d(X; \mathbb{Q})$ is a ring with unit $e_X^0 \in H_{orb}^0(X; \mathbb{Q})$ under \cup_{orb} , where e_X^0 is the Poincaré dual to the fundamental class $[X]$. In particular, \cup_{orb} is associative.
 - Restricted to each $H_{orb}^d(X; \mathbb{Q}) \times H_{orb}^{2n-d}(X; \mathbb{Q}) \rightarrow H_{orb}^{2n}(X; \mathbb{Q})$,

$$(2.5) \quad \int_X \alpha \cup_{orb} \beta = \langle \alpha, \beta \rangle_{orb}.$$

- The cup product \cup_{orb} is invariant under deformations of J .
- When X is of integral degree shifting numbers, the total orbifold cohomology group $H_{orb}^*(X; \mathbb{Q})$ is integrally graded, and we have supercommutativity

$$(2.6) \quad \alpha_1 \cup_{orb} \alpha_2 = (-1)^{\deg \alpha_1 \cdot \deg \alpha_2} \alpha_2 \cup_{orb} \alpha_1.$$

- Restricted to the nontwisted sectors, i.e., the ordinary cohomologies $H^*(X; \mathbb{Q})$, the cup product \cup_{orb} equals the ordinary cup product on X .

We remark that there is an analogous construction using Dolbeault cohomology groups; for details see [CR1].

3 Good Map and Pull-Back Bundle

Now we come to one of the main issues in the construction of orbifold Gromov-Witten invariants. Recall that if $f : X \rightarrow X'$ is a C^∞ map between manifolds and E is a smooth vector bundle over X' , then there is a smooth pull-back vector bundle f^*E over X and a bundle morphism $\bar{f} : f^*E \rightarrow E$ which covers the map f . However, if instead we have a C^∞ map \tilde{f} between orbifolds X and X' , and an orbibundle E over the orbifold X' , the question whether there is a pull-back orbibundle E^* over X and an orbibundle morphism $\tilde{f} : E^* \rightarrow E$ covering the map \tilde{f} is a quite complicated issue: E^* might not exist, or even if it exists it might not be unique. Traditionally, a neighborhood of a smooth map into a manifold is described by smooth sections of the pull-back of the tangent bundle of the manifold. Hence understanding this question became the very first step in describing

the moduli spaces of pseudo-holomorphic maps, or more precisely, the very first step in order to understand what would be the corresponding notion of “stable map” in the orbifold case.

By a C^∞ map between orbifolds X and X' we mean an equivalence class of collections of local smooth liftings between local uniformizing systems of a continuous map from X to X' . This notion is equivalent to the notion of V -maps in [S], where the notion of orbifold was first introduced under the name V -manifold. A brief review of orbifolds is given in [CR1], and a self-contained, elementary discussion of various aspects of differential geometry and global analysis on orbifolds is contained in [CR2].

Now we describe our key concept: the notion of *good map*. Let $\tilde{f} : X \rightarrow X'$ be a C^∞ map between orbifolds X and X' whose underlying continuous map is denoted by f . Let $\mathcal{U} = \{U_\alpha; \alpha \in \Lambda\}$ be an open cover of X and $\mathcal{U}' = \{U'_\alpha; \alpha \in \Lambda\}$ be an open cover of the image $f(X)$ in X' , which satisfy the following conditions:

1. Each U_α (resp. U'_α) is uniformized by $(V_\alpha, G_\alpha, \pi_\alpha)$ (resp. by $(V'_\alpha, G'_\alpha, \pi'_\alpha)$).
2. If $U_\alpha \subset U_\beta$ (resp. $U'_\alpha \subset U'_\beta$), then there is a collection of smooth open embeddings $(V_\alpha, G_\alpha, \pi_\alpha) \rightarrow (V_\beta, G_\beta, \pi_\beta)$ (resp. $(V'_\alpha, G'_\alpha, \pi'_\alpha) \rightarrow (V'_\beta, G'_\beta, \pi'_\beta)$), which are called *injections*.
3. For any point $p \in U_\alpha \cap U_\beta$ (resp. $p' \in U'_\alpha \cap U'_\beta$), there is a U_γ (resp. U'_γ) such that $p \in U_\gamma \subset U_\alpha \cap U_\beta$ (resp. $p' \in U'_\gamma \subset U'_\alpha \cap U'_\beta$).
4. Any inclusion $U_\alpha \subset U_\beta$ implies $U'_\alpha \subset U'_\beta$.
5. For each $\alpha \in \Lambda$, $f(U_\alpha) \subset U'_\alpha$. Moreover, there is a collection of local smooth liftings of f , $\{\tilde{f}_\alpha : V_\alpha \rightarrow V'_\alpha; \alpha \in \Lambda\}$ which defines the given C^∞ map \tilde{f} , such that any injection $i_{\beta\alpha} : (V_\alpha, G_\alpha, \pi_\alpha) \rightarrow (V_\beta, G_\beta, \pi_\beta)$ is assigned with an injection $\lambda(i_{\beta\alpha}) : (V'_\alpha, G'_\alpha, \pi'_\alpha) \rightarrow (V'_\beta, G'_\beta, \pi'_\beta)$ satisfying the following compatibility conditions:

