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G-MINIMALITY AND INVARIANT NEGATIVE SPHERES IN G-HIRZEBRUCH SURFACES

WEIMIN CHEN

Abstract. In this paper we initiate a study on the notion of G-minimality of four-manifolds equipped with an action of a finite group G. Our work shows that even in the case of cyclic actions on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, the comparison of G-minimality in the various categories (i.e., locally linear, smooth, symplectic) is already a delicate and interesting problem. In particular, we show that if a symplectic $\mathbb{Z}_n$-action on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ has an invariant locally linear topological (-1)-sphere, then it must admit an invariant symplectic (-1)-sphere, provided that $n=2$ or n is odd. For the case where $n>2$ and even, the same conclusion is true under a stronger assumption, i.e., the invariant (-1)-sphere is smoothly embedded. The techniques developed in this paper also find applications in the smooth classification of G-Hirzebruch surfaces. More precisely, we give a classification of G-Hirzebruch surfaces (each equipped with a homologically trivial, holomorphic $G=\mathbb{Z}_n$-action) up to orientation-preserving equivariant diffeomorphisms. The main technical issue encountered in the classification is to distinguish non-diffeomorphic G-Hirzebruch surfaces which have the same fixed-point set structure, and an interesting discovery of this paper is that a certain "equivariant Gromov-Taubes invariant", i.e., an invariant defined by counting certain embedded invariant negative two-spheres, can be used to distinguish such G-Hirzebruch surfaces. Finally, going back to the original question of G-minimality, we show that for $G=\mathbb{Z}_n$, a minimal rational G-surface is minimal as a symplectic G-manifold if and only if it is minimal as a smooth G-manifold.

1. Introduction

Minimality is a basic concept in four-manifold topology. An oriented smooth four-manifold is called minimal if it does not contain any smoothly embedded two-spheres of self-intersection -1; such a two-sphere is called a (-1)-sphere. If a four-manifold X contains a (-1)-sphere, then X is naturally diffeomorphic to a connected sum $X'\#\overline{\mathbb{CP}^2}$, where X' is called the blowdown of X along the (-1)-sphere. One can simplify a non-minimal four-manifold through a sequence of blowdowns until one gets a minimal four-manifold, and it is a fundamental question to understand how a four-manifold and its blowdowns are related, e.g., how their gauge theoretic invariants are related (cf. [10, 11]). When the four-manifolds and the (-1)-spheres in question are complex analytic or symplectic, the blowdown operation can be done in the corresponding category,

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giving naturally the notion of minimality in the complex analytic or the symplectic category. It is well-known that there are complex surfaces which are minimal in the complex analytic category but not minimal as smooth four-manifolds, i.e., the Hirzebruch surfaces F_r where r is odd and $r > 1$ (F_r is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ when r is odd). On the other hand, thanks to the deep work of Taubes on the equivalence of Seiberg-Witten and Gromov invariants of symplectic four-manifolds, the notions of minimality in the symplectic and smooth categories are equivalent (cf. [24, 18, 17]), which has had important consequences in four-manifold topology.

The notion of minimality can be extended naturally to the equivariant setting. An algebraic surface with a finite group G of automorphisms is called a minimal G-surface if it can not be blown down equivariantly. (We remark that the notion of minimal rational G-surfaces played a fundamental role in the modern approach to the classical problem of classifying finite subgroups of the plane Cremona group, i.e., the group of birational transformations of the projective plane, cf. [9].) One can similarly define the notion of symplectic minimality or smooth minimality in the equivariant setting. To be more concrete, a symplectic four-manifold with a symplectic G-action (resp. a smooth four-manifold with a smooth G-action) is called minimal if there does not exist any G-invariant set of disjoint union of symplectic (resp. smooth) (-1)-spheres; clearly one can blow down the G-manifold equivariantly if such a G-invariant set of (-1)-spheres is contained in the four-manifold. It is known that there are minimal G-Hirzebruch surfaces which are not minimal as smooth G-manifolds (cf. [26]), and it is a natural question as whether the notions of G-minimality are equivalent in the symplectic and smooth categories. (We shall restrict ourselves to the situation where the G-invariant set of (-1)-spheres can be consistently oriented such that the corresponding homology classes are preserved under the G-action; this more restrictive assumption is automatically satisfied in the symplectic or holomorphic category.)

First of all, a quick observation: recall that for a symplectic four-manifold which is neither rational nor ruled, the notions of minimality in all three categories (i.e., complex analytic, symplectic, or smooth) are equivalent; this continues to hold in the equivariant setting by the following two facts: (1) if a symplectic four-manifold which is neither rational nor ruled contains no J-holomorphic (-1)-spheres for some compatible almost complex structure J (not necessarily generic), then it must be minimal (cf. [6], Lemma 2.3), and (2) if a symplectic four-manifold contains an immersed symplectic sphere with nonnegative self-intersection whose pairing with the canonical class is less than -1, then it must be rational or ruled (due to McDuff, see [17], p.612). It follows easily from these two facts that for a symplectic four-manifold which is neither rational nor ruled, the notion of symplectic or holomorphic G-minimality is equivalent to the smooth minimality of the underlying manifold.

With the preceding understood, the only interesting case concerning various notions of G-minimality is the case of rational or ruled symplectic four-manifolds. While finite automorphism groups of rational surfaces have been studied extensively by algebraic geometers in connection with the plane Cremona group (cf. [9]), general symplectic finite group actions on a rational four-manifold remain largely unexplored except for the case of $\mathbb{CP}^2$ (cf. [5, 2, 4, 3]); see also [7].

In this paper, we shall take an initial step by focusing on the case of symplectic G-actions on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ where $G=\mathbb{Z}_n$ is a cyclic group of order n. Our work shows that even in the simple setting of $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, the comparison of G-minimality in the various categories is already quite a delicate and interesting problem. Furthermore, the techniques we developed in this paper, in the form of equivariant Gromov-Taubes invariants, also find applications in the smooth classification of G-Hirzebruch surfaces.

Before stating our theorems, we remark that one can also consider the notion of G-minimality in the category of locally linear topological G-manifolds. More concretely, let X be a topological 4-manifold equipped with a locally linear G-action. A G-invariant, topologically embedded surface $\Sigma\subset X$ is called locally linear if for any $z\in\Sigma$, there is a G_z-invariant neighborhood U_z of z in X such that $(U_z, U_z\cap\Sigma)$ is equivariantly homeomorphic to $(\mathbb{R}^4,\mathbb{R}^2)$ with a linear G_z-action (cf. Lashof-Rothenberg [16], p. 227); in particular, Σ is locally flat. It follows easily from Freedman-Quinn (cf. [13], Theorem 9.3A, p. 137) that a G-invariant locally linear surface has a G-equivariant normal bundle and hence a G-invariant regular neighborhood given by the corresponding disc bundle. With this understood, if X contains a G-invariant set of disjoint union of locally linear (-1)-spheres, then one can blow down X equivariantly in the category of locally linear topological G-manifolds. We say X is minimal as a topological G-manifold if no such a G-invariant set of disjoint union of locally linear (-1)-spheres exists in X.

Theorem 1.1. *Let $G=\mathbb{Z}_n$ be a finite cyclic group of order n where either $n=2$ or n is odd. Suppose a smooth G-action on $X=\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ admits a G-invariant, locally linear (-1)-sphere. Let ω be any given G-invariant symplectic form. Then for any generic G-invariant ω-compatible almost complex structure J, there is a G-invariant, J-holomorphic (-1)-sphere in X. In particular, X contains a G-invariant, ω-symplectic (-1)-sphere.*

Since a symplectic $\mathbb{Z}_n$-action on $\mathbb{CP}^2$ is equivariantly diffeomorphic to a linear action (cf. [2, 5]), Theorem 1.1 has the following corollary, where the case of pseudo-free holomorphic actions can be also deduced from Theorem 4.14 in [26].

A symplectic $\mathbb{Z}_n$-action on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, for $n=2$ or odd, is equivariantly diffeomorphic to an equivariant connected sum of a pair of linear actions on $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ if and only if it admits an invariant, locally linear (-1)-sphere.

We remark that the existence of a G-invariant J-holomorphic (-1)-sphere asserted in Theorem 1.1 is quite a subtle issue. It is not true for a non-generic J, as there are examples of minimal G-Hirzebruch surfaces which are not minimal as smooth G-manifolds. Moreover, since the existence of a G-invariant locally linear (-1)-sphere imposes certain constraints on the representations of G on the tangent spaces of the fixed points, which are not satisfied by a general symplectic (even holomorphic) $\mathbb{Z}_n$-action on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, Theorem 1.1 is not expected to be true for an arbitrary symplectic $\mathbb{Z}_n$-action. Finally, there are pairs of pseudo-free G-Hirzebruch surfaces with isomorphic local representations at the fixed points, such that exactly one of them contains a G-invariant locally linear (-1)-sphere (cf. [26]). This shows that in Theorem 1.1, one can not replace the assumption of existence of a G-invariant locally linear (-1)-sphere by any condition merely on the local representations at the fixed points (see Example 3.1 for more details). We remark that it is a consequence of the topological classification theorems of pseudo-free locally linear cyclic actions in [25, 26] that only one of the pseudo-free G-Hirzebruch surfaces in each such pair contains a G-invariant locally linear (-1)-sphere. Our approach in this paper offered an alternative proof of this fact, see Lemma 3.3 and Remark 3.4.

For $\mathbb{Z}_n$-actions where $n > 2$ and even, we need to impose a stronger assumption, i.e., the G-invariant (-1)-sphere is smoothly embedded.

Theorem 1.2. *Let $G = \mathbb{Z}_n$ be a cyclic group of order n. Suppose a smooth G-action on $X = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ admits a G-invariant smooth (-1)-sphere. Then for any G-invariant symplectic form ω, there is a G-invariant, ω-symplectic (-1)-sphere in X.*

As in the case of Theorem 1.1, one similarly has the following corollary.

A symplectic $\mathbb{Z}_n$-action on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is equivariantly diffeomorphic to an equivariant connected sum of a pair of linear actions on $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ if and only if it admits an invariant, smooth (-1)-sphere.

It is a natural question as whether a symplectic $\mathbb{Z}_n$-action on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ always possesses an invariant, topologically embedded (-1)-sphere which is not necessarily locally linear; note that without the locally-linear condition there are no additional constraints which must be satisfied by the local representations of the $\mathbb{Z}_n$-action. A particular interesting case is that of a piecewise linear (-1)-sphere, as such a (-1)-sphere has a regular neighborhood whose boundary is a smoothly embedded integral homology three-sphere.

Question 1.3. Consider an arbitrary symplectic $\mathbb{Z}_n$-action on $X = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

- (i) Is there always an invariant piecewise linear (-1)-sphere in X?
- (ii) Is there an equivariant decomposition of X into X_+, X_- along a smoothly embedded integral homology three-sphere Σ^3 such that X_+, X_- are an integral homology $\mathbb{CP}^2 \setminus B^4$, $\overline{\mathbb{CP}^2} \setminus B^4$ respectively?
- (iii) Is there an equivariant decomposition of X as described in (ii) such that Σ^3 is of contact type with respect to the $\mathbb{Z}_n$-invariant symplectic form on X?

Note that in (ii), (iii) of Question 1.3, if we replace the word "integral" by "rational", the answers to both questions are affirmative, cf. Lemma 2.3(2). Moreover, regarding (ii) of Question 1.3 in the non-equivariant setting, we should mention the work of Freedman-Taylor [14] on splitting smooth, simply connected four-manifolds along integral homology three-spheres.

Back to Theorems 1.1 and 1.2. What we have obtained therein can be paraphrased in terms of the non-vanishing of certain "equivariant Gromov-Taubes invariant", i.e., the invariant defined by counting G-invariant J-holomorphic (-1)-spheres for a G-invariant J which may be required to satisfy certain genericity conditions. In particular, what we proved in Theorems 1.1 and 1.2 asserts that the corresponding equivariant Gromov-Taubes invariant is non-zero as long as there is a G-invariant locally linear or smooth (-1)-sphere, which is a property of the G-action that depends only on the equivariant homeomorphism or diffeomorphism type of the G-action. Given the fact that there are non-homeomorphic or non-diffeomorphic G-Hirzebruch surfaces which have the same fixed-point set structure, this type of results may be turned into an effective method for distinguishing G-Hirzebruch surfaces that is more powerful than the basic method of using fixed-point set data, provided that we also have a corresponding vanishing theorem for such equivariant Gromov-Taubes invariants.

Let $G = \mathbb{Z}_n$ for a fixed integer n, we shall classify Hirzebruch surfaces F_r, which is equipped with a homologically trivial, holomorphic G-action, up to orientation-preserving equivariant diffeomorphisms. We will denote such a G-Hirzebruch surface by $F_r(a, b)$, where (a, b) is an ordered pair of integers mod n which completely determines the holomorphic G-action (cf. [26], §4, for the precise definition of $F_r(a, b)$). Given any two G-Hirzebruch surfaces $F_r(a, b)$ and $F_{r'}(a', b')$, there are six types of canonical equivariant diffeomorphisms (all orientation-preserving) between them if certain numerical conditions are satisfied by the triples (a, b, r) and (a', b', r'). (A detailed description of these canonical equivariant diffeomorphisms can be found at the beginning of Section 5.) Call the composition of a sequence of finitely many canonical equivariant diffeomorphisms a *standard* equivariant diffeomorphism. Then there is a complete set of numerical conditions for the triples (a, b, r) and (a', b', r'), such that there is a standard equivariant diffeomorphism between $F_r(a, b)$ and $F_{r'}(a', b')$ if and only if one of the numerical conditions is satisfied.

On the other hand, if $F_r(a, b)$ and $F_{r'}(a', b')$ are orientation-preservingly equivariantly diffeomorphic, then they must have isomorphic fixed-point set structures, which can be stated equivalently as one of a set of numerical conditions is satisfied by the triples (a, b, r) and (a', b', r'). This set of numerical conditions is strictly weaker than the set of conditions which guarantees a standard equivariant diffeomorphism between $F_r(a, b)$ and $F_{r'}(a', b')$. With this understood, the main task of our classification is to show, using the technique of equivariant Gromov-Taubes invariants, that one of the stronger set of conditions must be satisfied.

With the preceding understood, we shall formulate our classification as follows.

Theorem 1.4. *Two G-Hirzebruch surfaces are orientation-preservingly equivariantly diffeomorphic if and only if there is a standard equivariant diffeomorphism between them.*

We should point out that an analogous classification for pseudo-free G-Hirzebruch surfaces was obtained by Wilczynski (cf. [26], Theorem 4.2), where the classification is slightly different from the one in Theorem 1.4 in the sense that orientation-reversing equivariant differomorphisms are also allowed there. The result was a consequence of the topological classification theorems of pseudo-free locally linear cyclic actions on simply connected four-manifolds in [25, 26]. That approach is not readily extendable to non-pseudo-free actions without substantially additional work.

