December 7, 2004

# On a Theorem of Peters on Automorphisms of Kahler Surfaces

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# ON A THEOREM OF PETERS ON AUTOMORPHISMS OF KÄHLER SURFACES

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## Abstract

For any Kähler surface which admits no nonzero holomorphic vectorfields, we consider the group of holomorphic automorphisms which induce identity on the second rational cohomology. Assuming the canonical linear system is without base points and fixed components, C.A.M. Peters [12] showed that this group is trivial except when the Kähler surface is of general type and either $c_1^2 = 2c_2$ or $c_1^2 = 3c_2$ holds. Moreover, this group is a 2-group in the former case, and is a 3-group in the latter. The purpose of this note is to give further information about this group. In particular, we show that $c_1^2$ is divisible by the order of the group. Our argument is based on the results of C.H. Taubes in [14, 15] on symplectic 4-manifolds, which are applied here in an equivariant setting.


## 1. Introduction

Let $X$ be a Kähler surface with $H^0(X, T_X) = 0$, ie., $X$ admits no nonzero holomorphic vectorfields, and let $\mathrm{Aut}(X)^o \subset \mathrm{Aut}(X)$ be the subgroup of holomorphic automorphisms of $X$ which operates trivially on $H^2(X; \mathbb{Q})$. By a result of Lieberman [10], $\mathrm{Aut}(X)^o$ is finite.

In [12], C.A.M. Peters proved the following theorem concerning $\mathrm{Aut}(X)^o$.

**Theorem** (Peters) *Let $X$ be a Kähler surface with $H^0(X, T_X) = 0$. Suppose $|K_X|$ is without base points and fixed components. Then $g \in Aut(X)^o$ is trivial unless $X$ is a surface of general type and either*

(i) *$c_1^2 = 2c_2$, and $|g|$ is a power of 2, or*
(ii) *$c_1^2 = 3c_2$, $|g|$ is a power of 3 and moreover, $g$ acts trivially on $H^*(X; \mathbb{Q})$.*

The purpose of this note is to give further information about $\mathrm{Aut}(X)^o$ for the two exceptional cases in Peters' theorem. In particular, we show that $c_1^2$ must be divisible by the order of $\mathrm{Aut}(X)^o$. Before stating our theorem, we first have a digression on free actions of a finite group on Riemann surfaces.

Let $G$ be a finite group and $\Sigma_m$ be a Riemann surface of genus $m$ such that $G$ acts freely on $\Sigma_m$ via orientation-preserving homeomorphisms, and let $\Sigma_n \equiv \Sigma_m/G$ be the quotient Riemann surface which has genus $n$. Then the following are easily seen:

(a) $\chi(\Sigma_m) = |G| \cdot \chi(\Sigma_n)$, or equivalently, $m - 1 = |G| \cdot (n - 1)$
(b) $G = \pi_1(\Sigma_n)/\pi_1(\Sigma_m)$ where $\pi_1(\Sigma_m)$ is naturally regarded as a normal subgroup of $\pi_1(\Sigma_n)$ under the regular covering $\Sigma_m \to \Sigma_n$

With the preceding understood, we introduce the following terminology. For any finite group $G$, we will call the minimal genus of a Riemann surface which admits a free $G$-action the

*Date*: November 18, 2013.

*free genus* of $G$. The free genus of a finite group $G$ is closely related to the minimal number of generators of $G$ (which is simply the rank of $G$ when $G$ is abelian).

**Lemma** *Let $r$ be the minimal number of generators of $G$, and let $[x]$ be the greatest integer less than or equal to $x$. Then*

$$([2^{-1}(r+1)]-1)\cdot|G|+1 \leq \text{free genus of } G \leq (r-1)\cdot|G|+1.$$

**Proof** The left hand side of the inequality follows easily from the assertions (a), (b) above. As for the right hand side, we appeal to the following construction which was pointed out to us by J. McCarthy, compare also [8].

Let $c_1, \cdots, c_r$ be a set of generators of $G$. Then $G$ can be realized as a quotient group of

$$\pi_1(\Sigma_r) = \langle x_1, y_1, \cdots, x_r, y_r \mid x_1y_1x_1^{-1}y_1^{-1}\cdots x_ry_rx_r^{-1}y_r^{-1} = 1\rangle$$

under the homomorphism $x_i \mapsto c_i$, $y_i \mapsto 1$, $i = 1, \cdots, r$. Let $\Sigma_\delta \to \Sigma_r$ be the corresponding regular covering (note that $G$ is finite). Then $(\delta - 1) = (r-1)\cdot|G|$, and the free genus of $G$ is less than or equal to $\delta$. The lemma follows immediately.