$$(3.1) \quad \tilde{f}_\beta \circ i_{\beta\alpha} = \lambda(i_{\beta\alpha}) \circ \tilde{f}_\alpha \quad \forall \alpha, \beta \in \Lambda$$

and

$$(3.2) \quad \lambda(i_{\gamma\beta} \circ i_{\beta\alpha}) = \lambda(i_{\gamma\beta}) \circ \lambda(i_{\beta\alpha}) \quad \forall \alpha, \beta, \gamma \in \Lambda.$$

Definition 3.1: We call such a $(\mathcal{U}, \mathcal{U}', \{\tilde{f}_\alpha\}, \lambda)$ a compatible system of the C^∞ map \tilde{f} . A C^∞ map is said to be good if it admits a compatible system.

Lemma 3.2: Let $pr : E \rightarrow X'$ be an orbibundle over X' . For any C^∞ good map $\tilde{f} : X \rightarrow X'$ with a compatible system $\xi = (\mathcal{U}, \mathcal{U}', \{\tilde{f}_\alpha\}, \lambda)$, there is a canonically defined pull-back orbibundle $pr : E_{\tilde{f}, \xi}^* \rightarrow X$ with an orbibundle morphism $\tilde{f}_{\tilde{f}, \xi} : E_{\tilde{f}, \xi}^* \rightarrow E$ which covers \tilde{f} .

Definition 3.3: Two compatible systems ξ_1, ξ_2 of a C^∞ good map $\tilde{f} : X \rightarrow X'$ are said to be isomorphic if for any orbibundle E over X' there is an orbibundle isomorphism $\phi : E_{\tilde{f}, \xi_1}^* \rightarrow E_{\tilde{f}, \xi_2}^*$ such that

$$(3.3) \quad \tilde{f}_{\tilde{f}, \xi_1} = \tilde{f}_{\tilde{f}, \xi_2} \circ \phi.$$

Example 3.4a: Not every C^∞ map is good, as shown in the following example: consider an effective linear representation of a finite group (\mathbf{R}^n, G) . Let H^g be the linear subspace of fixed points

of an element $1 \neq g \in G$. Then the centralizer $C(g)$ of g in G acts on H^g , and $(H^g, C(g)/K_g)$ is an effective linear representation, where $K_g \subset C(g)$ is the kernel of the action of $C(g)$ on H^g . Suppose $H^g \neq \{0\}$ and there is no homomorphism $\lambda : C(g)/K_g \rightarrow C(g)$ such that $\pi \circ \lambda$ is the identity homomorphism, where $\pi : C(g) \rightarrow C(g)/K_g$ is the projection, then the continuous map $H^g/C(g) \rightarrow \mathbf{R}^n/G$ induced by inclusion $H^g \hookrightarrow \mathbf{R}^n$ is a C^∞ map which is not a good one.

Example 3.4b: There could be non-isomorphic compatible systems of the same C^∞ map, as shown in the following example: Let $X = \mathbf{C} \times \mathbf{C}/G$ where $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ acting on $\mathbf{C} \times \mathbf{C}$ in the standard way. For the C^∞ map $\mathbf{C}/\mathbf{Z}_2 \rightarrow X$ defined by the inclusion $\mathbf{C} \times \{0\} \hookrightarrow \mathbf{C} \times \mathbf{C}$, there are two non-isomorphic compatible systems $(\tilde{f}_i, \lambda_i) : (\mathbf{C}, \mathbf{Z}_2) \rightarrow (\mathbf{C} \times \mathbf{C}, G)$, $i = 1, 2$, where $\lambda_1(1) = (1, 0)$ and $\lambda_2(1) = (1, 1)$.