Now we discuss the technical aspect of this paper. It is well-known that in the J-holomorphic curve theory in dimension four, the presence of J-holomorphic curves of negative self-intersection causes considerable complications in the analysis of singularity or intersection patterns of J-holomorphic curves. One basic approach to get around this issue is to work with generic almost complex structures. The basic fact is that for a generic J, the only J-holomorphic curves of negative self-intersection are (-1)-spheres. Therefore, by working with the corresponding minimal symplectic four-manifolds and by working with generic almost complex structures, one can avoid the issue of J-holomorphic curves of negative self-intersection. With this said, however, in various applications of J-holomorphic curves one is often forced to work with non-generic almost complex structures. See Li-Zhang [20] and McDuff-Opshtein [21] for the recent articles on this topic.

For J-holomorphic curves in the equivariant setting (or more generally the orbifold setting), one has to work with G-invariant almost complex structures. Even though one can choose generic G-invariant J, these almost complex structures are not generic in the usual sense; in particular, one has to face the presence of J-holomorphic curves of negative self-intersection. In this situation, information about local representations at the fixed points of the G-action becomes extremely important in analyzing singularity or intersection patterns of G-invariant J-holomorphic curves.

With the preceding understood, two technical results of this paper concerning J-holomorphic curves in $X=\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ or $\mathbb{S}^2\times\mathbb{S}^2$ for an *arbitrary* J are worth mentioning. To describe the result for $X=\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, let ω be any given symplectic form on X and write the canonical class $c_1(K_\omega)=-3e_0+e_1$ for some basis $e_0,e_1\in H^2(X)$ such that $e_0^2=-e_1^2=1$ and $e_0\cdot e_1=0$. Then one has the following alternative:

For any ω-compatible J, either e_1 is represented by an embedded J-holomorphic two-sphere, or X admits a fibration by embedded J-holomorphic two-spheres in the class e_0-e_1, together with a J-holomorphic section C_0 of odd, negative self-intersection.

This result is the content of Lemma 2.3; there is a corresponding result for $X=\mathbb{S}^2\times\mathbb{S}^2$ given in Lemma 2.4. Section 2 is devoted to the proofs of these lemmas.

With Lemma 2.3 in hand and assuming J is G-invariant, the main task in the proof of Theorems 1.1 and 1.2 is to show that, with the existence of a G-invariant topological or smooth (-1)-sphere, one can force the J-holomorphic section C_0 from Lemma 2.3 to have self-intersection -1 when J is chosen to be a certain generic G-invariant almost complex structure. Sections 3 and 4 are occupied by these discussions, with Section 3 devoted to the case of pseudo-free actions and Section 4 to non-pseudo-free actions which need a different approach.

In Section 5 we extend the techniques developed in Sections 3 and 4, and show that with the existence of a G-invariant smooth $(-r)$-sphere where $r\geq 0$ and relatively small, one can force the J-holomorphic section C_0 from Lemma 2.3 or 2.4 to have self-intersection $-r$ when J is chosen to be a certain generic G-invariant almost complex structure. With this understood, the main task in proving Theorem 1.4 is to use the corresponding equivariant Gromov-Taubes invariant to distinguish G-Hirzebruch surfaces $F_r(a,b)$ and $F_{r+n}(a,b)$ (which have the same fixed-point set structure), showing that for exactly one of the G-Hirzebruch surfaces, the corresponding equivariant Gromov-Taubes invariant is non-vanishing. This is the content of Proposition 5.3.

Finally, we return to our original question about symplectic and smooth minimality of symplectic G-manifolds. In Section 6 we consider the question of minimality of minimal rational G-surfaces in the category of symplectic (resp. smooth or even locally linear topological) G-manifolds. Using some general facts about minimal rational G-surfaces, we show that such a G-surface must be minimal as a topological G-manifold, unless it is a conic bundle with singular fibers or a Hirzebruch surface. Furthermore, specializing to the case of $G = \mathbb{Z}_n$, we show that a minimal G-conic bundle with singular fibers must be minimal as a topological G-manifold (cf. Proposition 6.2). Combining this result with Theorem 1.2, we obtain the following theorem.

Theorem 1.5. *Let $G = \mathbb{Z}_n$ be a cyclic group of order n. Then a minimal rational G-surface is minimal as a symplectic G-manifold if and only if it is minimal as a smooth G-manifold.*

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2. Preliminary lemmas

We first consider the case where $X = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. Fix a basis e_0, e_1 of $H^2(X)$ such that

$$e_0^2 = 1, e_1^2 = -1, \text{ and } e_0 \cdot e_1 = 0.$$

The following lemma shows that such a basis is unique up to a sign change.

Lemma 2.1. *Suppose $f_0, f_1 \in H^2(X)$ such that $f_0^2 = -f_1^2 = 1$. Then*

$$f_0 = \pm e_0, f_1 = \pm e_1.$$

Proof. Write $f_0 = ae_0 + be_1$ for $a, b \in \mathbb{Z}$. Then $1 = f_0^2 = a^2 - b^2 = (a-b)(a+b)$. It follows easily that $a = \pm 1$ and $b = 0$. The claim for f_1 follows similarly. □

Let ω be any given symplectic form on X, and let K_ω be the canonical bundle. Then $c_1^2(K_\omega) = 8$ implies easily that $c_1(K_\omega) = \pm 3e_0 \pm e_1$. Without loss of generality, we assume that $c_1(K_\omega) = -3e_0 + e_1$. Note that with this choice, e_1 is represented by a ω-symplectic (-1)-sphere (cf. [18], Theorem A) which, by Lemma 2.1, is the only such class. In particular, $\omega(e_1) > 0$. We shall also consider the "fiber" class $F = e_0 - e_1$. We claim that $\omega(F) > 0$ as well; in particular, $\omega(e_0) > \omega(e_1)$. To see this, we blow down (X, ω) along the symplectic (-1)-sphere representing e_1, and denote the resulting symplectic four-manifold by (X', ω'). Then X' is a symplectic $\mathbb{CP}^2$, with $c_1(K_{\omega'}) = -3e_0$ where we naturally identify e_0 as a class in X'. Then e_0 can be represented by an embedded, ω'-symplectic two-sphere S in X' passing through the center of the blowdown operation. The proper transform of S in X is an embedded, ω-symplectic two-sphere representing F, hence $\omega(F) > 0$ as claimed.

The following lemma should be well-known to the experts. However, we did not find the exact statements of the lemma in the literature, hence for the sake of completeness, we include it here.

Lemma 2.2. *Let SW_X denote the Seiberg-Witten invariant of (X, ω) defined in the Taubes chamber. Then*

$$SW_X(e_1) = \pm 1, \text{ and } SW_X(F) = \pm 1.$$

Proof. Since $b_1(X) = 0$, the wall-crossing number is ± 1 (cf. [15]). Consequently,

$$|SW_X(e_1) \pm SW_X(K_\omega - e_1)| = 1, \text{ and } |SW_X(F) \pm SW_X(K_\omega - F)| = 1.$$

The lemma follows by showing that $SW_X(K_\omega - e_1) = SW_X(K_\omega - F) = 0$. The key point is that $\omega(K_\omega - e_1) = -3\omega(e_0) < 0$ and

$$\omega(K_\omega - F) = -4\omega(e_0) + 2\omega(e_1) < -2\omega(e_1) < 0,$$

so that if any of $SW_X(K_\omega - e_1)$, $SW_X(K_\omega - F)$ is nonzero, by Taubes' theorem [24] the corresponding class is represented by pseudo-holomorphic curves, contradicting the above negativity of the symplectic areas. The lemma is proved.

□

Now let J be any given ω-compatible almost complex structure on X. By Taubes' theorem [24], there is a finite set of J-holomorphic curves $\{C_i | i \in I\}$, such that $e_1 = \sum_{i \in I} m_i C_i$, where $m_i > 0$.

Lemma 2.3. *One has the following alternative: (1) The set $\{C_i | i \in I\}$ consists of a single element C_0 which is an embedded J-holomorphic two-sphere, and $e_1 = C_0$, or (2) X admits a $\mathbb{S}^2$-fibration over $\mathbb{S}^2$ such that each fiber is an embedded J-holomorphic two-sphere in the class F, and furthermore, the set $\{C_i | i \in I\} = \{C_0\} \sqcup \{C_i | i \in I_0\}$ where C_0 and each C_i are a section and a fiber of the $\mathbb{S}^2$-fibration respectively, and $e_1 = C_0 + \sum_{i \in I_0} m_i C_i$.*

Proof. We consider first the case where for each $i \in I$, $C_i^2 < 0$. If for all $i \in I$, $C_i^2 \leq -2$, then the adjunction formula would imply that $c_1(K_\omega) \cdot C_i \geq -2 - C_i^2 \geq 0$, which would then give

$$c_1(K_\omega) \cdot e_1 = \sum_{i \in I} m_i c_1(K_\omega) \cdot C_i \geq 0.$$

But $c_1(K_\omega) \cdot e_1 = -1$, which is a contradiction. Hence there must be a $C_0 \in \{C_i | i \in I\}$ such that $C_0^2 = -1$. Lemma 2.1 implies that $C_0 = e_1$, from which it follows easily that $\{C_i | i \in I\}$ consists of a single element C_0. The adjunction formula implies that C_0 is an embedded two-sphere. This belongs to case (1).

For case (2) let $I_0 = \{i \in I | C_i^2 \geq 0\}$ and assume that I_0 is nonempty. For each $i \in I_0$, we write $C_i = a_i e_0 - b_i e_1$ for $a_i, b_i \in \mathbb{Z}$, and note that $C_i^2 \geq 0$ is equivalent to $a_i^2 - b_i^2 \geq 0$. On the other hand, since $SW_X(F) = \pm 1$, F can be represented by J-holomorphic curves by Taubes' theorem [24]. By positivity of intersection of J-holomorphic curves, together with the fact that $C_i^2 \geq 0$, $i \in I_0$, we see that $F \cdot C_i \geq 0$ is true for each $i \in I_0$, which can be translated into $a_i - b_i \geq 0$. It follows easily that $a_i + b_i \geq 0$ and $a_i > 0$ for each $i \in I_0$.

Now we set $\Theta = \sum_{i \in I \setminus I_0} m_i C_i$. Note that for each $i \in I_0$, $\Theta \cdot C_i \geq 0$ by the positivity of intersection of J-holomorphic curves. It follows easily that $\Theta^2 < 0$ as $e_1^2 = -1 < 0$.

Writing $\Theta = a_0 e_1 - b_0 e_1$ for some $a_0, b_0 \in \mathbb{Z}$, we then have

$$e_1 = (a_0 + \sum_{i \in I_0} m_i a_i) e_0 - (b_0 + \sum_{i \in I_0} m_i b_i) e_1,$$

which gives $a_0 + \sum_{i \in I_0} m_i a_i = 0$ and $b_0 + \sum_{i \in I_0} m_i b_i + 1 = 0$. Consequently, unless $a_i - b_i = 0$ for all $i \in I_0$, we must have

$$a_0 + b_0 = -\sum_{i \in I_0} m_i (a_i + b_i) - 1 < 0 \text{ and } a_0 - b_0 = -\sum_{i \in I_0} m_i (a_i - b_i) + 1 \leq 0,$$

implying $\Theta^2 = a_0^2 - b_0^2 \geq 0$ which is a contradiction. Hence $C_i = a_i F$ for all $i \in I_0$. By the adjunction inequality, $C_i^2 + c_1(K_\omega) \cdot C_i + 2 \geq 0$, which implies that for each $i \in I_0$, $a_i = 1$ and C_i is an embedded J-holomorphic two-sphere with self-intersection 0.

To prove the rest of the assertions in case (2), we let $\mathcal{M}$ be the moduli space of embedded J-holomorphic two-spheres with self-intersection 0 representing the class F. Then $\mathcal{M}$ is smooth (cf. [22], Lemma 3.3.3) and has dimension 2, and we have just shown that $\mathcal{M} \neq \emptyset$ (in fact, $C_i \in \mathcal{M}$ for each $i \in I_0$). Furthermore, $\mathcal{M}$ must be compact, because if $F = \sum_j m_j' C_j'$ for some J-holomorphic curves C_j' with multiplicity m_j', then for any $i \in I_0$, $0 = F \cdot C_i = \sum_j m_j' C_j' \cdot C_i$. Note that $C_j' \cdot C_i \geq 0$ for all j, from which it follows that $C_j' \cdot C_i = 0$ for all j, and that each C_j' is a positive multiple of F. It follows easily that $\{C_j'\}$ consists of a single element C_j' with $m_j' = 1$. Furthermore, the adjunction formula implies that C_j' is an embedded two-sphere, hence $C_j' \in \mathcal{M}$. By Gromov compactness, $\mathcal{M}$ is compact.

The J-holomorphic curves in $\mathcal{M}$ gives rise to a $\mathbb{S}^2$-fibration over $\mathbb{S}^2$ structure on X. Since $1 = e_1 \cdot F = (\Theta + \sum_{i \in I_0} m_i C_i) \cdot F = \Theta \cdot F$, we see immediately that $\{C_i | i \in I \setminus I_0\}$ consists of a single element C_0 with multiplicity $m_0 = 1$, and $C_0 \cdot F = 1$. The latter implies easily that C_0 is a section of the $\mathbb{S}^2$-fibration on X. Finally, it is clear that $e_1 = C_0 + \sum_{i \in I_0} m_i C_i$. This finishes the proof of the lemma.

□

We remark that it is easily seen that C_0^2 and e_1^2 have the same parity. Consequently, C_0 is an embedded J-holomorphic two-sphere with odd, negative self-intersection.

Lemma 2.3 has an analog in the case of $X = \mathbb{S}^2 \times \mathbb{S}^2$ which will be used in the proof of Theorem 1.4 in Section 5. The proof is similar, so we shall only sketch it here. Let ω be any given symplectic structure on X. Then there is a basis $e_1, e_2 \in H^2(X)$ where $e_1^2 = e_2^2 = 0$ and $e_1 \cdot e_2 = 1$, such that the canonical class $c_1(K_\omega) = -2e_1 - 2e_2$ (cf. [19]). Observe that $[\omega]^2 > 0$ implies that $\omega(e_1)$, $\omega(e_2)$ are non-zero and have the same sign. Together with the fact that $c_1(K_\omega) \cdot [\omega] < 0$, it implies that both $\omega(e_1)$, $\omega(e_2)$ are positive. Finally, an argument involving wall-crossing as in Lemma 2.2 shows that $SW_X(e_i) = \pm 1$ for $i = 1, 2$.

Let J be any given ω-compatible almost complex structure on X. By Taubes' theorem [24], for any $j = 1, 2$, e_j is represented by J-holomorphic curves. Without loss of generality, we only consider the case of e_1. Then there is a finite set of J-holomorphic curves $\{C_i | i \in I\}$ such that $e_1 = \sum_{i \in I} m_i C_i$ for some $m_i > 0$.