□

Now we state our main result.

**Theorem** *Let $X$ be a Kähler surface as in Peters' theorem such that $Aut(X)^o$ is nontrivial. Then the following conclusions hold.*

(a) *Each $g \in Aut(X)^o$ has order 2 or 3. In particular, in the case of $c_1^2 = 2c_2$, $Aut(X)^o$ is an elementary abelian 2-group (ie., product of copies of $\mathbb{Z}_2$).*
(b) *$c_1^2$ is divisible by the order of $Aut(X)^o$.*
(c) *$c_1^2 \geq \max(\text{free genus of } Aut(X)^o - 1, |Aut(X)^o|)$.*

The canonical map of minimal surfaces of general type was systematically studied by A. Beauville in [3], particularly for the case where the arithmetic genus is relatively large. Based on Beauville's theorem, J.-X. Cai in [4] showed that for a minimal surface $X$ of general type with $\chi(\mathcal{O}_X) > 188$, $\mathrm{Aut}(X)^o$ is either cyclic of order less than 5, or is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover, for the case where $|K_X|$ has no base points or fixed components, it was shown that Beauville's theorem directly implies that when $\chi(\mathcal{O}_X) \geq 31$, the order of $\mathrm{Aut}(X)^o$ is less than 5 (cf. [4], page 347). Combined with this observation, we arrived at the following

**Corollary** *Let $X$ be a Kähler surface as in Peters' theorem with $Aut(X)^o$ nontrivial. Then the following must hold.*

(a) *For the case of $c_1^2 = 2c_2$, $Aut(X)^o$ is an elementary abelian 2-group of rank $\leq 6$.*
(b) *For the case of $c_1^2 = 3c_2$, $Aut(X)^o$ is a 3-group of order $\leq 243$.*

The proof of our main result, which is given in the next section, is divided into two parts. In Part 1 we give a proof for part (a) of the theorem, which is a refined version of Peters' argument in [12], and as in [12], is based on application of $G$-index theorems and the Miyaoka-Yau inequality $c_1^2 \leq 3c_2$. Part 2 is concerned with (b) and (c). The new ingredient here is the application of the results of C.H. Taubes in [14, 15] in an equivariant context. (See also [5, 6].) More concretely, we showed that there is a finite set of disjoint, embedded surfaces $\{C_i\}$ in $X$ such that (1) each $C_i$ lies in the complement of the exceptional orbits of $\mathrm{Aut}(X)^o$,

(2) each $C_i$ is invariant under the action of $\mathrm{Aut}(X)^o$, and (3) $c_1^2 = \sum_i(\mathrm{genus}(C_i) - 1)$ holds, from which (b) and (c) of the theorem follow.

**Acknowledgments**

Benefits from joint work [6] with S. Kwasik and helpful communications with J. McCarthy are gratefully acknowledged. The author is partially supported by NSF grant DMS-0304956.

## 2. The proof

**Part 1**. We start with the following lemma about the local representation of $g \in \mathrm{Aut}(X)^o$ at a fixed point, which is Lemma 2 in Peters [12]. We wish to point out that the order of $g$ is not necessarily prime here (which is assumed in [12]). We set $\mu_p \equiv \exp(\frac{2\pi i}{p})$ below.

**Lemma 2.1** *Let $x \in X$ be a fixed point of $1 \neq g \in Aut(X)^o$. Then $x$ is an isolated fixed point and the action of $g$ near $x$ is given by $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ for some integer $k$, where $p = |g|$, and $k$ is relatively prime to $p$.*

**Proof** The proof is the same as in [12]. Because $|K_X|$ is without base points and fixed components, there exists a holomorphic 2-form $\omega$ such that $\omega(x) \neq 0$. On the other hand, since $g$ acts trivially on $H^2(X;\mathbb{Q})$, $g^*\omega = \omega$, which implies the lemma.

□

The next lemma gives a lower bound on the number of fixed points of a certain orientation-preserving self-diffeomorphism of a 4-manifold having only isolated fixed points (compare Lemma 3 of [12]).