In general it is not only difficult to determine whether a given C^∞ map is good or not, but also difficult to classify compatible systems of an arbitrary good map by equivalence up to isomorphism. Nevertheless, it is clear that a good map together with an isomorphism class of compatible systems is the object we ought to deal with in the orbifold quantum cohomology theory.

We end this section with a discussion on the case when the domain of a good map is a 2-dimensional orbifold.

Definition-Construction 3.5:

1. Given a marked Riemann surface (Σ, \mathbf{z}) where $\mathbf{z} = (z_1, \dots, z_k)$ is the set of marked points, we can give a unique orbifold structure to Σ by assigning to each marked point z_i an integer $m_i \geq 1$ (note that $m_i = 1$ is allowed for convenience). We will call $(\Sigma, \mathbf{z}, \mathbf{m})$ an orbifold marked Riemann surface, where $\mathbf{m} = (m_1, \dots, m_k)$ is the set of assigned integers, called multiplicities.
2. An orbifold nodal Riemann surface is a marked nodal Riemann surface with the following data: (i) each irreducible component is an orbifold marked Riemann surface (here a nodal point is considered marked on an irreducible component); (ii) two identified nodal points are assigned with the same multiplicity.

Convention-Definition 3.6:

1. Note that, in the definition of compatible systems, the compatibility conditions (3.1), (3.2) give rise to a collection of homomorphisms $\lambda_\alpha, \alpha \in \Lambda$, between local groups G_α and G'_α , such that each local smooth lifting $\tilde{f}_\alpha : V_\alpha \rightarrow V'_\alpha$ is λ_α -equivariant. For a good C^∞ map whose domain is an orbifold marked Riemann surface, we require that each λ_α be a monomorphism for any of its compatible systems.
2. A good C^∞ map with a compatible system from an orbifold nodal Riemann surface into an orbifold X is a collection of good C^∞ maps with compatible systems defined on its irreducible components which satisfies the following compatibility condition: for each pair of identified nodal points z_ν and z_ω , the homomorphisms λ_{z_ν} and λ_{z_ω} between local groups, which are determined by the corresponding compatible systems, satisfy the equation

$$(3.4) \quad \lambda_{z_\nu}(x) \cdot \lambda_{z_\omega}(x) = 1_{G_p}$$

in G_p , where $p \in X$ is the image of the identified nodal points z_ν and z_ω under the good C^∞ map, and x is a generator of the local cyclic group at z_ν and z_ω (z_ν and z_ω have the same multiplicity, hence the same local cyclic group).

Finally we observe that each good C^∞ map with a compatible system from an orbifold nodal Riemann surface with k marked points into an orbifold X determines a point in the space \tilde{X}^k , where $\tilde{X} = \sqcup_{(g) \in T} X_{(g)}$ (cf. (2.1)), as follows: let the underlying continuous map be f and for each marked point z_i , $i = 1, \dots, k$, let x_i be the positive generator of the cyclic local group at z_i , and λ_{z_i} be the homomorphism determined by the given compatible system, then the determined point in \tilde{X}^k is

$$(3.5) \quad ((f(z_1), (\lambda_{z_1}(x_1))_{G_{f(z_1)}}), \dots, (f(z_k), (\lambda_{z_k}(x_k))_{G_{f(z_k)}})).$$

Let $\mathbf{x} = (X_{(g_1)}, \dots, X_{(g_k)})$ be a connected component in \tilde{X}^k . We say that a good map with a compatible system is of *type* \mathbf{x} if the point (3.5) it determines in \tilde{X}^k lies in the component \mathbf{x} .

4 Orbifold Stable Maps

We start with the definition of *pseudo-holomorphic map* from a Riemann surface into an almost complex orbifold.

Definition 4.1: A pseudo-holomorphic map from a Riemann surface (Σ, j) into an almost complex orbifold (X, J) is a continuous map $f : \Sigma \rightarrow X$ which satisfies the following conditions:

1. For any point $z \in \Sigma$, there is a disc neighborhood D_z of z with a branched covering map $br_z : \tilde{D}_z \rightarrow D_z$ given by $w \rightarrow w^{m_z}$ (here $m_z = 1$ is allowed).
2. Let $p = f(z)$. There is a local uniformizing system (V_p, G_p, π_p) of X at p and a local smooth lifting $\tilde{f}_z : \tilde{D}_z \rightarrow V_p$ of f in the sense that $f \circ br_z = \pi_p \circ \tilde{f}_z$.
3. \tilde{f}_z is pseudo-holomorphic, i.e., $d\tilde{f}_z \circ j = J \circ d\tilde{f}_z$.