Lemma 2.4. *One has the following alternative: (1) The set $\{C_i | i \in I\}$ consists of a single element C_0 which is an embedded J-holomorphic two-sphere, and $e_1 = C_0$, or*

(2) X admits a $\mathbb{S}^2$-fibration over $\mathbb{S}^2$ such that each fiber is an embedded J-holomorphic two-sphere in the class e_2, and furthermore, the set $\{C_i|i \in I\} = \{C_0\} \sqcup \{C_i|i \in I_0\}$ where C_0 and each C_i are a section and a fiber of the $\mathbb{S}^2$-fibration respectively, and $e_1 = C_0 + \sum_{i\in I_0} m_i C_i$.

Proof. We set $I_0 = \{i \in I | C_i^2 \geq 0\}$. Then as we argued in the previous lemma, if $I_0 = \emptyset$, i.e., $C_i^2 < 0$ for all $i \in I$, then in the present case as X is even, $C_i^2 \leq -2$ for all $i \in I$, so that $c_1(K_\omega) \cdot C_i \geq 0$ for all $i \in I$. This would contradict $c_1(K_\omega) \cdot e_1 = -2$, hence we must have $I_0 \neq \emptyset$.

Then there are two possibilities: (1) $I_0 = I$, or (2) $I_0 \neq I$. In the former case, it is easily seen that $C_i^2 = 0$ for all i, as $e_1^2 = 0$, and furthermore, since e_1 is primitive, the set $\{C_i | i \in I\}$ must consist of a single element C_0 such that $e_1 = C_0$. By the adjunction formula, C_0 is an embedded J-holomorphic two-sphere.

In the latter case where $I \setminus I_0 \neq \emptyset$, we set $\Theta = \sum_{i \in I\setminus I_0} m_i C_i$. Then by a similar argument as in Lemma 2.3, we have $\Theta^2 \leq 0$. Moreover, if $\Theta^2 = 0$, we must have $\Theta \cdot C_i = 0$ and $C_i^2 = 0$ for all $i \in I_0$.

To proceed further, for each $i \in I_0$ we write $C_i = a_i e_1 + b_i e_2$ where $a_i, b_i \in \mathbb{Z}$. Then $C_i^2 \geq 0$ is equivalent to $a_i b_i \geq 0$. Now $0 < \omega(C_i) = a_i \omega(e_1) + b_i \omega(e_2)$ implies immediately that $a_i, b_i \geq 0$.

Now we write $\Theta = a_0 e_1 + b_0 e_2$ for some $a_0, b_0 \in \mathbb{Z}$. Then

$$e_1 = (a_0 + \sum_{i \in I_0} m_i a_i) e_1 + (b_0 + \sum_{i\in I_0} m_i b_i) e_2.$$

If there is an $a_i > 0$, then $a_0 = 1 - \sum_{i \in I_0} m_i a_i \leq 0$. On the other hand, $b_0 = -\sum_{i\in I_0} m_i b_i \leq 0$, which implies that $\Theta^2 = 2a_0 b_0 \geq 0$. Since $\Theta^2 \leq 0$, we must have $\Theta^2 = 0$, which means either $a_0 = 0$ or $b_0 = 0$. We claim this is a contradiction. To see it, recall that there is an $a_i > 0$, and $\Theta \cdot C_i = 0$ for all $i \in I_0$. It follows easily that $b_0 = 0$. On the other hand, $0 < \omega(\Theta) = a_0 \omega(e_1)$, so that $a_0 > 0$ must be true. But this contradicts $a_0 = 1 - \sum_{i\in I_0} m_i a_i \leq 0$, hence our claim follows.

This shows that $a_i = 0$ for all $i \in I_0$. Furthermore, as in the proof of Lemma 2.3, the adjunction inequality implies that $b_i = 1$ for all $i \in I_0$. Hence for each $i \in I_0$, $C_i = e_2$ and is an embedded J-holomorphic two-sphere of self-intersection 0.

Similarly, $1 = e_1 \cdot e_2 = \Theta \cdot e_2$ implies that $\{C_i | i \in I \setminus I_0\}$ consists of a single element C_0 with multiplicity $m_0 = 1$, and $C_0 \cdot e_2 = 1$. Furthermore, the existence of C_i, $i \in I_0$, gives rise to a $\mathbb{S}^2$-fibration over $\mathbb{S}^2$ structure on X, where each fiber is an embedded J-holomorphic two-sphere in the class e_2, and C_0 is a section of the $\mathbb{S}^2$-fibration. Finally, we note that $e_1 = C_0 + \sum_{i\in I_0} m_i C_i$. This finishes off the proof.

□

We note that it follows easily from the proof that C_0 is an embedded J-holomorphic two-sphere with even, negative self-intersection.

We end this section with the following remarks. Suppose $X = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is given a smooth G-action and ω is a G-invariant symplectic form on X. We first note that the G-action must be homologically trivial in view of Lemma 2.1 and the fact that $\omega(e_0) > 0$ and $\omega(e_1) > 0$.

Suppose the almost complex structure J in Lemma 2.3 is chosen to be G-invariant. Then in case (1) C_0 must be G-invariant. This is because for any $g \in G$, $(g \cdot C_0) \cdot C_0 = C_0^2 = -1$, so that $g \cdot C_0$ and C_0 can not be distinct J-holomorphic curves. A similar argument shows that in case (2), C_0 is also G-invariant, and the $\mathbb{S}^2$-fibration on X is G-invariant.

Note that Theorems 1.1 and 1.2 follow immediately if Lemma 2.3(1) is true. Hence without loss of generality, we shall assume case (2) of the lemma in the next two sections.

3. Invariant (-1)-spheres of pseudo-free actions

In this section, we give a proof for Theorem 1.1 assuming the action is pseudo-free. Our goal is to show that by picking generic G-invariant J, one could manage to force the J-holomorphic two-sphere C_0 from Lemma 2.3(2) to have self-intersection -1. Note that so far we have not used the assumption that X contains a G-invariant locally linear (-1)-sphere. On the other hand, note also that for a G-Hirzebruch surface F_r with r odd, such a G-invariant fibration with a G-invariant section of odd, negative self-intersection exists automatically. In order to better understand the role played by the G-invariant locally linear (-1)-sphere, we begin with the following example.

Example 3.1. Recall that for an orientation-preserving, locally linear $\mathbb{Z}_n$-action on an oriented four-manifold, the representation on the tangent space at a fixed point is given by a pair of integers mod n after fixing a generator of $\mathbb{Z}_n$. The pair of weights is called the rotation numbers, which is uniquely determined up to a change of order or a simultaneous change of sign. When the $\mathbb{Z}_n$-action preserves an almost complex structure (e.g. being symplectic or holomorphic), the complex structure on the tangent spaces picks up a canonical sign so that the rotation numbers are uniquely determined only up to a change of order in this case.

With the preceding understood, let $F_r(a,b)$ be the G-Hirzebruch surface F_r with a pseudo-free cyclic automorphism group G, such that after fixing an appropriate generator of G, the rotation numbers at the four isolated fixed points are $(a, \pm b)$, $(-a, \pm(b+ra))$, with the second number in each pair standing for the weight in the fiber direction (cf. [26], §4, for the precise definition). We shall consider a pair of examples: $F_1(1,3)$ and $F_{11}(3,1)$, with $G = \mathbb{Z}_7$. Clearly, $F_1(1,3)$ contains a G-invariant holomorphic (-1)-sphere, i.e., the zero-section E_0. Moreover, $F_1(1,3)$ and $F_{11}(3,1)$ have the same set of rotation numbers. To see this, we note that the rotation numbers of $F_1(1,3)$ are $(1,\pm 3)$, $(-1,\pm 4)$, while the rotation numbers of $F_{11}(3,1)$ are $(3,\pm 1)$, $(-3,\pm 34) = (4, \mp 1)$. After a simultaneous change of sign on $(3,-1)$ and $(4,1)$, the rotation numbers of $F_{11}(3,1)$ match up exactly with the rotation numbers of $F_1(1,3)$ as unordered pairs.

We claim that $F_{11}(3,1)$ does not contain any G-invariant locally linear (-1)-sphere (while $F_1(1,3)$ contains a G-invariant holomorphic (-1)-sphere). The reason is that, if $F_{11}(3,1)$ contain a G-invariant locally linear (-1)-sphere, we can equivariantly blow down both $F_{11}(3,1)$ and $F_1(1,3)$ to get two locally linear, pseudo-free $\mathbb{Z}_7$-actions on $\mathbb{CP}^2$ (cf. [12]), which have the same rotation numbers. By Theorem 4.1 in [25], the two $\mathbb{Z}_7$-actions on $\mathbb{CP}^2$ are equivariantly homeomorphic. This then implies that $F_{11}(3,1)$

and $F_1(1,3)$ are equivariantly homeomorphic. However, by Theorem 4.2(2) in [26], $F_1(1,3)$, $F_{11}(3,1)$ are equivariantly diffeomorphic to $F_7(1,3)$ and $F_7(3,1)$ respectively, and by Theorem 4.11(2) in [26], $F_7(1,3)$ and $F_7(3,1)$ are not equivariantly homeomorphic, which is a contradiction.

With the preceding understood, the assumption on the existence of a G-invariant locally linear (-1)-sphere in Theorem 1.1 enters in the proof in such a way that it gives an alternative proof that $F_{11}(3,1)$ does not contain any G-invariant locally linear (-1)-sphere. See Lemma 3.3 and Remark 3.4 for more details.

Let C be the G-invariant locally linear (-1)-sphere in X. Since the G-action is pseudo-free, the induced action on C must be effective and C contains exactly two fixed points of the G-action on X. We shall orient C so that the class of C equals e_1, and with this choice of orientation on C, the rotation numbers of the G-action at the two fixed points contained in C can be written as unordered pairs $(1,a)$ and $(-1,a+1)$ for some $a \in \mathbb{Z}$ mod n after fixing an appropriate generator of G, where the second number in each pair stands for the weight in the normal direction. (Note that no simultaneous change of sign is allowed here as the orientation of C is fixed.) We denote by q_1, q_2 the fixed points whose rotation numbers are $(1,a)$, $(-1,a+1)$ respectively. We shall fix the above generator of G for the rest of this section and in the next one, with all the rotation numbers or weights in reference of this generator of G. Finally, we observe that since the G-action is pseudo-free, the order n of G must be odd as both a and $a+1$ are co-prime to n and one of them is even.

Lemma 3.2. *There is a G-equivariant complex line bundle E over X such that (i) $c_1(E) = e_1$, (ii) the weights of the G-action on the fibers E_{q_1}, E_{q_2} are a and $a+1$ respectively, and are zero at the other fixed points of the G-action.*

Proof. We shall define E in a neighborhood of C first. To this end, we consider the following $\mathbb{Z}_n$-action on $\mathbb{CP}^2$:

$$\mu \cdot [z_0 : z_1 : z_2] = [z_0 : \mu z_1 : \mu^{-a} z_2], \text{ where } \mu = \exp(2\pi/n).$$

Note that the local representations at the fixed points $[1:0:0], [0:1:0]$ are $(1,-a)$ and $(-1,-a-1)$ respectively.

Consider the complex line bundle E on $\mathbb{CP}^2$, where $E = \mathbb{S}^5 \times_{\mathbb{S}^1} \mathbb{C}$, with the $\mathbb{S}^1$-action on $\mathbb{S}^5 \times \mathbb{C}$ given by

$$\lambda \cdot (z_0, z_1, z_2, t) = (\lambda z_0, \lambda z_1, \lambda z_2, \lambda^{-1} t), \quad \lambda \in \mathbb{S}^1, \quad t \in \mathbb{C}.$$

There is a specific lifting of the $\mathbb{Z}_n$-action on $\mathbb{CP}^2$ to E, given on $\mathbb{S}^5 \times \mathbb{C}$ by

$$\mu \cdot (z_0, z_1, z_2, t) = (\mu^a z_0, \mu^{a+1} z_1, z_2, t).$$

It is easy to see that the weight of the action on the fiber at $[1:0:0]$ is a, on the fiber at $[0:1:0]$ is $a+1$, and on the fiber at $[0:0:1]$ is 0.

Now we give $\mathbb{CP}^2$ the opposite orientation and consider E as a G-equivariant complex line bundle over $\overline{\mathbb{CP}^2}$. The G-equivariant section of E given by $(z_0, z_1, z_2) \mapsto (z_0, z_1, z_2, z_2^{-1})$ has a pole at $z_2 = 0$ and defines a trivialization of E in the complement of it. If we fix an orientation-preserving identification between a G-invariant regular neighborhood of C in X and a G-invariant regular neighborhood of $z_2 = 0$ in

$\overline{\mathbb{CP}^2}$, E defines a G-equivariant complex line bundle in a G-invariant neighborhood of C which can be extended trivially and G-equivariantly over the rest of X.

It remains to verify that E has the properties (i) and (ii). The latter is clear from the construction of E. As for (i), we note that E as a complex line bundle over $\mathbb{CP}^2$ admits a non-vanishing section with a pole at the complex line $z_2 = 0$, so that $c_1(E)$ is Poincaré dual to the negative of the class of complex lines in $\mathbb{CP}^2$. After reversing the orientation of the manifold, $c_1(E)$ is Poincaré dual to the class of complex lines, from which it follows easily that $c_1(E) = C = e_1$.

□

The assumption on the existence of a G-invariant locally linear (-1)-sphere comes into play through the following lemma.

Lemma 3.3. *For any G-invariant ω-compatible almost complex structure J on X, the rotation numbers determined by the corresponding complex structure on the tangent spaces are $(1, a)$ and $(-1, a+1)$ at q_1, q_2 respectively, and $(1, -a)$ and $(-1, -a-1)$ at the other two fixed points.*

Proof. We first verify the assertion of the lemma for the fixed points q_1, q_2. We shall accomplish this by computing the virtual dimension of the moduli space of Seiberg-Witten equations associated to E, which is denoted by $d(E)$, and show that it is integral only when the rotation numbers at q_1, q_2 are as claimed.

The formula for $d(E)$ is given in [4], Appendix A (see also [3], Lemma 3.3), according to which

$$d(E) = \frac{1}{n}(c_1(E)^2 - c_1(E) \cdot c_1(K_\omega)) + I_{q_1} + I_{q_2} = I_{q_1} + I_{q_2},$$

where I_{q_1}, I_{q_2} are contributions from the fixed points q_1, q_2. There are no contributions from the other fixed points because the weights of the G-action on the corresponding fibers of E are zero by Lemma 3.2.

Suppose the rotation numbers at q_1 are $(1, a)$, then I_{q_1} is given by

$$I_{q_1} = \frac{1}{n} \sum_{x=1}^{n-1} \frac{2(\mu^{ax} - 1)}{(1 - \mu^{-x})(1 - \mu^{-ax})}, \text{ where } \mu = \exp(2\pi/n).$$

Similarly, if the rotation numbers at q_2 are $(-1, a+1)$, then

$$I_{q_2} = \frac{1}{n} \sum_{x=1}^{n-1} \frac{2(\mu^{(a+1)x} - 1)}{(1 - \mu^{x})(1 - \mu^{-(a+1)x})}.$$

One can easily check that $I_{q_1} = -I_{q_2}$, and it follows that $d(E) = 0$ in this case.