**Lemma 2.2** *Let $f : M \to M$ be an orientation-preserving, periodic, self-diffeomorphism of a 4-manifold which has only isolated fixed points. If $f$ induces identity on $H^2(M;\mathbb{Q})$, then the number of fixed points of $f$ is bounded below by the Euler characteristic of $M$.*

**Proof** Let $n$ be the number of fixed points of $f$. Then by the Lefschetz fixed point theorem,

$$n = \sum_{k=0}^{4}(-1)^k \mathrm{Trace}(f^*|H^k(M;\mathbb{Q})) = 2 + b_2 - \mathrm{Trace}(f^*|H^1(M;\mathbb{Q})) - \mathrm{Trace}(f^*|H^3(M;\mathbb{Q})),$$

because $f$ is periodic, orientation-preserving, and induces identity on $H^2(M;\mathbb{Q})$. On the other hand, each $H^k(M;\mathbb{Q})$, $k = 1, 3$, is decomposed into $f^*$-invariant direct sum $\oplus_i V_i$, where $V_i$ is either 1-dimensional with $f^*|V_i = \pm 1$, or 2-dimensional with $f^*|V_i$ being a rotation in $V_i$. In any event, $\mathrm{Trace}(f^*|V_i) \leq \dim V_i$, and consequently, $\mathrm{Trace}(f^*|H^k(M;\mathbb{Q})) \leq b_k$ for $k = 1, 3$. The lemma follows immediately.

□

Now we are ready to give a proof for part (a) of the theorem. To this end, we first recall the following version of the $G$-signature theorem for a cyclic group action of $G$ on a 4-manifold $M$, which has only isolated fixed points (cf. [9], Equation (12) on page 177)

$$|G| \cdot \mathrm{sign}(M/G) = \mathrm{sign}(M) + \sum_m \mathrm{def}_m.$$

Here in the above formula, $m \in M$ is running over the set of exceptional orbits, and $\mathrm{def}_m$ stands for the signature defect. For $m$ with isotropy subgroup of order $p$ and local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{kq} z_2)$ where $k, q$ are relatively prime to $p$, the signature defect is given by the following formula (cf. [9], Equation (19) on page 179)

$$\mathrm{def}_m = I_{p,q} \equiv \sum_{k=1}^{p} \frac{(1+\mu_p^k)(1+\mu_p^{kq})}{(1-\mu_p^k)(1-\mu_p^{kq})}.$$

The following lemma computes $I_{p,q}$ for the case of $q = -1$ (see also Lemma 2.3 in [6]).

**Lemma 2.3** $I_{p,-1} = \frac{1}{3}(p-1)(p-2)$.

**Proof** $I_{p,q}$ can be computed in terms of Dedekind sum $s(q,p)$ (cf. [9], page 92), where

$$s(q,p) = \sum_{k=1}^{p} ((\frac{k}{p}))\ ((\frac{kq}{p}))$$

with

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

(Here $[x]$ stands for the greatest integer less than or equal to $x$.)

In fact, Equation (24) in [9], page 180, gives

$$I_{p,q} = -4p \cdot s(q,p),$$

with the expression $6p \cdot s(q,p)$ given by (cf. [9], Equations (10) and (9) on page 94)

$$6p \cdot s(q,p) = (p-1)(2pq - q - \frac{3p}{2}) - 6f_p(q),$$

where $f_p(q) = \sum_{k=1}^{p-1} k[\frac{kq}{p}]$. Since $f_p(-1) = \sum_{k=1}^{p-1} k \cdot (-1) = \frac{1}{2}(1-p)p$, one obtains

$$I_{p,-1} = \frac{1}{3}(p-1)(p-2)$$

as claimed.

□

With the preceding preparation, we now give a proof for part (a) of the theorem. Suppose to the contrary that $g \in \mathrm{Aut}(X)^o$ has order other than 2 or 3. Then by Peters' theorem, and by passing to a suitable power of $g$, we may assume without loss of generality that $|g| = p^2$, where $p = 2$ or 3. Now consider the cyclic action of $G \equiv \langle g \rangle$ on the Kähler surface $X$. Since $G$ operates trivially on $H^2(X; \mathbb{Q})$, the $G$-signature theorem implies

$$(p^2 - 1) \cdot \mathrm{sign}(X) = \sum_{m \in X^g} \mathrm{def}_m + \sum_{m \in X^{g^p}} \mathrm{def}_m.$$

By Lemma 2.1 and Lemma 2.3, $\mathrm{def}_m \geq 0$ for any $m \in X^{g^p}$, and by Lemma 2.2, $|X^g| \geq c_2$. It follows easily that (cf. Lemma 2.3)

$$\frac{1}{3}(p^2-1)(c_1^2 - 2c_2) \geq \frac{1}{3}(p^2-1)(p^2-2)c_2,$$

which contradicts the Miyaoka-Yau inequality $c_1^2 \leq 3c_2$ (cf. [2]). Hence part (a) of the theorem.