Remarks 4.2:

1. When (X, J) is a complex orbifold, i.e., J is integrable, a pseudo-holomorphic map $f : (\Sigma, j) \rightarrow (X, J)$ is just a holomorphic map from (Σ, j) into the analytic space (X, J) .
2. For each pseudo-holomorphic map $f : (\Sigma, j) \rightarrow (X, J)$, there is a subset of finitely many points $\{z_1, z_2, \dots, z_k\} \subset \Sigma$ such that for any $z \in \Sigma \setminus \{z_1, z_2, \dots, z_k\}$ the multiplicity m_z in Definition 4.1-1 equals one (cf. [HW]). We will consider pseudo-holomorphic maps from marked Riemann surfaces into (X, J) . As a convention we will always mark these points $\{z_1, z_2, \dots, z_k\}$ where the multiplicity is greater than one.

Given a pseudo-holomorphic map f from a marked Riemann surface (Σ, \mathbf{z}) into (X, J) , where $\mathbf{z} = (z_1, \dots, z_k)$ is a set of finitely many distinct marked points on Σ , there is an orbifold structure on Σ with singular set contained in \mathbf{z} such that f can be lifted to a good C^∞ map \tilde{f} . A crucial technical result is summarized in the following

Lemma 4.3: For any pseudo-holomorphic map f from a Riemann surface Σ of genus g with k marked points $\mathbf{z} = (z_1, z_2, \dots, z_k)$ into (X, J) , there are finitely many orbifold structures on Σ whose singular set is contained in \mathbf{z} , and for each of these orbifold structures there are finitely many pairs (\tilde{f}, ξ) , where \tilde{f} is a good map whose underlying map is f , and ξ is an isomorphism class of compatible systems of \tilde{f} . The total number is bounded from above by a constant $C(X, g, k)$ depending only on X, g, k .

Definition 4.4: An orbifold stable map from a marked nodal Riemann surface into an almost complex orbifold (X, J) consists of the following data:

1. A continuous map from the marked nodal Riemann surface into (X, J) whose restriction to each irreducible component is pseudo-holomorphic.
2. An orbifold structure on the marked nodal Riemann surface so that it becomes an orbifold nodal Riemann surface, and a good map with a compatible system from the orbifold nodal Riemann surface into (X, J) with the given underlying continuous map.
3. Stability condition: on each S^2 or T^2 component which is mapped into a point in X there are at least three or one special points (marked or nodal).

There is an obvious equivalence relation amongst the set of orbifold stable maps. We denote by $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ the set of all equivalence classes of orbifold stable maps of genus g , k marked points, type \mathbf{x} and homology class A into (X, J) .

Remark 4.5: In the algebraic setting of Deligne-Mumford stack, a related notion which is called twisted stable map was discussed in [AV]. Their twisted stable map was described in the language of category and functor. Our good map was formulated in elementary differential-geometric language. From the first sight, two notions look quite different. However, D. Abramovich kindly informed us that they are actually equivalent [A].

Remark 4.6: If $f : \Sigma \rightarrow X$ is a pseudo-holomorphic map whose image intersects the singular locus of X at only finitely many points, then there is a unique choice of orbifold structure on Σ together with a unique (\tilde{f}, ξ) , where \tilde{f} is a good map with an isomorphism class of compatible systems ξ whose underlying continuous map is f . If the image of f lies completely inside the singular locus, there could be different choices, and they are regarded as different points in the moduli space.

Definition 4.7:

1. An orbifold X is symplectic if there is a closed 2-form ω on X whose local liftings are non-degenerate.
2. A projective orbifold is a complex orbifold which is a projective variety as an analytic space.

The usual Gromov Compactness Theorem for pseudo-holomorphic maps combined with Lemma 4.3 gives the following

Proposition 4.8: Suppose that X is a symplectic or projective orbifold. The moduli space of orbifold stable maps $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ is a compact metrizable space under a natural topology, whose “virtual dimension” is $2d$, where

$$d = c_1(TX) \cdot A + (\dim_{\mathbf{C}} X - 3)(1 - g) + k - \iota(\mathbf{x}).$$

Here $\iota(\mathbf{x}) := \sum_{i=1}^k \iota_{(g_i)}$ for $\mathbf{x} = (X_{(g_1)}, \dots, X_{(g_k)})$.