Suppose the rotation numbers at q_1 are $(-1, -a)$ instead, then

$$I_{q_1} = \frac{1}{n} \sum_{x=1}^{n-1} \frac{2(\mu^{ax} - 1)}{(1 - \mu^{x})(1 - \mu^{ax})} = -\frac{2}{n} \sum_{x=1}^{n-1} \frac{1}{(1 - \mu^{x})} = -\frac{n-1}{n}.$$

If the rotation numbers at q_2 are still $(-1, a+1)$, then (cf. [3], Example 3.4)

$$I_{q_2} = \frac{1}{n} \sum_{x=1}^{n-1} \frac{2(\mu^{(a+1)x} - 1)}{(1 - \mu^{x})(1 - \mu^{-(a+1)x})} = \frac{1}{n} \sum_{x=1}^{n-1} \frac{2\mu^{(a+1)x}}{1 - \mu^{x}} = \frac{-(n-1) + 2a}{n},$$

where a is the unique integer satisfying $0 \leq a < n$ for the given congruence mod n class. With this,

$$d(E) = I_{q_1} + I_{q_2} = -\frac{n-1}{n} + \frac{-(n-1)+2a}{n} = \frac{2(a+1-n)}{n},$$

which is non-integral because n is odd and $a+1 \neq 0 \pmod n$. One can similarly verify that in all other cases, i.e., when the rotation numbers are $(-1,-a),(1,-a-1)$, or $(1,a),(1,-a-1)$, $d(E)$ is non-integral. This proves the assertion for q_1, q_2.

For the rest of the fixed points, we use the same strategy but with consideration of some different G-equivariant complex line bundles. Recall that, as we argued in Example 3.1, it is easily seen that the existence of the G-invariant locally linear (-1)-sphere C implies that X is equivariantly homeomorphic to the G-Hirzebruch surface $F_1(1,a)$ such that the class $e_1 \in H^2(X)$ is sent to the class of the (-1)-section in $F_1(1,a)$ under the equivariant homeomorphism. Furthermore, since the complex conjugation on $\mathbb{CP}^2$ defines an orientation-preserving involution τ which acts as -1 on the second cohomology, it follows easily that, with a further application of τ if necessary, one can arrange to have the class $e_0 \in H^2(X)$ sent to the class of the $(+1)$-section in $F_1(1,a)$. Consequently, the fiber class $F = e_0 - e_1 \in H^2(X)$ is sent to the fiber class of $F_1(1,a)$ under the equivariant homeomorphism.

With the preceding understood, we consider the G-equivariant complex line bundle L on X defined as follows. Let $\pi : F_1(1,a) \to B = \mathbb{S}^2$ be the holomorphic $\mathbb{S}^2$-fibration, and let F_1, F_2 be the two invariant fibers containing the fixed points with rotation numbers $(1,\pm a)$, $(-1,\pm(a+1))$ respectively, and let $b_i = \pi(F_i) \in B$, $i = 1,2$. Note that the fixed point q_i, for $i = 1,2$, is contained in the preimage of F_i in X; denote the other fixed point contained in the preimage of F_i by q_i', $i = 1,2$.

There is a G-equivariant complex line bundle L' on B, such that the weight of the G-action on the fiber L'_{b_1} equals $+1$ and the weight on the fiber L'_{b_2} equals 0, and L' has degree 1. With this understood, the G-bundle L on X is the pull-back of $\pi^*(L')$ via the equivariant homeomorphism from X to $F_1(1,a)$; it has weight $+1$ on the fibers at q_1, q_1', and weight 0 on the fibers at q_2, q_2'. Moreover, $c_1(L) = F$.

The virtual dimension of the moduli space of Seiberg-Witten equations associated to L is

$$d(L) = \frac{1}{n}(c_1(L)^2 - c_1(L)\cdot c_1(K_\omega)) + I_{q_1} + I_{q_1'} = \frac{2}{n} + I_{q_1} + I_{q_1'},$$

where $I_{q_1}, I_{q_1'}$ are contributions from the fixed points q_1, q_1'. There are no contributions from q_2, q_2' because the weights of the G-action on the corresponding fibers of L are zero. Since the rotation numbers at q_1 are $(1,a)$ and the weight of the G-action is $+1$ on L_{q_1},

$$I_{q_1} = \frac{1}{n}\sum_{x=1}^{n-1} \frac{2(\mu^x - 1)}{(1-\mu^{-x})(1-\mu^{-ax})} = \frac{2}{n}\sum_{x=1}^{n-1} \frac{\mu^x}{1-\mu^{-ax}} = \frac{n-1-2b}{n},$$

where $ab = 1 \pmod n$ and $0 < b < n$ (cf. [3], Example 3.4). Since $(1,-a)$ and $(-1,a)$ are the same as unordered pairs when $a = 1 \pmod n$, we shall assume without loss of generality that $a \neq 1 \pmod n$ in the calculations below.

Suppose the rotation numbers at q_1' are $(1,-a)$. Then with the weight of the G-action being $+1$ on $L_{q_1'}$, we have

$$I_{q_1'} = \frac{1}{n}\sum_{x=1}^{n-1} \frac{2(\mu^x-1)}{(1-\mu^{-x})(1-\mu^{ax})} = \frac{2}{n}\sum_{x=1}^{n-1}\frac{\mu^x}{1-\mu^{ax}} = \frac{n-1-2b'}{n},$$

where $ab' = -1 \pmod n$ and $0 < b' < n$ (cf. [3], Example 3.4). Observing that $b + b' = 0 \pmod n$, we have

$$d(L) = \frac{2}{n} + \frac{n-1-2b}{n} + \frac{n-1-2b'}{n} = \frac{2(n-b-b')}{n} \in \mathbb{Z}.$$

However, if the rotation numbers at q_1' are $(-1,a)$ instead, we have

$$I_{q_1'} = \frac{1}{n}\sum_{x=1}^{n-1} \frac{2(\mu^x-1)}{(1-\mu^{x})(1-\mu^{-ax})} = -\frac{2}{n}\sum_{x=1}^{n-1}\frac{1}{1-\mu^{-ax}} = -\frac{n-1}{n}.$$

In this case,

$$d(L) = \frac{2}{n} + \frac{n-1-2b}{n} - \frac{n-1}{n} = \frac{2(1-b)}{n},$$

which is non-integral because $b \neq 1$. Hence the rotation numbers at q_1' must be $(1,-a)$. A similar argument proves that the rotation numbers at q_2' must be $(-1,-a-1)$, which finishes off the proof.

□

Remark 3.4. If we apply Lemma 3.3 to $F_{11}(3,1)$, we see immediately that $F_{11}(3,1)$ contains no G-invariant locally linear (-1)-spheres, because the rotation numbers are

$$(3,1), (3,-1), (4,-1), (4,1).$$

With $a = 3$, we find that the second and the last pairs have the wrong sign.

With the above preparation on the rotation numbers, we now prove that, by assuming J is generic, the J-holomorphic section C_0 of the $\mathbb{S}^2$-fibration in Lemma 2.3(2) must be a (-1)-sphere.

To this end, we recall the formula for the virtual dimension of the moduli space of J-holomorphic curves in a 4-orbifold, applied here to the curve C_0/G in the 4-orbifold X/G. Let $f : \Sigma \to X/G$ be a J-holomorphic parametrization of C_0/G, where Σ is the orbifold Riemann two-sphere with two orbifold points z_i, $i = 1, 2$, of order n, and let $(\hat{f}_{z_i}, \rho_{z_i}) : (D_i, \mathbb{Z}_n) \to (V_i, G)$ be a local representative of f at z_i, where the action of $\rho_{z_i}(\mu)$ on V_i is given by

$$\rho_{z_i}(\mu)\cdot(w_1, w_2) = (\mu^{m_{i,1}}w_1, \mu^{m_{i,2}}w_2), \text{ with } \mu = \exp(2\pi/n),\ 0 \leq m_{i,1}, m_{i,2} < n.$$

Then the virtual dimension of the moduli space of J-holomorphic curves at C_0/G is given by $2d$, where

$$d = -\frac{c_1(K_\omega)\cdot C_0}{n} + 1 - \sum_{i=1}^{2}\frac{m_{i,1}+m_{i,2}}{n}.$$

See [8], Lemma 3.2.4. With this understood, we recall that the transversality theorem (cf. [3], Lemma 1.10) for the moduli space of J-holomorphic curves implies that $d \geq 0$ for a generic J.

For our purpose, we would like to express d in terms of the self-intersection number of C_0. To this end, we apply the adjunction formula ([2], Theorem 3.1) to C_0/G, and as C_0 is embedded (equivalently, C_0/G is embedded), we obtain

$$\frac{1}{2n}(C_0^2 + c_1(K_\omega) \cdot C_0) + 1 = 2(\frac{1}{2} - \frac{1}{2n}) = 1 - \frac{1}{n},$$

which gives the desired expression for d:

$$d = \frac{1}{n}C_0^2 + \frac{2}{n} + 1 - \sum_{i=1}^{2} \frac{m_{i,1} + m_{i,2}}{n}$$

In order to compute d, we need to locate the two fixed points on C_0. By looking at the rotation numbers, it is clear that q_1, q_1' and q_2, q_2' are contained in two distinct invariant fibers of the $\mathbb{S}^2$-fibration in Lemma 2.3(2). It follows easily that the following are the only possibilities for the fixed points on C_0:

$$\text{(a) } q_1, q_2 \in C_0; \quad \text{(b) } q_1, q_2' \in C_0; \quad \text{(c) } q_1', q_2 \in C_0; \quad \text{(d) } q_1', q_2' \in C_0.$$

With this understood, the weights $(m_{i,1}, m_{i,2})$, $i = 1, 2$, in the formula for d can be read off from the rotation numbers (since C_0 is embedded and is a section) and are given correspondingly as follows:

(a) $(m_{1,1}, m_{1,2}) = (1, a)$, $(m_{2,1}, m_{2,2}) = (1, n - a - 1)$;
(b) $(m_{1,1}, m_{1,2}) = (1, a)$, $(m_{2,1}, m_{2,2}) = (1, a + 1)$;
(c) $(m_{1,1}, m_{1,2}) = (1, n - a)$, $(m_{2,1}, m_{2,2}) = (1, n - a - 1)$;
(d) $(m_{1,1}, m_{1,2}) = (1, n - a)$, $(m_{2,1}, m_{2,2}) = (1, a + 1)$

where a satisfies the inequality $0 < a < n - 1$. Correspondingly, we have

$$(a)\ d = \frac{C_0^2 + 1}{n},\ (b)\ d = \frac{C_0^2 + n - 2a - 1}{n},\ (c)\ d = \frac{C_0^2 + 2a + 1 - n}{n},\ (d)\ d = \frac{C_0^2 - 1}{n}.$$

If J is generic, we have $d \geq 0$, so that with the fact that $C_0^2 < 0$, we obtain

$$(a)\ C_0^2 = -1,\ (b)\ C_0^2 = -n + 2a + 1,\ (c)\ C_0^2 = -2a - 1 + n,$$

and case (d) is a contradiction. Furthermore, since n is odd, (b) and (c) can be ruled out by the fact that C_0^2 is odd (cf. Lemma 2.3(2)). This shows that C_0 is a G-invariant J-holomorphic (-1)-sphere when J is generic. Thus Theorem 1.1 is proved for the case of pseudo-free actions.

4. Invariant (-1)-spheres of non-pseudo-free actions

For non-pseudo-free actions, the argument in the previous section broke down in a couple of places. One of them is Lemma 3.3, where a theorem of Wilczynski [25] asserting that a pseudo-free locally linear action on $\mathbb{CP}^2$ is equivalent to a linear action was used. Perhaps the most serious obstacle in the non-pseudo-free case is the failure of the argument for ruling out the cases (b) and (c) at the end of the proof. The argument relies on the fact that the order n of G is odd, which is no longer true for a non-pseudo-free action in general.

In this section, we shall give a different proof for Lemma 3.3 for non-pseudo-free actions, and rescue the argument of ruling out (b) and (c) for the case of $n > 2$ and even by exploiting the smoothness assumption of the invariant (-1)-sphere C.

For the first part, we continue to assume that C is a G-invariant locally linear (-1)-sphere in X. If there exists a $g \in G$ which acts trivially on C, then C, as a 2-dimensional fixed component of g, is naturally a smooth, ω-symplectic (-1)-sphere, and Theorems 1.1 and 1.2 are trivially true. So without loss of generality, we assume the induced action of G on C is effective. Then as in the previous section, C contains exactly two fixed points of the G-action on X. We shall orient C so that the class of C equals e_1, and with this choice of orientation on C, the rotation numbers of the G-action at the two fixed points contained in C, continued to be denoted by q_1, q_2, can be written as unordered pairs $(1, a)$ and $(-1, a+1)$ for some $a \in \mathbb{Z}$ mod n after fixing an appropriate generator $\mu \in G$, with the second number in each pair standing for the weight in the normal direction. The only difference from the pseudo-free case is that the weights a, $a+1$ are no longer required to be co-prime to n, and consequently, n may be an even integer in this case. Finally, since the pseudo-free case has been dealt with in the previous section, we shall consider exclusively the non-pseudo-free case, i.e., there is a 2-dimensional fixed component Σ of some element $\kappa \in G$.

Our first observation is that the induced G-action on the base $\mathbb{S}^2$ of the $\mathbb{S}^2$-fibration from Lemma 2.3(2) must be effective. To see this, suppose to the contrary that there is an element $h \in G$ which acts trivially on the base $\mathbb{S}^2$. Then h must fix two J-holomorphic sections, denoted by E_0, E_∞, of the $\mathbb{S}^2$-fibration. Furthermore, it is easy to see that every fixed point of G is contained in E_0 or E_∞; in particular, $q_1, q_2 \in E_0 \cup E_\infty$. Now note that the weights of the action of h at q_1, q_2 can be written as (k, ka) and $(-k, k(a+1))$ for some $k \neq 0 \pmod n$. However, as $q_1, q_2 \in E_0 \cup E_\infty$, we must have $ka = k(a+1) = 0 \pmod n$, which implies $k = 0 \pmod n$, a contradiction. Hence the claim follows. As a consequence, the $\mathbb{S}^2$-fibration has exactly two G-invariant fibers, with Σ being one of them; in particular, $\Sigma = F = e_0 - e_1$. We denote the other G-invariant fiber by Σ'.