**Part 2**. We begin by recalling the relevant theorems of Taubes in [14, 15], which are cast here in an equivariant setting. See also [5, 6].

Let $(M, \omega)$ be a symplectic 4-manifold, and $G$ be a finite group acting on $(M, \omega)$ via symplectomorphisms. Denote by $b_G^{2,+}$ the dimension of the maximal subspace of $H^2(M; \mathbb{Q})$ over which the cup product is positive and the induced action of $G$ is identity.

For any given $G$-equivariant $\omega$-compatible almost complex structure $J$, consider the associated $G$-equivariant Riemannian metric $g = \omega(\cdot, J(\cdot))$. There is a canonical $G$-$Spin^{\mathbb{C}}$ structure on $M$ such that the associated $U(2)$ $G$-bundles are $S_+^0 = \mathbb{I} \oplus K^{-1}$ and $S_-^0 = T^{0,1}M$, where $\mathbb{I}$ is the trivial $G$-bundle and $K = \det T^{1,0}M$ is the canonical bundle. Note that in this setup, the associated Seiberg-Witten equations for a pair $(A, \psi)$

$$D_A\psi = 0 \text{ and } P_+F_A = \frac{1}{4}\tau(\psi \otimes \psi^*) + \mu$$

are $G$-equivariant, where $A$ is a $G$-equivariant $U(1)$-connection on $\det S_+^0$ and $\psi$ is a $G$-equivariant smooth section of $S_+^0$, and $\mu$ is some fixed $G$-equivariant, imaginary valued self-dual 2-form. According to Taubes [14], there is a canonical (up to gauge equivalence) connection $A_0$ on $K^{-1} = \det S_+^0$, such that if we set $u_0 \equiv (1, 0) \in \Gamma(\mathbb{I} \oplus K^{-1})$, then for any $r > 0$, $(A_0, \sqrt{r}u_0)$ (which is clearly $G$-equivariant) satisfies the Seiberg-Witten equations with $\mu = -\frac{i}{4}r \cdot \omega + P_+F_{A_0}$. Moreover, $(A_0, \sqrt{r}u_0)$ is the only solution (up to gauge equivalence) when $r > 0$ is sufficiently large, which is also non-degenerate. Thus assuming $b_G^{2,+} \geq 2$, if we define a $G$-equivariant Seiberg-Witten invariant as an algebraic count of the solutions to the $G$-equivariant Seiberg-Witten equations, Taubes' theorem in [14] implies that the corresponding invariant equals $\pm 1$ for the canonical $G$-$Spin^{\mathbb{C}}$ structure $S_+^0 \oplus S_-^0$.

Consequently, as in the non-equivariant case (assuming $b_G^{2,+} \geq 2$), the $G$-equivariant Seiberg-Witten invariant equals $\pm 1$ for the $G$-$Spin^{\mathbb{C}}$ structure $K \otimes (S_+^0 \oplus S_-^0)$. This means that for any $r > 0$, there is a solution $(A, \psi)$ to the $G$-equivariant Seiberg-Witten equations with $\mu = -\frac{i}{4}r \cdot \omega + P_+F_{A_0}$. Particularly, $(A, \psi)$ is $G$-equivariant. Now write $\psi = \sqrt{r}(\alpha, \beta) \in \Gamma(K \oplus \mathbb{I})$ (note: $K \otimes S_+^0 = K \oplus \mathbb{I}$). Then according to Taubes [15], the zero set $\alpha^{-1}(0)$ converges as $r \to \infty$ to a finite set of $J$-holomorphic curves $\{C_i\}$, such that for some integers $n_i > 0$, the canonical class $c_1(K)$ is Poincaré dual to the fundamental class of $\sum_i n_i C_i$. The crucial observation here is that $\cup_i C_i$ is $G$-invariant. This is because $\alpha$ is a $G$-equivariant section of $K$, and as $r \to \infty$, $\alpha^{-1}(0)$ converges to $\cup_i C_i$ with respect to the natural distance function on $M$.