5 Orbifold Quantum Cohomology

For any component $\mathbf{x} = (X_{(g_1)}, \dots, X_{(g_k)})$, there are k evaluation maps (cf. (3.5))

$$(5.1) \quad e_i : \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x}) \rightarrow X_{(g_i)}, \quad i = 1, \dots, k.$$

For any set of cohomology classes $\alpha_i \in H^{*-2\iota(g_i)}(X_{(g_i)}; \mathbf{Q}) \subset H_{orb}^*(X; \mathbf{Q})$, $i = 1, \dots, k$, the orbifold Gromov-Witten invariant is defined as the virtual integral

$$(5.2) \quad \Psi_{(g,k,A,\mathbf{x})}^{X,J}(\alpha_1^{l_1}, \dots, \alpha_k^{l_k}) = \int_{\overline{\mathcal{M}}_{g,k}(X,J,A,\mathbf{x})}^{vir} \prod_{i=1}^k c_1(L_i)^{l_i} e_i^* \alpha_i,$$

where L_i is the line bundle generated by cotangent space of the i -th marked point.

When $g = 0$ and $A = 0$, the moduli space $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ admits a very nice and elementary description, based on which we gave an elementary construction of genus zero, degree zero orbifold Gromov-Witten invariants in [CR1]. Even in this case, virtual integration is needed where there is an obstruction bundle. The orbifold cup product (cf. Theorem 2.3) is defined through these orbifold Gromov-Witten invariants. In the general case, we need to use the full scope of the virtual integration machinery developed by [FO], [LT], [Ru] and [Sie].

Singularities of an orbifold impose additional difficulties in carrying out virtual integration in the orbifold case. Due to the presence of singularities, even on a closed orbifold, the function of injective radius of the exponential map does not have a positive lower bound. As a consequence, it is not known that a neighborhood of a (good) C^∞ map into an orbifold can be completely described by C^∞ sections of the pull-back tangent bundle via the exponential map. Our approach is a combination of techniques developed in the smooth case. We first construct a local Kuranishi neighborhood for each stable map in $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$, then find finitely many stable maps whose local Kuranishi neighborhoods (although they may have different dimensions) can be patched together to form a “global virtual neighborhood” of $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ (cf. [FO]). This is similar to the constructions of [FO], [LT]. We carry out the virtual integration over this “global virtual neighborhood” by constructing a system of compatible “Thom forms” (cf. [Ru]).

When X has a symplectic torus action, the “global virtual neighborhood” can be constructed so that it respects this torus action. The localization theory can be extended to the case of virtual integration. We leave this to another paper.

Main results of this work are summarized in the following

Theorem 5.1: *Let X be a closed symplectic or projective orbifold. The orbifold Gromov-Witten invariants defined in (5.2) satisfy the quantum cohomology axioms of Witten-Ruan for ordinary Gromov-Witten invariants (cf. [Ru1]) except that in the Divisor Axiom, the divisor class is required to be in the nontwisted sector (i.e. in $H^2(X; \mathbf{Q})$). In the formulation of axioms, the ordinary cup product is replaced by the orbifold cup product \cup_{orb} (cf. Theorem 2.3).*

As a consequence, we have

Theorem 5.2: *Let X be a closed symplectic or projective orbifold. With suitable coefficient ring \mathcal{C} , the small quantum product and the big quantum product are well-defined on $H_{orb}^*(X; \mathbf{Q}) \otimes \mathcal{C}$, and have properties similar to those of the ordinary quantum cohomology.*

6 Closing Remarks

What we have accomplished so far is just a tip of iceberg! For example, it is still a difficult problem to compute orbifold quantum cohomology. This requires developing new machinery such as localization and surgery techniques. There are two topics whose natural home should be orbifold. They are birational geometry and mirror symmetry. For birational geometry, recent results in algebraic geometry show that birational transformation can be decomposed as a sequence of wall-crossings in GIT-quotients [AKMW], [HK], [W1], [W2]. The latter is naturally in the orbifold

category. For mirror symmetry, the Calabi-Yau 3-folds in most of the known examples are crepant resolutions of Calabi-Yau orbifolds. Therefore, it is more natural to consider mirror symmetry for orbifolds. Moreover, the second author believes that orbifold quantum cohomology is different from the quantum cohomology of crepant resolutions. How to formulate mirror symmetry in the category of Calabi-Yau orbifolds seems to be an extremely interesting problem. Suppose we can do all of these, we are still working only in the so-called type II string theory. There are orbifold versions for other types of string theory (such as heterotic string theory) as well. The amount of new mathematics we can unearth is unimaginable!

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