Before we proceed further, recall that the intersection theory works for locally flat, topologically embedded surfaces in a topological 4-manifold (cf. Freedman-Quinn [13]). With this understood, observe that $C \cdot \Sigma = e_1 \cdot F = 1$, so that $C \cap \Sigma \neq \emptyset$. This implies that either q_1 or q_2 must be contained in Σ. Without loss of generality, we assume $q_1 \in \Sigma$. Writing $\kappa = \mu^k$ where μ is the fixed generator of G, we see that the weights of the action of κ at q_1 are (k, ka), implying $ka = 0 \pmod n$. Consequently, the weights of the action of κ at q_2 are $(-k, k(a+1)) = (-k, k)$, which implies that q_2 is not contained in Σ. Hence q_1 is the only intersection point of Σ and C. We further notice that the action of μ on the complex vector space $(T_{q_1}X, J)$ has two distinct eigenspaces, which are $T_{q_1}C$ and $T_{q_1}\Sigma$, so that C and Σ must intersect transversely and positively at q_1. (Here we used the fact that C is locally linear.) It follows easily that with respect to the complex structure determined by J, the rotation numbers at q_1 are $(1, a)$, with the second number a being the weight in the fiber direction. Finally, note that q_2 must be contained in Σ'.

It turns out that for the rest of the arguments, it is more convenient to divide the discussions according to the following two scenarios :

(i) Neither Σ nor Σ' is fixed by G, i.e., $0 < a < n-1$; in particular, $n \neq 2$.
(ii) Either Σ or Σ' is fixed by G, i.e., either $a = 0$ or $a = n-1$.

Case (i): Neither Σ nor Σ' is fixed by G. In this case, each of Σ, Σ' contains another fixed point of G, which we continue to denote by q_1', q_2' respectively. Since Σ has a trivial normal bundle in X, it follows easily that the rotation numbers at q_1' are $(1,-a)$ with respect to the complex structure determined by J. It remains to determine the rotation numbers at q_2, q_2' with respect to the complex structure determined by J.

Let $\pi : X \to B = \mathbb{S}^2$ be the $\mathbb{S}^2$-fibration, and set $b = \pi(\Sigma)$, $b' = \pi(\Sigma') \in B$. Note that B has a natural orientation determined by the orientation of X and the orientation of the fibers given by J. With respect to this orientation, the induced G-action on B has weight $+1$ at b, so that the weight at b' must be -1. Consequently, with respect to the complex structure determined by J, the weight of the G-action on X must be -1 in the normal direction of the G-invariant fiber Σ'. With this understood, we claim that the weight of the G-action at q_2 must be $a+1$ in the fiber direction. To see this, note that the rotation numbers at q_2 determined by J are either $(-1, a+1)$ or $(1, -a-1)$. Our claim is clear in the former case. Assume the latter is true. If $n > 2$, then we must have $-a-1 = -1 \pmod n$, which gives $a = 0 \pmod n$, and furthermore, the weight in the fiber direction must be 1, which equals $a+1 \pmod n$. If $n = 2$, then $a = 0 \pmod 2$ must be true. Moreover, the weight in the fiber direction is 1, which can be also written as $a+1 \pmod 2$. This shows that the rotation numbers at q_2 are $(-1, a+1)$ with the second entry being the weight in the fiber direction. Finally, it follows easily that the rotation numbers at q_2' are $(-1, -a-1)$.

The following lemma summarizes the discussion, which corresponds to Lemma 3.3.

Lemma 4.1. *For any G-invariant ω-compatible almost complex structure J on X, the rotation numbers determined by the corresponding complex structure on the tangent spaces are $(1,a)$ and $(-1,a+1)$ at q_1, q_2, and $(1,-a)$ and $(-1,-a-1)$ at the other two fixed points q_1', q_2' respectively.*

With this in hand, one can argue similarly as in the pseudo-free case that for a generic G-invariant J, the self-intersection number of C_0 falls into three possibilities:

$$(a)\ C_0^2 = -1,\ (b)\ C_0^2 = -n + 2a + 1, \text{ or } (c)\ C_0^2 = -2a - 1 + n$$

according to (a) $q_1, q_2 \in C_0$, (b) $q_1, q_2' \in C_0$, or (c) $q_1', q_2 \in C_0$. We finish off the proof of Theorem 1.1 in case (i) (i.e., neither Σ nor Σ' is fixed by G) by observing that $n \neq 2$ so that n is odd.

For Theorem 1.2 (continuing with the assumption that neither Σ nor Σ' is fixed by G), we shall only consider the case where $n > 2$ and even, and assume that the G-invariant (-1)-sphere C is smoothly embedded. The smoothness assumption on C will play an essential role in our argument. First, we observe

Lemma 4.2. *Let $n > 2$ be an even integer. Suppose a linear $\mathbb{Z}_n$-action on $\mathbb{R}^4$ preserves a complex structure J on $\mathbb{R}^4$. Then every $\mathbb{Z}_n$-invariant 2-dimensional subspace of $\mathbb{R}^4$ is J-invariant.*

Proof. If the $\mathbb{Z}_n$-action on the complex vector space $(\mathbb{R}^4, J)$ has two distinct eigenspaces (i.e., with distinct weights), then the corresponding 2-dimensional real subspaces are the only ones which are $\mathbb{Z}_n$-invariant. The lemma follows easily in this case.

Suppose the $\mathbb{Z}_n$-action on $(\mathbb{R}^4, J)$ is given by a complex scalar multiplication (i.e., with the same weight). Then one can choose appropriate coordinates so that a generator g of $\mathbb{Z}_n$ and the complex structure J may be represented by matrices

$$g = \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} & 0 & 0 \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & 0 & 0 \\ 0 & 0 & \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ 0 & 0 & \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} \text{ and } J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let L be any $\mathbb{Z}_n$-invariant 2-dimensional subspace of $\mathbb{R}^4$. If $n = 4m$, then $J = g^m$ so that for any vector $v \in L$, $Jv = g^m \cdot v \in L$. If $n = 4m+2$, then for any vector $v \in L$, $g \cdot v + g^{2m} \cdot v = 2\sin\frac{2\pi}{n} \cdot Jv$. Since $\sin\frac{2\pi}{n} \neq 0$ as $n > 2$, one has $Jv \in L$ as well. This proves that L is J-invariant.

□

As a corollary of Lemma 4.2, we see that in the case of $n > 2$ and even, the tangent spaces of C at q_1, q_2 are J-invariant for any G-invariant ω-compatible almost complex structure J on X. Furthermore, since C intersects Σ positively, the symplectic form ω is positive on C near q_1. We claim that the same is true at q_2 if the intersection of C and Σ' is transverse. To see this, suppose the intersection of C and Σ' is transverse and negative at q_2. Then note that the G-action on $C \cap \Sigma' \setminus \{q_2\}$ is free so that by a small, equivariant perturbation of C supported in the complement of $\{q_2\}$, one may assume C and Σ' intersect transversely. It follows then that

$$1 = e_1 \cdot F = C \cdot \Sigma' = -1 \pmod{n},$$

contradicting the assumption $n > 2$. Hence the claim follows.

The above discussion has the following consequence: for any fixed G-invariant ω-compatible almost complex structure J_0 on X, we can fix a G-invariant, ω-compatible almost complex structure $\hat{J}_0$ in a neighborhood of q_1, q_2 with the following significance:

- $\hat{J}_0$ agrees with J_0 at q_1, q_2;
- C is $\hat{J}_0$-holomorphic near q_1, and when C intersects Σ' transversely, it is also $\hat{J}_0$-holomorphic near q_2.

Lemma 4.3. *Let J_0 be any given G-invariant ω-compatible almost complex structure. Then for any G-invariant J which equals $\hat{J}_0$ in a neighborhood of q_1, q_2, the intersection number of C with the J-holomorphic section C_0 satisfies the following congruence equation*

- $C \cdot C_0 = a \pmod{n}$ *if* $q_1, q_2' \in C_0$*;*
- $C \cdot C_0 = -a - 1 \pmod{n}$ *if* $q_1', q_2 \in C_0$*.*

Proof. We first consider the case where $q_1, q_2' \in C_0$. The local intersection number of C and C_0 at q_1, both being $\hat{J}_0$-holomorphic near q_1, can be determined as follows. By work of Micallef and White (cf. [23], Theorems 6.1 and 6.2), for any C or C_0, there is a C^1-coordinate chart $\Psi : V \to \mathbb{C}^2$ near q_1, such that the J-holomorphic curve may be parametrized by a holomorphic map of the form $z \mapsto (z, f(z))$ (here we also use the fact that C and C_0 are embedded near q_1). Furthermore, since C and C_0 are tangent at q_1, there are $f(z), f_0(z)$ such that C, C_0 are parametrized by $z \mapsto (z, f(z))$, $z \mapsto (z, f_0(z))$ with respect to the same chart. There are two possibilities: (1) C, C_0 are distinct near q_1, (2) $C \equiv C_0$ near q_1. In case (1), the local intersection number of C, C_0 equals the order of vanishing of $f(z) - f_0(z)$ at $z = 0$ (cf. [23]), which equals $a \pmod n$ because C, C_0 are G-invariant and the weight of the G-action in the normal direction equals a. In case (2), the local intersection number is not well-defined. However, we may remedy this by performing a small, equivariant perturbation to C so that it is represented by the graph of $z \mapsto g(z)$ near q_1 for some $g(z) \neq f(z)$. Then the local intersection number becomes well-defined and it equals $a \pmod n$.

On the other hand, the G-action on $C \cap C_0 \setminus \{q_1\}$ is free, so that by a small, equivariant perturbation of C away from q_1, one can arrange C and C_0 intersect transversely, so that the contribution to $C \cdot C_0$ that are from the intersection points other than q_1 equals $0 \pmod n$. The lemma follows easily for this case.

When $q_1', q_2 \in C_0$, the lemma follows by the same argument if C and Σ' intersect transversely. Suppose C and Σ' are tangent at q_2. Then the weights of the G-action on $T_{q_2}X$ must be the same in all complex directions; in particular, $a + 1 = -1 \pmod n$. On the other hand, since $n > 2$ and the weight of the G-action on $T_{q_2}C$ and $T_{q_2}\Sigma'$ is the same (which is -1), the orientations on $T_{q_2}C$ and $T_{q_2}\Sigma'$ must coincide. Consequently, the local intersection number of C and C_0 at q_2 must be $+1$ because C_0 is a J-holomorphic section. Then it follows that $C \cdot C_0 = 1 \pmod n$ as we argued in the previous case. But this is the same as $C \cdot C_0 = -a - 1 \pmod n$ because $a + 1 = -1 \pmod n$. Hence the lemma follows.

□

The following lemma finishes off the proof of Theorem 1.2 in case (i), i.e., neither Σ nor Σ' is fixed by G.

Lemma 4.4. *For any fixed G-invariant J_0, the J-holomorphic section C_0 is a (-1)-sphere for any generic G-invariant J which equals $\hat{J}_0$ in a neighborhood of q_1, q_2.*

Proof. Let $\delta = C \cdot C_0$. Then the fact that $C = e_1$, $F = e_0 - e_1$ and $C_0 \cdot F = 1$ implies that $C_0 = (\delta+1)e_0 - \delta e_1$, which then gives $C_0^2 = 2\delta + 1$. By Lemma 4.3, $\delta = a \pmod n$ if $q_1, q_2' \in C_0$, and $\delta = -a - 1 \pmod n$ if $q_1', q_2 \in C_0$. Consequently, $C_0^2 = 2a + 1 \pmod{2n}$ in the former case, and $C_0^2 = -2a - 1 \pmod{2n}$ in the latter case.

Now we recall that the transversality theorem for moduli space of J-holomorphic curves continues to hold even if we consider a more restrictive class of G-invariant almost complex structures J, i.e., those which equal to $\hat{J}_0$ in a fixed neighborhood of q_1, q_2, because no J-holomorphic curves can lie completely in such a neighborhood. Consequently, for a generic such J, we continue to have $C_0^2 = -n + 2a + 1$ if $q_1, q_2' \in C_0$ and $C_0^2 = -2a - 1 + n$ if $q_1', q_2 \in C_0$. But this contradicts what was obtained in the previous paragraph, hence $C_0^2 = -1$ must be true. This finishes off the proof.

□

Case (ii): Either Σ or Σ' is fixed by G. Without loss of generality, we shall only consider the case where Σ is fixed, which corresponds to $a = 0$; the other case is completely analogous.

We note that in this case, the fixed point q_1' is not well-defined. However, q_2' is well-defined and the rotation numbers at q_2, q_2' continue to be $(-1, a+1) = (-1, 1)$, $(-1, -a-1) = (-1, -1)$ respectively (with respect to almost complex structure J).

Let x, y be the two fixed points on C_0, where $x \in \Sigma$, and $y = q_2$ or q_2'. Then it follows easily that the weights $(m_{1,1}, m_{1,2})$ at x equals $(1, 0)$ and the weights $(m_{2,1}, m_{2,2})$ at y equals $(1, n-1)$ or $(1, 1)$, depending on whether $y = q_2$ or q_2'. Correspondingly, when J is a generic G-invariant almost complex structure, we have

$$C_0^2 = -1 \text{ if } y = q_2, \text{ and } C_0^2 = -n+1 \text{ if } y = q_2'.$$

Theorem 1.1 follows immediately, observing that if $n = 2$ or n is odd, C_0^2 must be -1.

For Theorem 1.2 where $n > 2$ and even, we can eliminate the case where $y = q_2'$ as follows. If $x = q_1$, then Lemma 4.3 still applies and we get $C \cdot C_0 = a = 0 \pmod{n}$. If $x \neq q_1$, then $C \cdot C_0 = 0 \pmod{n}$ holds automatically. Then, in any event, we have, as in Lemma 4.4, $C_0^2 = 1 \pmod{2n}$ which contradicts $C_0^2 = -n+1$. This finishes off the proof for Theorem 1.2.

5. Smooth classification of G-Hirzebruch surfaces

We fix a generator $\mu \in G = \mathbb{Z}_n$, and let $F_r(a, b)$ be the Hirzebruch surface F_r equipped with a homologically trivial, holomorphic G-action with the following fixed-point set structure. Note that such a G-action has two invariant fibers, denoted by F_0, F_1, and leaves the zero-section E_0 and the infinity-section E_1 invariant also. (Here our convention is from [26] that $E_0 \cdot E_0 = -r$.) We set $x_{ij} = F_i \cap E_j$ for $i, j = 0, 1$, which are fixed points of the G-action. With this understood, the integers $(a, b) \pmod{n}$ are the rotation numbers at x_{00} with respect to the complex structure on F_r, with the second number in the pair being the weight of the action in the fiber direction. The rotation numbers at the other fixed points x_{01}, x_{10}, x_{11} are $(a, -b)$, $(-a, b + ra)$, and $(-a, -b - ra)$. See Wilczynski [26], Theorem 4.1. We note that the integers a, b, n must satisfy $\gcd(a, b, n) = 1$, and furthermore,

- $\gcd(a, n) = 1$ if and only if E_0, E_1 have trivial isotropy;
- $\gcd(b, n) = 1$ if and only if F_0 has trivial isotropy; and
- $\gcd(b + ra, n) = 1$ if and only if F_1 has trivial isotropy.