With the preceding understood, the following lemma gives certain regularity about the $J$-holomorphic curves $\{C_i\}$ for a generic $G$-equivariant $J$, provided that $(M, \omega)$ and the $G$-action satisfy some further conditions.

**Lemma 2.4** *Suppose $(M, \omega)$ is minimal, and the action of $G$ is pseudofree such that for any $m \in M$, the representation of the isotropy subgroup $G_m$ on the tangent space of $m$ is contained in $SL_2(\mathbb{C})$ with respect to a $G$-equivariant, $\omega$-compatible almost complex structure $J_0$. Then for a generic $G$-equivariant $J$, the $J$-holomorphic curves $\{C_i\}$ are disjoint, embedded, all contained in the complement of the exceptional orbits, with $n_i = 1$ for any $C_i$ such that $C_i^2 > 0$.*

**Proof** The non-equivariant version of this result was due to Taubes [15], whose proof was based on transversality for moduli spaces of $J$-holomorphic maps and the adjunction formula. The proof of Lemma 2.4 is a somewhat equivariant version of that in Taubes [15].

Let $H \subset G$ be any subgroup (here $H$ is allowed to be trivial), and let $\Sigma$ be a Riemann surface which admits a holomorphic $H$-action. We shall consider the transversality problem for the moduli space of pseudoholomorphic maps $f : \Sigma \to M$ where $f$ is equivariant, ie., $f \circ h = h \circ f$ for any $h \in H$.

To this end, we put the problem in the Fredholm framework as follows. Fix a sufficiently large $r > 0$, we consider the space

$$[\Sigma; M]^H \equiv \{f : \Sigma \to M \mid f \text{ is of } C^r \text{ class, and } f \circ h = h \circ f, \forall h \in H\},$$

and the space $\mathcal{J}$ of $G$-equivariant, $\omega$-compatible almost complex structures $J$ of $C^r$ class, which equals $J_0$ in a fixed neighborhood of the exceptional orbits. The tangent space $T_f$ of $[\Sigma; M]^H$ at $f$ is the Banach space of equivariant $C^r$-sections of the $H$-bundle $f^*TM$, and the tangent space $T_J$ of $\mathcal{J}$ at $J$ is the Banach space of equivariant $C^r$-sections $A$ of the $G$-bundle End $TM$, which obeys (1) $AJ + JA = 0$, (2) $A^t = A$ (here the transpose $A^t$ is taken with respect to the metric $\omega(\cdot, J(\cdot))$), and (3) $A$ vanishes in a fixed neighborhood of the exceptional orbits.

We will also need to consider the moduli space $\mathcal{M}_\Sigma^H$ of $H$-equivariant complex structures $j$ on $\Sigma$, which is generally a finite dimensional complex orbifold. For technical reason, we will cover $\mathcal{M}_\Sigma^H$ by countably many open sets of form $U = \hat{U}/G_U$, where $\hat{U}$ is a complex manifold and $G_U$ is a finite group, and work instead with each $\hat{U}$. Denote by $\mathcal{U}$ any one of the $\hat{U}$'s.

With the preceding understood, for any $(f, J, j) \in [\Sigma; M]^H \times \mathcal{J} \times \mathcal{U}$, let $\mathcal{E}_{(f,J,j)}$ be the Banach space of equivariant $C^{r-1}$-sections $s$ of the $H$-bundle $\mathrm{Hom}(T\Sigma, f^*TM)$ which obeys $s \circ j = -J \circ s$. Then there is a Banach bundle $\mathcal{E} \to [\Sigma; M]^H \times \mathcal{J} \times \mathcal{U}$ whose fiber at $(f, J, j)$ is $\mathcal{E}_{(f,J,j)}$. The zero set of the smooth section $\underline{L} : [\Sigma; M]^H \times \mathcal{J} \times \mathcal{U} \to \mathcal{E}$, where

$$\underline{L}(f, J, j) = df + J \circ df \circ j,$$

consists of triples $(f, J, j)$ such that $f$ is equivariant (ie., $f \circ h = h \circ f, \forall h \in H$), and is $J$-holomorphic with respect to the complex structure $j$ on $\Sigma$. We consider the subspace

$$\widetilde{\mathcal{M}}_{H,\Sigma,\mathcal{U}} \equiv \{(f, J, j) \in \underline{L}^{-1}(0) \mid f \text{ is nonconstant and not multiply covered}\},$$

and state the promised transversality result in the following

**Claim 1** *The subspace $\widetilde{\mathcal{M}}_{H,\Sigma,\mathcal{U}} \subset [\Sigma; M]^H \times \mathcal{J} \times \mathcal{U}$ is a Banach submanifold.*