Let $F_{r'}(a', b')$ be another G-Hirzebruch surface with the corresponding invariant fibers and sections and the fixed points denoted by F_i', E_i', and x_{ij}' respectively. Under appropriate numerical conditions on (a, b, r) and (a', b', r'), there are six types, $c_1, c_2, \cdots, c_6$, of canonically defined, orientation-preserving, equivariant diffeomorphisms between $F_r(a, b)$ and $F_{r'}(a', b')$, which we describe below. (See also related discussions in Wilczynski [26].)

Type c_1. Suppose $a' = -a$, $b' = -b$, and $r' = r$. Then there is an equivariant diffeomorphism $c_1 : F_r(a, b) \to F_{r'}(a', b')$, which sends F_i to F_i', E_i to E_i', and which induces $z \mapsto \bar{z}$ between the bases and the fibers of F_r and $F_{r'}$.

Type c_2. Suppose $a'=-a$, $b'=b+ra$, and $r'=r$. Then there is an equivariant diffeomorphism $c_2: F_r(a,b)\rightarrow F_{r'}(a',b')$, which sends F_0 to F_1' and F_1 to F_0', E_i to E_i', and which induces $z\mapsto z^{-1}$ between the bases of F_r and $F_{r'}$.

Type c_3. Suppose $a'=a$, $b'=-b$, and $r'=-r$. Then there is an equivariant diffeomorphism $c_3: F_r(a,b)\rightarrow F_{r'}(a',b')$, which sends F_i to F_i', E_0 to E_1', E_1 to E_2', and which induces $z\mapsto z^{-1}$ between the fibers of F_r and $F_{r'}$.

Type c_4. Suppose $r'=r=0$, $a'=b$, and $b'=a$. Then there is an equivariant diffeomorphism $c_4: F_r(a,b)\rightarrow F_{r'}(a',b')$, which switches the fibers and sections between F_r and $F_{r'}$, sending F_0 to E_0', F_1 to E_1', and E_0 to F_0', E_1 to F_1'.

For the types c_5, c_6, we assume that $\gcd(a,n)=\gcd(a',n)=1$.

Type c_5. Suppose $a'=a$, $b'=b$, and $r'=r \pmod{2n}$. Then there is an equivariant diffeomorphism $c_5: F_r(a,b)\rightarrow F_{r'}(a',b')$, sending the fixed points x_{ij} to x_{ij}', $i,j=0,1$. To see this, we shall explain that there is a diffeomorphism between the quotient orbifolds, $\hat{c}_5: F_r(a,b)/G\rightarrow F_{r'}(a',b')/G$, which are orbifold $\mathbb{S}^2$-bundles over an orbifold $\mathbb{S}^2$ with two singular points z_0, z_1 of order n. Moreover, the orbifold $\mathbb{S}^2$-bundles are induced, under the canonical embedding $\mathbb{S}^1\subset SO(3)$, from the principal orbifold $\mathbb{S}^1$-bundles of Euler number $-r/n$, $-r'/n$ respectively. Since $\gcd(a,n)=\gcd(a',n)=1$, we may assume without loss of generality that $a=a'=1$. Then the Seifert invariants of the two bundles, which are the same, are (n,β_0), (n,β_1) at z_0, z_1 where $\beta_0=b \pmod{n}$, $\beta_1=-b-r \pmod{n}$. With this understand, let $e, e'\in\mathbb{Z}$ such that

$$-\frac{r}{n}=\frac{\beta_0}{n}+\frac{\beta_1}{n}+e,\quad -\frac{r'}{n}=\frac{\beta_0}{n}+\frac{\beta_1}{n}+e',$$

then $r'-r=(e-e')\cdot n$, which implies $e'=e \pmod 2$ because $r'=r \pmod{2n}$. Since $\pi_1 SO(3)=\mathbb{Z}_2$, $e'=e \pmod 2$ implies that the two orbifold $\mathbb{S}^2$-bundles are isomorphic, which gives the diffeomorphism $\hat{c}_5: F_r(a,b)/G\rightarrow F_{r'}(a',b')/G$.

Type c_6. Suppose $a'=a$, $b'=b$, and $r'a'=-2b-ra \pmod{2n}$. Then there is an equivariant diffeomorphism $c_6: F_r(a,b)\rightarrow F_{r'}(a',b')$, sending the fixed points x_{0j} to x_{0j}', $j=0,1$, and x_{10} to x_{11}', x_{11} to x_{10}'. To see this, note that switching x_{10} and x_{11} means applying $z\mapsto z^{-1}$ to a neighborhood of F_1, which has the effect of changing the sign of β_1 in the Seifert invariant. Therefore, there is a diffeomorphism from $F_r(a,b)$ to $F_{\tilde{r}}(a,b)$ which switches x_{10} and x_{11}, where $\tilde{r}$ satisfies

$$-\frac{\tilde{r}}{n}=\frac{\beta_0}{n}+\frac{-\beta_1}{n}+e.$$

It follows that $\tilde{r}-r=2\beta_1=-2(b+r) \pmod{2n}$, which gives $\tilde{r}=-2b-r \pmod{2n}$. Note that $r'=\tilde{r} \pmod{2n}$, so that there is a $c_5: F_{\tilde{r}}(a,b)\rightarrow F_{r'}(a',b')$. Consequently, there is an equivariant diffeomorphism $c_6: F_r(a,b)\rightarrow F_{r'}(a',b')$ as claimed. (Compare also the relevant discussions in Wilczynski [26].)

With the preceding preparations, we shall derive in the next two lemmas a set of numerical conditions which must be satisfied by the triples (a,b,r) and (a',b',r') (modulo the relations from the canonical equivariant diffeomorphisms c_1 through c_6) if there is an orientation-preserving equivariant diffeomorphism between $F_r(a,b)$ and $F_{r'}(a',b')$.

Lemma 5.1. *Suppose* $\gcd(a,n)=\gcd(a',n)=1$. *If* $F_r(a,b)$ *is orientation-preservingly equivariantly diffeomorphic to* $F_{r'}(a',b')$, *then after replacing* $F_r(a,b)$, $F_{r'}(a',b')$ *by a* G*-Hirzebruch surface (continuously denoted by* $F_r(a,b)$, $F_{r'}(a',b')$ *for simplicity) which is equivariantly diffeomorphic to* $F_r(a,b)$, $F_{r'}(a',b')$ *by a sequence of canonical equivariant diffeomorphisms of types* c_i, $1\leq i\leq 6$, *one of the following must be true: (i)* $F_r(a,b)=F_{r'}(a',b')$, *or (ii)* $a'=a$, $b'=b$, *and* $r'=r-n$, *or (iii)* $a'=b$, $b'=a$ *and* $r'=r=n$, *where* $a\neq\pm b$.

Proof. First, consider the case where $r\neq 0 \pmod n$ and $2b+ra\neq 0 \pmod n$. The key observation is that in this case, a is the unique number among a,b such that either a or $-a$ shows up in all four pairs of the rotation numbers, i.e., $(a,\pm b)$ and $(-a,\pm(b+ra))$. It follows easily from the assumption that $F_r(a,b)$ is orientation-preservingly equivariantly diffeomorphic to $F_{r'}(a',b')$ that $r'\neq 0 \pmod n$ and $2b'+r'a'\neq 0 \pmod n$ must also hold, and that $a'=\pm a$. Furthermore, observe that either $b\neq 0 \pmod n$ or $b+ra\neq 0 \pmod n$, which means that at most one of F_0,F_1 is fixed under the G-action. We assume without loss of generality that $b\neq 0 \pmod n$. Then x_{00} and x_{01}, being isolated fixed points, must be sent to the fixed-points x'_{ij} for some i,j under the equivariant diffeomorphism. After replacing $F_{r'}(a',b')$ by a G-Hirzebruch surface (continue to be denoted by $F_{r'}(a',b')$ for simplicity) which is equivariantly diffeomorphic to $F_{r'}(a',b')$ by a sequence of canonical equivariant diffeomorphisms of types c_i, $i=2,3$, one can arrange so that x_{00} is sent to x'_{00} under the equivariant diffeomorphism from $F_r(a,b)$ to $F_{r'}(a',b')$. Then the assumption $r\neq 0 \pmod n$ and $2b+ra\neq 0 \pmod n$ implies that x_{01} must be sent to x'_{01}. With an application of c_1 if necessary, we may arrange to have $a'=a$, $b'=b$. Finally, if $b+ra=0 \pmod n$, then $b'+r'a'=0 \pmod n$ must also be true, and if $b+ra\neq 0 \pmod n$, with an application of c_6 if necessary, we may arrange to have $b+ra=b'+r'a' \pmod n$. In any event, $r'=r \pmod n$ is satisfied. With a further application of c_5, we have either $F_r(a,b)=F_{r'}(a',b')$, or $r'=r-n$. Note that when $n\neq 2$ and x_{1j} are isolated, x_{1j} is sent to x'_{1j} under the equivariant diffeomorphism from $F_r(a,b)$ to $F_{r'}(a',b')$.

Suppose $r=0 \pmod n$ or $2b+ra=0 \pmod n$. Then $r'=0 \pmod n$ or $2b'+r'a'=0 \pmod n$ must also hold. With an application of c_6 to both $F_r(a,b)$ and $F_{r'}(a',b')$ if necessary, one may assume $r'=r=0 \pmod n$. If $b=0 \pmod n$, then we must also have $b'=0 \pmod n$, and with a further application of c_1, we may arrange to have $a=a'$, $b=b'$. If $b\neq 0 \pmod n$, then $\{x_{ij}\}$ are the only fixed points, and will be sent to $\{x'_{ij}\}$ under the equivariant diffeomorphism. With an application of c_2,c_3 to $F_{r'}(a',b')$ if necessary, one can arrange to have x_{00} sent to x'_{00} and have $(a,b)=\pm(a',b')$ as unordered pairs. Assume first that $a\neq\pm b$. Then with an application of c_1 if necessary, we have either $a'=b$, $b'=a$, in which case x_{01} is sent to x'_{10}, or $a'=a$, $b'=b$, in which case x_{01} is sent to x'_{01}. In the former case, if $r=0 \pmod{2n}$ or $r'=0 \pmod{2n}$, one may apply c_4 and c_5 to arrange to have $a'=a$, $b'=b$. In the latter case, note that x_{1j} is sent to x'_{1j} when $n\neq 2$. Assume $a=\pm b$. Then with an application of c_1 if necessary, we may arrange to have $a'=a$, $b'=b$. It follows easily that we either have $F_r(a,b)=F_{r'}(a',b')$, or $a'=a$, $b'=b$ with $r'=r-n$, or $a'=b$, $b'=a$ and $r'=r=n$ with $a\neq\pm b$.

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We remark that from the proof it is clear that when $F_r(a,b) \neq F_{r'}(a',b')$, there is an orientation-preserving equivariant diffeomorphism from $F_r(a,b)$ to $F_{r'}(a',b')$, which, in the case of $a'=a$, $b'=b$ and $r'=r-n$, sends F_i to F_i' if the fibers are fixed, and sends x_{ij} to x'_{ij} when $n \neq 2$ if the fixed points are isolated. Moreover, in the case of $a'=b$, $b'=a$ and $r'=r=n$, where $a \neq \pm b$, the equivariant diffeomorphism sends x_{ij} to x'_{ij} for $i=j$ and sends x_{ij} to x'_{ji} for $i \neq j$.

Lemma 5.2. *Suppose* $\gcd(a,n) \neq 1$. *If $F_r(a,b)$ is orientation-preservingly equivariantly diffeomorphic to $F_{r'}(a',b')$, then after replacing $F_{r'}(a',b')$ by a G-Hirzebruch surface (continuously denoted by $F_{r'}(a',b')$ for simplicity) which is equivariantly diffeomorphic to $F_{r'}(a',b')$ by a sequence of canonical equivariant diffeomorphisms of types c_i, $1 \leq i \leq 6$, one has either $F_r(a,b) = F_{r'}(a',b')$, or $r=0$,* $\gcd(a',n)=1$ *and* $r'=0 \pmod n$.

Proof. First, consider the case where $r \neq 0$. Then since $\gcd(a,n) \neq 1$, the sections E_0, E_1 of $F_r(a,b)$ have nontrivial isotropy. On the other hand, E_0, E_1 have nonzero self-intersections, so that they can not be mapped to fibers of $F_{r'}(a',b')$ with nontrivial isotropy. Consequently, the sections E_0', E_1' of $F_{r'}(a',b')$ must also have nontrivial isotropy, and furthermore, $r' = \pm r$ and $a' = \pm a$. On the other hand, observe that $b \neq 0 \pmod n$ and $b + ra \neq 0 \pmod n$ because $\gcd(a,n) \neq 1$ but $\gcd(a,b,n)=1$. (In other words, there are no fixed fibers.) Finally, note that c_2, c_3 act transitively on the set $\{x_{ij}\}$, so that with an application of c_2, c_3 to $F_{r'}(a',b')$ if necessary, we may assume that the equivariant diffeomorphism from $F_r(a,b)$ to $F_{r'}(a',b')$ sends x_{00} to x'_{00}. This particularly implies $(a',b') = \pm(a,b)$ as unordered pairs. With a further application of c_1, we obtain $(a',b') = (a,b)$ as ordered pairs (note that $a \neq \pm b$ because otherwise, $\gcd(a,b,n) \neq 1$, and that $a' = \pm a$). Finally, $r'=r$, because E_0 must be sent to E_0', hence $F_r(a,b) = F_{r'}(a',b')$ if $r \neq 0$.

Next, we assume $r=0$. If $\gcd(a',n) \neq 1$, then r' must be 0, so that with an application of c_4 if necessary, we may assume the equivariant diffeomorphism from $F_r(a,b)$ to $F_{r'}(a',b')$ sends sections to sections. It follows easily that $F_r(a,b) = F_{r'}(a',b')$ up to a sequence of canonical equivariant diffeomorphisms.

Finally, consider the case where $r=0$ and $\gcd(a',n)=1$. We observe that $b'+r'a' = \pm b' \pmod n$ must be true because it is true for $F_r(a,b)$, and $F_r(a,b)$ is equivariantly diffeomorphic to $F_{r'}(a',b')$. Consequently, either $r'a' = 0 \pmod n$, or $2b'+r'a' = 0 \pmod n$. In the latter case, an application of c_6 will reduce it to the former case, which is equivalent to $r' = 0 \pmod n$. This proves the lemma.

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Note that in the latter case of Lemma 5.2, we can apply c_4 to $F_r(a,b)$, and with a further application of c_5 to $F_{r'}(a',b')$ if necessary, we may arrange to have either $F_r(a,b) = F_{r'}(a',b')$, or $a'=a$, $b'=b$, $r'=r-n$, and $\gcd(a,n)=1$. Furthermore, there is an orientation-preserving equivariant diffeomorphism from $F_r(a,b)$ to $F_{r'}(a',b')$, which sends F_i to F_i' if the fibers are fixed, and sends x_{ij} to x'_{ij} when $n \neq 2$ if the fixed points are isolated.

With the preceding understood, the main technical result is summarized in the following proposition.

Proposition 5.3. *Suppose the G-actions are non-pseudo-free and $n \neq 2$ unless r, r' are even. Then there are no orientation-preserving equivariant diffeomorphisms from $F_r(a,b)$ to $F_{r'}(a',b')$, which send F_i to F_i' if the fibers are fixed, and send x_{ij} to x_{ij}' when $n \neq 2$ if the fixed points are isolated, where $a'=a$, $b'=b$, $r'=r-n$, and* $\gcd(a,n)=\gcd(a',n)=1$.

Proof. Suppose to the contrary, there is such an orientation-preserving equivariant diffeomorphism $f: F_r(a,b) \to F_{r'}(a',b')$. We first observe that r, r' must have the same parity, so that n is necessarily even. Without loss of generality, we assume $a'=a=1$. Furthermore, with an application of c_3 to both $F_r(a,b)$ and $F_{r'}(a',b')$ if necessary, we may assume $0 \leq 2b' = 2b \leq n$.

Next, we claim that with an application of c_5, c_6 to both G-Hirzebruch surfaces if necessary, one can arrange so that r' and r satisfy the following constraints:

$$0 \leq r' < n, \quad b + r' \leq n, \quad \text{and } n \leq r = r' + n < 2n.$$

To see this, note that with c_5, we may assume $0 \leq r, r' < 2n$, and assuming without loss of generality that $r' < r$, we have $0 \leq r' < n \leq r = r'+n < 2n$. If $b + r' \leq n$, then we are done. Suppose $b + r' > n$. Then we apply c_6 to both G-Hirzebruch surfaces and replace r, r' by $\tilde{r} = 4n - 2b - r$ and $\tilde{r}' = 2n - 2b - r'$ respectively. Note that $\tilde{r}' - \tilde{r} = -2n - (r' - r) = -n$, so that $\tilde{r}' = \tilde{r} - n$ continues to hold. We will show that the conditions $0 < \tilde{r}'$ and $b + \tilde{r}' < n$ are satisfied, with which $n \leq \tilde{r} = \tilde{r}' + n < 2n$ follows easily. With this understood, $0 < \tilde{r}'$ follows from $2b \leq n$ and $r' < n$, and $b + \tilde{r}' < n$ follows from the assumption $b + r' > n$. Hence the claim.

With the preceding preparation, we shall denote $F_r(a,b)$ by X and let $C \subset X$ be the pre-image of the holomorphic $(-r')$-section $E_0' \subset F_{r'}(a',b')$ under f. Then C is a G-invariant, smoothly embedded two-sphere in X with self-intersection $-r'$. Let J_0 be the G-invariant complex structure on $X = F_r(a,b)$, and let ω_0 be a fixed G-invariant Kähler form. Our goal is to first show that for a certain generic G-invariant, ω_0-compatible J, there is a G-invariant, embedded J-holomorphic two-sphere $\tilde{C}$ with self-intersection $-r'$. On the other hand, by choosing a sequence of such J which converges to J_0, the corresponding J-holomorphic $(-r')$-spheres will converge to a cusp-curve C_∞ by Gromov compactness. Carefully analyzing C_∞ will lead to a contradiction to the complex geometry of J_0, which proves that f should not exist.

We begin our proof by giving an orientation to C. Since the G-action is non-pseudo-free, it follows easily that either F_0 or F_1 has nontrivial isotropy. Without loss of generality, we assume F_0 has nontrivial isotropy. Then it follows that under f, F_0 is mapped to F_0'. As a consequence, we see that C intersects F_0 transversely. With this understood, we shall orient C so that $C \cdot F_0 = 1$.

Before we proceed further, we shall fix the following notations which are compatible with those used in Lemmas 2.3 and 2.4. For $X = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, let $e_0, e_1 \in H^2(X)$ be a basis such that $c_1(K_{\omega_0}) = -3e_0 + e_1$; in particular, $F_0 = F_1 = e_0 - e_1$. For $X = \mathbb{S}^2 \times \mathbb{S}^2$, we choose a basis $e_1, e_2 \in H^2(X)$ such that $c_1(K_{\omega_0}) = -2e_1 - 2e_2$, and moreover, $F_0 = F_1 = e_2$.

As in Section 4, it is more convenient to divide our discussions according to the following two scenarios :

(i) Neither F_0 nor F_1 is fixed by G, i.e., $0<b<n-r'$; in particular, $n\neq 2$.
(ii) Either F_0 or F_1 is fixed by G, i.e., either $b=0$ or $b+r'=0$ or n.

Case (i): Neither F_0 nor F_1 is fixed by G. In this case, f sends x_{ij} to x'_{ij} for all i,j; in particular, C contains x_{00} and x_{01}. Furthermore, we remark that since the space of G-invariant ω_0-compatible J is contractible, the rotation numbers at x_{0j}, x_{1j}, which are $(1,\pm b)$, $(-1,\pm(b+r'))$ respectively respect to J_0, remain unchanged with respect to any other ω_0-compatible J. In particular, the weight of the action on C is $+1$, -1 at x_{00} and x_{01} respectively.

Since $n>2$ and even, Lemma 4.2 is true so that the tangent space of C at x_{0j} is J_0-invariant for $j=0,1$. As in Section 4, one can construct a G-invariant, ω_0-compatible, almost complex structure $\hat{J}_0$ in a neighborhood of x_{00}, x_{10}, such that

- $\hat{J}_0$ agrees with J_0 at x_{00},x_{10};
- for any $i=0,1$, if C intersects the G-invariant fiber F_i transversely at x_{i0}, then C is $\hat{J}_0$-holomorphic near x_{i0}.

The following lemma is a generalization of Lemma 4.4.

Lemma 5.4. *For any generic G-invariant J equaling $\hat{J}_0$ in a neighborhood of x_{00},x_{10}, there is an embedded G-invariant J-holomorphic two-sphere $\tilde{C}$ containing x_{00}, x_{10} such that $\tilde{C}$ and C are homologous; in particular, $\tilde{C}^2=-r'$.*

Proof. For any given G-invariant J equaling $\hat{J}_0$ in a neighborhood of x_{00},x_{10}, we apply Lemmas 2.3 and 2.4 to it. We further note that, in the present situation, since F_0 has nontrivial isotropy, it is automatically J-holomorphic, so that the $\mathbb{S}^2$-fibration structure on X always exists, and we may consider C_0 as a J-holomorphic section even in case (1) of the lemmas.

As we have seen before, there are four possibilities for the fixed points on C_0:

(a) $x_{00},x_{10}\in C_0$; (b) $x_{00},x_{11}\in C_0$; (c) $x_{01},x_{10}\in C_0$; (d) $x_{01},x_{11}\in C_0$.

Moreover, the weights $(m_{i,1},m_{i,2})$, $i=1,2$, in the formula for the virtual dimension $2d$ of the moduli space of J-holomorphic curves at C_0 can be read off from the rotation numbers and are given correspondingly as follows:

(a) $(m_{1,1},m_{1,2})=(1,b)$, $(m_{2,1},m_{2,2})=(1,n-b-r')$;
(b) $(m_{1,1},m_{1,2})=(1,b)$, $(m_{2,1},m_{2,2})=(1,b+r')$;
(c) $(m_{1,1},m_{1,2})=(1,n-b)$, $(m_{2,1},m_{2,2})=(1,n-b-r')$;
(d) $(m_{1,1},m_{1,2})=(1,n-b)$, $(m_{2,1},m_{2,2})=(1,b+r')$

where b,r' satisfy the inequalities $0<b<n-r'$. Correspondingly, we have

$$(a)\ d=\frac{C_0^2+r'}{n},\ (b)\ d=\frac{C_0^2+n-2b-r'}{n},\ (c)\ d=\frac{C_0^2+2b+r'-n}{n},\ (d)\ d=\frac{C_0^2-r'}{n}.$$

We choose J to be a generic G-invariant almost complex structure equaling $\hat{J}_0$ in a neighborhood of x_{00},x_{10}. Then $d\geq 0$, so that with $C_0^2\leq 0$, we obtain

$$(a)\ C_0^2=-r',\ (b)\ C_0^2=-n+2b+r',\ (c)\ C_0^2=-2b-r'+n,$$

and case (d) is a contradiction unless $r'=0$, and in this case $C_0^2=0$.

In order to rule out cases (b), (c), we observe that Lemma 4.3 continues to hold here, i.e., $C \cdot C_0 = b \pmod{n}$ if $x_{00}, x_{11} \in C_0$ and $C \cdot C_0 = -b - r' \pmod{n}$ if $x_{01}, x_{10} \in C_0$. With this understood, we need to discuss separately according to $X = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ or $X = \mathbb{S}^2 \times \mathbb{S}^2$.

Suppose $X = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. We write $C = (u+1)e_0 - ue_1$, $C_0 = (v+1)e_0 - ve_1$ for some $u, v \in \mathbb{Z}$. Then $C_0^2 + C^2 = 2(u+v+1)$, and $C \cdot C_0 = u+v+1$. Consequently, in case (b), $u+v+1 = b \pmod{n}$ so that $C_0^2 = 2(u+v+1) - C^2 = 2b + r' \pmod{2n}$, and in case (c), $u+v+1 = -b-r' \pmod{n}$, so that $C_0^2 = -2b - r' \pmod{2n}$. It follows easily that in both cases we reached a contradiction.

Suppose $X = \mathbb{S}^2 \times \mathbb{S}^2$. We write $C = e_1 + ue_2$, $C_0 = e_1 + ve_2$ for some $u, v \in \mathbb{Z}$. Then $C_0^2 + C^2 = 2(u+v)$ and $C \cdot C_0 = u+v$. The same argument shows that $C_0^2 = 2b + r' \pmod{2n}$ in case (b) and $C_0^2 = -2b - r' \pmod{2n}$ in case (c), which is a contradiction.

Note that in case (a), the above calculations also show that $C^2 = C_0^2$ implies that C, C_0 are homologous. In this case, $\tilde{C} = C_0$, and the lemma follows.

Finally, suppose $r' = 0$ and case (d) occurs. Then C_0 gives rise to a $\mathbb{S}^2$-fibration on X whose fibers are embedded J-holomorphic two-spheres in the class of C_0, which is G-invariant because G fixes the class of C_0. It follows easily that there is a G-invariant fiber containing the fixed-points x_{00}, x_{10}, which is the desired J-holomorphic curve $\tilde{C}$. ☐

We shall next construct a sequence of suitable G-invariant ω_0-compatible almost complex structures converging to J_0. To this end, let g_0 be the Kähler metric and $\hat{g}_0 = \omega_0(\cdot, \hat{J}_0 \cdot)$ be the metric associated to $\hat{J}_0$. Fix a large enough $k_0 > 0$ and let ρ be a cutoff function such that $\rho(t) \equiv 1$ for $t \leq 1$ and $\rho(t) \equiv 0$ for $t \geq 2$, and $0 \leq \rho(t) \leq 1$ and $|\rho'(t)| \leq 100$. Then for any integer $k > k_0$, we define a G-invariant metric g_k on X by

$$g_k(x) = g_0(x) + (\rho(k|x - x_{00}|) + \rho(k|x - x_{10}|))(\hat{g}_0(x) - g_0(x)), \quad \forall x \in X.$$

Here $|x - x_{i0}|$ is the distance from x to x_{i0}, measured with respect to the Kähler metric g_0. We choose k_0 large enough so that (i) $|x - x_{i0}| \leq \frac{3}{k_0}$ is contained in the neighborhood of x_{i0} where $\hat{J}_0$ is defined, and (ii) $\max\{k_0|x - x_{i0}|, i = 0, 1\} > 2$. Then it follows easily that (1) g_k equals $\hat{g}_0$ in a neighborhood of each x_{i0} and converges to g_0 in C^0-topology as $k \to \infty$, and (2) the C^1-norm of g_k is uniformly bounded by a constant depending only on g_0, ω_0 and $\hat{J}_0$. With this understood, we let J_k be the ω_0-compatible almost complex structure determined by g_k. Then it is clear that J_k is G-invariant, and J_k converges to J_0 in C^0-topology as $k \to \infty$, and the C^1-norm of J_k is uniformly bounded by a constant depending only on J_0, ω_0 and $\hat{J}_0$.

We apply Lemma 5.4 to J_k and for each $k > k_0$, pick a generic G-invariant J'_k from Lemma 5.4, such that the C^1-norm of $J'_k - J_k$ is bounded by $1/k$. Then $\{J'_k\}$ is a sequence of G-invariant ω_0-compatible almost complex structures such that

- J'_k converges to J_0 in C^0-topology as $k \to \infty$,
- the C^1-norm of J'_k is uniformly bounded by a constant depending only on J_0, ω_0 and $\hat{J}_0$, and

- there is a G-invariant J'_k-holomorphic $(-r')$-sphere, denoted by C_k, which contains x_{00} and x_{10}.

Next we shall apply the Gromov Compactness Theorem to the sequence $\{C_k\}$. To this end, let $\mathbb{CP}^1$ be given the standard G-action with fixed points $0, \infty$, and let j_0 be the G-invariant complex structure on $\mathbb{CP}^1$. Let $f_k : \mathbb{CP}^1 \to X$ be a G-equivariant (J_k, j_0)-holomorphic map which parametrizes C_k. Then by the Gromov Compactness Theorem, after re-parametrization if necessary, a subsequence of f_k (still denoted by f_k for simplicity) converges to a "cusp-curve" $f_\infty : \Sigma \to X$ in the following sense as $k \to \infty$, where $\Sigma = \sum_\nu \Sigma_\nu$ is a nodal Riemann two-sphere.

- The maps f_k converges to f_∞ locally in Hölder $C^{1,\alpha}$-norm for some $\alpha > 0$ (this follows from the fact that the C^1-norm of J'_k is uniformly bounded and by the standard elliptic estimates, e.g. cf. [22]).
- There is no energy loss, i.e., $(f_\infty)_*[\Sigma] = C$.
- The map f_∞ is (J_0, j_0)-holomorphic (this is due to the fact that J'_k converges to J_0 in C^0-topology).

In the present situation, since each f_k is G-equivariant, the convergence $f_k \to f_\infty$ is also G-equivariant. In particular, there is a G-action on the nodal Riemann two-sphere $\Sigma = \sum_\nu \Sigma_\nu$, with respect to which $f_\infty : \Sigma \to X$ is G-equivariant. It is important to note that the G-action on Σ and the map f_∞ have the following properties:

- Each component Σ_ν is either G-invariant or is in a free G-orbit.
- For any two distinct G-invariant components Σ_ν, Σ_ω such that $z_0 \in \Sigma_\nu \cap \Sigma_\omega$ which is a fixed-point of G, if $g_\nu, g_\omega \in G$ are the elements which act by a rotation of angle $2\pi/n$ in a neighborhood of $z_0 \in \Sigma_\nu$ and $z_0 \in \Sigma_\omega$ respectively, then $g_\omega = g_\nu^{-1}$.