The proof of Claim 1 is based on the same analysis as in the non-equivariant setting (cf. Proposition 3.2.1 in [11]). Recall that the key point in the argument is that one is able to construct an $A \in T_J$ to kill the cokernel of $D\underline{L}_{(f,J,j)}|T_f$, where $D\underline{L}_{(f,J,j)}$ is the linearization of $\underline{L}$ at $(f, J, j)$. In the present, equivariant context, one needs to construct an $A \in T_J$ where there is an additional requirement that $A$ must be $G$-equivariant. But it is easily seen that this can be done, because for an open, dense subset of $\Sigma$, its image under $f$ lies in the complement of the exceptional orbits, where the action of $G$ is free.

Now consider the projection $\pi : \widetilde{\mathcal{M}}_{H,\Sigma,\mathcal{U}} \to \mathcal{J}$, which is a Fredholm map between Banach manifolds. By the Sard-Smale theorem [13], there is a Baire set $\mathcal{J}_{reg} \subset \mathcal{J}$, such that for

$J \in \mathcal{J}_{reg}$, the differential $d\pi$ is surjective along $\pi^{-1}(J)$. For any $J \in \mathcal{J}_{reg}$, we set

$$\widetilde{\mathcal{M}}^J_{H,\Sigma,\mathcal{U}} \equiv \pi^{-1}(J) = \{(f,j) \mid (f,J,j) \in \underline{L}^{-1}(0)\},$$

which is a finite dimensional manifold when nonempty.

Since the set of data $\{(H,\Sigma,\mathcal{U})\}$ is countable, it follows easily that there is a Baire set $\mathcal{J}_0 = \cap \mathcal{J}_{reg}$, such that for any $J \in \mathcal{J}_0$, which particularly may be chosen smooth, $\widetilde{\mathcal{M}}^J_{H,\Sigma,\mathcal{U}}$ is a finite dimensional manifold when nonempty for all $(H,\Sigma,\mathcal{U})$.

The next crucial step in the argument is to compute the dimension of $\widetilde{\mathcal{M}}^J_{H,\Sigma,\mathcal{U}}$. To this end, we observe that the dimension of $\widetilde{\mathcal{M}}^J_{H,\Sigma,\mathcal{U}}$ at $(f,j)$ is given by the sum of the index of $D\underline{L}_{(f,J,j)}|T_f$ with the (real) dimension of the moduli space $\mathcal{M}^H_\Sigma$ of $H$-equivariant complex structures on $\Sigma$. The index of $D\underline{L}_{(f,J,j)}|T_f$ can be computed by the Riemann-Roch theorem for orbit spaces in Atiyah-Singer [1], or more generally, the index formula for Cauchy-Riemann type operators over orbifold Riemann surfaces, cf. Lemma 3.2.4 of [7].

We begin by introducing some notations. Given any $(f,j) \in \widetilde{\mathcal{M}}^J_{H,\Sigma,\mathcal{U}}$, we pick for each exceptional orbit in $\Sigma$ a point $z_i$ from it, where $i = 1,2,\cdots,k$. We denote by $m_i$ the order of isotropy at $z_i$, and to each $z_i$, we assign a pair of *rotation numbers* $(m_{i,1}, m_{i,2})$ with $0 < m_{i,1}, m_{i,2} < m_i$ as follows: let $h_i \in H$ be the unique element in the isotropy subgroup at $z_i$ whose action near $z_i$ is given by a counterclockwise rotation of angle $\frac{2\pi}{m_i}$, then the action of $h_i$ on the tangent space of $p_i \equiv f(z_i)$ is given by $(\xi_1,\xi_2) \mapsto (\mu_{m_i}^{m_{i,1}}\xi_1, \mu_{m_i}^{m_{i,2}}\xi_2)$. Clearly, the rotation numbers $(m_{i,1}, m_{i,2})$ depend only on the exceptional orbit which $z_i$ lies in. Finally, we set $\Gamma \equiv \Sigma/H$, which is an orbifold Riemann surface with orbifold points $w_i = [z_i]$ of orders $m_i$, $i = 1,2,\cdots,k$. We denote by $g_{|\Gamma|}$ the genus of the underlying surface of $\Gamma$.

With these notations, the index of $D\underline{L}_{(f,J,j)}|T_f$ is given by $2d_{(f,j)}$ where $d_{(f,j)} \in \mathbb{Z}$ and

$$d_{(f,j)} = \frac{1}{|H|}c_1(TM)\cdot f_*([\Sigma]) + 2 - 2g_{|\Gamma|} - \sum_{i=1}^k \frac{m_{i,1}+m_{i,2}}{m_i}.$$

On the other hand, the moduli space $\mathcal{M}^H_\Sigma$ of $H$-equivariant complex structures on $\Sigma$ can be identified with the moduli space of complex structures on the marked Riemann surface $(\Gamma, \{w_i\})$. Thus we have

$$\dim_{\mathbb{C}} \mathcal{M}^H_\Sigma = \begin{cases} 0 & \text{if } g_{|\Gamma|} = 0, k \le 3, \\ k-3 & \text{if } g_{|\Gamma|} = 0, k > 3, \\ 1 & \text{if } g_{|\Gamma|} = 1, k = 0, \\ k-1 & \text{if } g_{|\Gamma|} = 1, k > 0, \\ 3g_{|\Gamma|} - 3 + k & \text{if } g_{|\Gamma|} \ge 2. \end{cases}$$

Now here is the crucial consequence of the assumption we made in Lemma 2.4 that for any $m \in M$, the representation of the isotropy subgroup $G_m$ on the tangent space of $m$ is contained in $SL_2(\mathbb{C})$ with respect to a $G$-equivariant, $\omega$-compatible almost complex structure $J_0$: for any $i = 1,2,\cdots,k$, the rotation numbers $(m_{i,1}, m_{i,2})$ obey $m_{i,1} + m_{i,2} = m_i$.

Now assuming $\widetilde{\mathcal{M}}^J_{H,\Sigma,\mathcal{U}} \neq \emptyset$, we see that $d_{(f,j)} + \dim_{\mathbb{C}} \mathcal{M}^H_\Sigma \geq 0$ must hold for any $(f,j) \in \widetilde{\mathcal{M}}^J_{H,\Sigma,\mathcal{U}}$. With $m_{i,1} + m_{i,2} = m_i$, this gives

$$\begin{array}{ll} (i)\ |H|^{-1}c_1(TM)\cdot f_*([\Sigma]) + 2 - k \geq 0 & \text{if } g_{|\Gamma|} = 0, k \leq 3, \\ (ii)\ |H|^{-1}c_1(TM)\cdot f_*([\Sigma]) - 1 \geq 0 & \text{if } g_{|\Gamma|} = 0, k > 3, \\ (iii)\ |H|^{-1}c_1(TM)\cdot f_*([\Sigma]) + 1 \geq 0 & \text{if } g_{|\Gamma|} = 1, k = 0, \\ (iv)\ |H|^{-1}c_1(TM)\cdot f_*([\Sigma]) - 1 \geq 0 & \text{if } g_{|\Gamma|} = 1, k > 0, \\ (v)\ |H|^{-1}c_1(TM)\cdot f_*([\Sigma]) + g_{|\Gamma|} - 1 \geq 0 & \text{if } g_{|\Gamma|} \geq 2. \end{array}$$

Furthermore, note that in cases (i) and (iii), the complex structure $j$ has an automorphism group of complex dimension $3-k$ and 1 respectively, so that in each of these two cases we have a sharper inequality

$$\begin{array}{ll} (i')\ |H|^{-1}c_1(TM)\cdot f_*([\Sigma]) - 1 \geq 0 & \text{if } g_{|\Gamma|} = 0, k \leq 3, \\ (iii')\ |H|^{-1}c_1(TM)\cdot f_*([\Sigma]) \geq 0 & \text{if } g_{|\Gamma|} = 1, k = 0. \end{array}$$

With these inequalities in hand, it now comes to the following observation.

**Claim 2** *$c_1(TM)\cdot C_i \leq 0$ for each of the $J$-holomorphic curves in $\{C_i\}$.*

To see this, note that $c_1(K)\cdot C_i = \sum_s n_s C_s \cdot C_i \geq n_i C_i^2$, so that if $c_1(TM)\cdot C_i > 0$, one has $C_i^2 \leq c_1(K)\cdot C_i < 0$, and from the adjunction formula, $C_i$ must be an embedded $(-1)$-sphere, contradicting the minimality assumption on $(M,\omega)$. Hence Claim 2.

Now back to the proof of Lemma 2.4. Fix any $J \in \mathcal{J}_0$, we consider the $J$-holomorphic curves $\{C_i\}$. For any $C_i$, let $H_i \subset G$ be the subgroup which leaves $C_i$ invariant, and let $f_i : \Sigma_i \to M$ be an equivariant $J$-holomorphic map parametrizing $C_i$. Then by Claim 2 there are only two possibilities for $C_i$: (1) $\Gamma_i \equiv \Sigma_i/H_i$ is of genus one and $H_i$ acts on $\Sigma_i$ freely, and $c_1(K)\cdot C_i = 0$, (2) the underlying surface of $\Gamma_i \equiv \Sigma_i/H_i$ has genus $g_{|\Gamma_i|} \geq 2$, and $c_1(K)\cdot C_i \leq |H_i|\cdot(g_{|\Gamma_i|} - 1)$. Moreover, it follows easily that in case (1), $C_i$ is an embedded torus with $C_i^2 = 0$, which is disjoint from the rest of the $J$-holomorphic curves and is in the complement of the exceptional orbits.

As for case (2), note that $g_{\Sigma_i} - 1 \geq |H_i|\cdot(g_{|\Gamma_i|} - 1)$ with equality iff $H_i$ acts on $\Sigma_i$ freely. On the other hand, by the adjunction formula,

$$2(g_{\Sigma_i} - 1) \leq C_i^2 + c_1(K)\cdot C_i \leq \frac{1}{n_i}c_1(K)\cdot C_i + c_1(K)\cdot C_i \leq (\frac{1}{n_i} + 1)\cdot |H_i|\cdot(g_{|\Gamma_i|} - 1),$$

which implies that $n_i = 1$, $H_i$ acts freely on $\Sigma_i$, and $c_1(K)\cdot C_i = C_i^2$. It follows easily that in this case, $C_i$ is embedded with genus $C_i^2 + 1 \geq 2$, disjoint from the rest of the $J$-holomorphic curves, and lies in the complement of the exceptional orbits. Lemma 2.4 is thus proved.

□

Now we give a proof for parts (b) and (c) of the theorem. First of all, observe that the geometric genus $p_g(X)$ is nonzero because the linear system $|K_X|$ is nonempty, so that we have $b_2^+(X) \geq 2$. Let $\omega$ be a Kähler form on $X$ which is equivariant under $\mathrm{Aut}(X)^o$. We apply Lemma 2.4 to $(X,\omega)$ with $G = \mathrm{Aut}(X)^o$ (note that $X$ is minimal, and $b_G^{2,+} = b_2^+ \geq 2$, so that with Lemma 2.1, the assumptions in Lemma 2.4 are satisfied), and obtain a set of $J$-holomorphic curves $\{C_i\}$ as in Lemma 2.4, with

$$c_1^2 = (\sum_i n_i C_i)\cdot(\sum_i n_i C_i) = \sum_i n_i^2 C_i^2.$$

The fact that $c_1^2 > 0$ implies that there is at least one $C_i$ with $C_i^2 > 0$, and since $n_i = 1$ for any $C_i$ with $C_i^2 > 0$, we have

$$c_1^2 = \sum_i C_i^2 = \sum_i (\text{ genus}(C_i) - 1).$$

Now recall that $\mathrm{Aut}(X)^o$ operates on $H^2(X;\mathbb{Q})$ trivially, and $\cup_i C_i$ is invariant under $\mathrm{Aut}(X)^o$. Thus for any $g \in \mathrm{Aut}(X)^o$, $g \cdot C_i$ is disjoint from $C_i$ if $g \cdot C_i \neq C_i$, which can occur only when $C_i^2 = 0$. This implies that $\mathrm{Aut}(X)^o$ leaves $C_i$ invariant for any $C_i$ with $C_i^2 > 0$. Consequently, for any $C_i$ with $C_i^2 > 0$, genus$(C_i) - 1$ is divisible by $|\mathrm{Aut}(X)^o|$, which implies that $c_1^2$ is divisible by $|\mathrm{Aut}(X)^o|$, and moreover, genus$(C_i)$ is greater than or equal to the free genus of $\mathrm{Aut}(X)^o$, which implies part (c) of the theorem.

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