(There is an equivalent formulation of the above statements in terms of the Orbifold Gromov Compactness Theorem, see [8].)

We proceed by finding what kind of possible components the J_0-holomorphic cusp-curve f_∞ has, which are J_0-holomorphic two-spheres in X. To this end, recall that X comes with a J_0-holomorphic $\mathbb{CP}^1$-fibration over $\mathbb{CP}^1$ which is G-invariant. The J_0-holomorphic two-spheres in X are either fibers or sections of this fibration. There are two G-invariant fibers F_0, F_1, containing x_{00}, x_{10} respectively, and two G-invariant sections E_0, E_1 which has self-intersection $-r$ and r respectively, and these are the only G-invariant J_0-holomorphic two-spheres in X (cf. Wilczynski [26], §4).

The following observation greatly simplifies the analysis of the components of the cusp-curve f_∞, that is,

$$2(f_\infty)_*[\Sigma] \cdot E_1 = 2C \cdot E_1 = C^2 + E_1^2 = -r' + r = n.$$

(We have seen it in the proof of Lemma 5.4.) An immediate consequence of this is that there are no components $f_\infty(\Sigma_\nu)$ which are not G-invariant, because the G-orbit of such a component will contribute at least n to the intersection number $(f_\infty)_*[\Sigma] \cdot E_1$, which contradicts $2(f_\infty)_*[\Sigma] \cdot E_1 = n$. Furthermore, E_1 can not show up in the cusp-curve either because $E_1^2 = r \geq n$. Consequently, the only J_0-holomorphic two-spheres which are possibly allowed in the cusp-curve f_∞ are the $(-r)$-section E_0 and the invariant fibers F_0, F_1.

Next we show that if F_i, $i = 0$ or 1, shows up in the cusp-curve f_∞, the multiplicity must be at least n, which is also not allowed by $2(f_\infty)_*[\Sigma] \cdot E_1 = n$. To see this, suppose without loss of generality that the component $f_\infty : \Sigma_\nu \to X$ has image F_0. Then the homology class $(f_\infty)_*[\Sigma_\nu] = m_\nu F_0$ for some $m_\nu > 0$ such that $m_\nu = b \pmod n$ because the weight of the G-action in the direction of fiber F_0 equals b at x_{00}. With this understood, let $z_0 \in \Sigma_\nu$ such that $f_\infty(z_0) = x_{01}$. Then it follows easily that there must be another component $f_\infty : \Sigma_\omega \to X$ with $z_0 \in \Sigma_\omega$. Furthermore, it must also have image F_0 because the other two allowable J_0-holomorphic two-spheres E_0 and F_1 do not contain x_{01}. Let $m_\omega > 0$ be the multiplicity of $(f_\infty)_*[\Sigma_\omega]$ in F_0. Then $m_\omega = -b \pmod n$, because the relation $g_\omega = g_\nu^{-1}$ in the Gromov Compactness Theorem we alluded to earlier, and the fact that the weight of the G-action in the direction of F_0 at x_{01} equals $-b$. This proves our claim that the multiplicity of F_0 in the cusp-curve f_∞ must be at least n as $m_\nu + m_\omega = 0 \pmod n$.

We conclude that E_0 is the only possible J_0-holomorphic curve in the cusp-curve f_∞. However, this is also a contradiction because $E_0 \cdot E_1 = 0$ but $(f_\infty)_*[\Sigma] \cdot E_1 \neq 0$. This completes the proof of Proposition 5.3 when neither F_0 nor F_1 is fixed by G.

Case (ii): Either F_0 or F_1 is fixed by G. Without loss of generality, we shall only consider the case where F_0 is fixed, which corresponds to $b = 0$. Note that both of F_0, F_1 are fixed by G if and only if $b = 0 = r'$; in particular, when $n = 2$, both F_0, F_1 are fixed because $b = 0 = r'$ in this case.

Let x_0, x_1 be the fixed points on C such that $x_0 \in F_0$. Then $x_1 = x_{10}$ unless F_1 is also fixed by G, in which case $x_1 \in F_1$. We shall fix a $\hat{J}_0$, which is a G-invariant, ω_0-compatible, integrable complex structure in a neighborhood of x_0 and x_1, such that

- $\hat{J}_0$ agrees with J_0 at x_0, x_1;
- for any $i = 0, 1$ such that C intersects F_i transversely, there are holomorphic coordinates z_1, z_2 (with respect to $\hat{J}_0$) such that C is given by $z_2 = 0$ and F_i is given by $z_1 = 0$.

Correspondingly, we have the following lemma in place of Lemma 5.4.

Lemma 5.5. *For any generic G-invariant J equaling $\hat{J}_0$ in a neighborhood of x_0 and x_1, there is an embedded G-invariant J-holomorphic two-sphere $\tilde{C}$ such that (1) $\tilde{C}$ and C are homologous, and (2) $x_{10} \in \tilde{C}$. In particular, $\tilde{C}^2 = -r'$.*

Proof. We apply Lemma 2.3 or 2.4 to any given G-invariant J which equals $\hat{J}_0$ in a neighborhood of x_0 and x_1, and note that since F_0 is fixed by G, it is automatically J-holomorphic, so that the $\mathbb{S}^2$-fibration structure on X always exists, and we may consider C_0 as a J-holomorphic section even in case (1) of the lemma.

Let y_0, y_1 be the fixed points on C_0 where $y_0 \in F_0$. Then $y_1 = x_{10}$ or x_{11} if $r' > 0$, and $y_1 \in F_1$ if $r' = 0$. When J is chosen to be a generic G-invariant almost complex structure, $d \geq 0$ implies that

$$(a)\ C_0^2 = -r' \text{ if } y_1 = x_{10},\ (b)\ C_0^2 = -n + r' \text{ if } y_1 = x_{11}, \text{ and } (c)\ C_0^2 = -n \text{ if } r' = 0.$$

Furthermore, case (b) or (c) can be ruled out by observing $C \cdot C_0 = 0 \pmod n$ as we argued before in Section 4 so that $C_0^2 = r' \pmod{2n}$, which contradicts the above equations. In particular, $r' \neq 0$, so that F_0, F_1 can not be both fixed by G. The lemma follows by taking $\tilde{C} = C_0$.

□

The rest of the argument is the same as in case (i), except there is one additional possibility. Let J_k' be a sequence of such generic almost complex structures converging to J_0 and C_k be the corresponding J_k'-holomorphic curves from Lemma 5.5 which converges to a cusp-curve f_∞. Denote by $x_0^{(k)}$, $x_1^{(k)}$ the fixed points on C_k. Then in case (i) $x_0^{(k)} = x_{00}$, $x_1^{(k)} = x_{10}$ for all k, however, in the present case, $x_0^{(k)}$ can be any point on F_0. In particular, there is the additional possibility that $x_0^{(k)}$ converges to x_{01} as $k \to \infty$. Let $f_\infty : \Sigma_\nu \to X$ be the corresponding component containing the limit point x_{01} of $x_0^{(k)}$. Then $(f_\infty)_*[\Sigma_\nu] = m_\nu F_0$ for some $m_\nu > 0$ such that $m_\nu = b \pmod n$. Since $b = 0$ we continue to have $m_\nu \geq n$. As in case (i), the existence of the cusp-curve f_∞ is a contradiction.

The proof of Proposition 5.3 is completed.

□

Proposition 5.3 has an analog for pseudo-free actions, where Lemma 5.4 requires a different approach as we have seen in the proof of Theorems 1.1 and 1.2. (Lemma 5.5 is irrelevant in the case of pseudo-free actions.) We choose to not repeat it here, and instead we refer to the corresponding results in Wilczynski [26].

Proof of Theorem 1.4

Suppose $F_r(a,b)$ and $F_{r'}(a',b')$ are orientation-preservingly equivariantly diffeomorphic. By Lemmas 5.1 and 5.2, after modifying $F_r(a,b)$ and $F_{r'}(a',b')$ by a sequence of canonical equivariant diffeomorphisms, we have either $F_r(a,b) = F_{r'}(a',b')$, in which case Theorem 1.4 follows, or $\gcd(a,n) = \gcd(a',n) = 1$ and one of the following occurs:

- $a' = a$, $b' = b$, and $r' = r - n$, or
- $a' = b$, $b' = a$ and $r' = r = n$, where $a \neq \pm b$. (Note that the G-actions are pseudo-free in this case.)

For pseudo-free actions and when $n > 2$, the above possibilities are ruled out by Lemma 4.9(4),(5) in Wilczynski [26]. For non-pseudo-free actions where $n > 2$ or $n = 2$ and r, r' are even, Proposition 5.3 can be used.

It remains to examine the case where $n = 2$ and either the G-actions are pseudo-free or r, r' are odd. Observe that in this case, with an application of c_2 if necessary, one can always arrange so that $b = b' = 1$. Thus as in the proof of Proposition 5.3, it suffices to examine $F_r(1,1)$ and $F_{r'}(1,1)$ for $(r', r) = (0,2)$ or $(1,3)$. In the former case, the two G-Hirzebruch surfaces are equivariantly diffeomorphic by c_6, and in the latter case, by $c_3 \circ c_6$. This finishes off the proof of Theorem 1.4.

6. Minimality of rational G-surfaces as G-manifolds

We begin by considering minimal rational G-surfaces X where G is a finite group in general. Our first lemma shows that we only need to look at the cases where X is a conic bundle with singular fibers or a Hirzebruch surface.

Lemma 6.1. *If X is a minimal rational G-surface which is not minimal as a topological G-manifold, then X must be a conic bundle with singular fibers or a Hirzebruch surface F_r with $r > 1$ and odd.*

Proof. Our first observation is that the dimension of $H^2(X;\mathbb{R})^G$ is at least 2. To see this, note that X has a G-invariant Kähler form ω_0 with $[\omega_0] \in H^2(X;\mathbb{R})^G$ and $[\omega_0]^2 > 0$, and on the other hand, the class E of the union of (-1)-spheres along which X is blown down also lies in $H^2(X;\mathbb{R})^G$ and $E^2 < 0$. It follows that $[\omega_0]$ and E are linearly independent, and consequently, the dimension of $H^2(X;\mathbb{R})^G$ is at least 2. With this understood, we recall the fact that for any minimal rational G-surfaces X, $Pic(X)^G$ has rank at most 2, cf. [9], Theorem 3.8. It follows that $Pic(X)^G$ must have rank 2. By Theorem 3.8 in [9] again, X must be either a conic bundle with singular fibers, or a Hirzebruch surface. Finally, we note that if X can be blown down G-equivariantly, the underlying manifold X must be topologically non-minimal. The lemma follows easily.

□

Now we specialize to the case where $G = \mathbb{Z}_n$ is a finite cyclic group. First, consider the case when X is a conic bundle with singular fibers. An element $\mu \in G$ which is a generator is called a de Jonquiéres element, and such elements have been classified by Blanc [1]. In particular, $n = 2m$ must be even, and $\tau = \mu^m$ is a de Jonquiéres involution, which has the following properties: The involution τ leaves each fiber of the conic bundle invariant, switches the two irreducible components in each singular fiber, and the fixed-point set of τ is an irreducible smooth bisection Σ with a hyperelliptic involution, such that the conic bundle projection defines the quotient map with ramification points equal to the singular points of the fibers. The de Jonquiéres element μ induces a permutation of the set of singular fibers. It follows easily that X as a $\langle\tau\rangle$-surface is also minimal.

Proposition 6.2. *Let X be a minimal G-conic bundle with singular fibers where $G = \mathbb{Z}_n$. Then X is minimal as a topological G-manifold.*

Proof. By the description of de Jonquiéres elements in [1], it suffices to consider only the case $G = \mathbb{Z}_2$ generated by a de Jonquiéres involution τ. Let Σ be the fixed-point set of τ, and let k be the number of singular fibers. Then $k = 2 + 2g$ where g is the genus of Σ. Note that $k \geq 4$ (cf. [9], Lemma 5.1), which implies that $g \geq 1$. Let F denote the fiber class of the conic bundle. Then F and Σ span $H^2(X;\mathbb{Q})^G$ over $\mathbb{Q}$ as $F \in Pic(X)^G$.

We shall first determine the intersection form of F, Σ.

Lemma 6.3. *$F^2 = 0$, $\Sigma \cdot F = 2$, and $\Sigma \cdot \Sigma = 2 + 2g$, where $g \geq 1$ is the genus of Σ.*

Proof. It is clear that $F^2 = 0$ and $\Sigma \cdot F = 2$ (note that Σ is a bisection). We shall prove that $\Sigma \cdot \Sigma = 2 + 2g$. To see this, let K_X be the canonical class of the rational surface X. Then $K_X \in Pic(X)^G$ implies that K_X is a linear combination of Σ and F. With $K_X \cdot F = -2$ (by the adjunction formula), $\Sigma \cdot F = 2$ and $F^2 = 0$, it follows easily that $K_X = -\Sigma + l \cdot F$ for some $l \in \mathbb{Z}$.

Now the adjunction formula $K_X\cdot\Sigma+\Sigma\cdot\Sigma+2=2g$ gives $2l+2=2g$, which implies that $l=g-1$.

To compute $\Sigma\cdot\Sigma$, note that

$$\Sigma\cdot\Sigma-4l=K_X^2=8-k=8-(2+2g)=6-2g,$$

which gives $\Sigma\cdot\Sigma=6-2g+4l=6-2g+4(g-1)=2+2g$.

□

Secondly, we show that X contains no G-invariant locally linear (-1)-spheres. Suppose to the contrary that there is such a (-1)-sphere C. Obviously the action of G on C is effective and orientation-preserving, so that C contains exactly two fixed points of G. Let m_1, m_2 be the weights of the G-action in the normal direction of C at the two fixed points. Then m_1, m_2 obey the following equation: $m_1+m_2\equiv -1 \pmod 2$. This implies that exactly one of m_1, m_2 equals 0 mod 2, and consequently, the G-action has an isolated fixed-point, which is a contradiction. Hence the claim.

As a corollary, if X is not minimal as a topological G-manifold, there must be two disjoint (-1)-spheres which are disjoint from the fixed-point set Σ, such that the $\mathbb{Z}_2$-action permutes the two (-1)-spheres. Denote their classes by E_1 and E_2 and let $E=E_1+E_2$. Then E satisfies the following conditions:

$$E\in H^2(X;\mathbb{Z})^G, E\cdot\Sigma=0, E^2=-2, \text{ and } F\cdot E=0 \pmod 2.$$

To see the last condition, let $\tau\in G$ be the involution. Then

$$F\cdot E=F\cdot(E_1+E_2)=F\cdot(E_1+\tau(E_1))=F\cdot E_1+\tau(F)\cdot E_1=2F\cdot E_1=0 \pmod 2.$$

We write $E=a\Sigma+bF$ for some $a,b\in\mathbb{Q}$. Then $E\cdot\Sigma=0$ gives $a\Sigma\cdot\Sigma+bF\cdot\Sigma=0$, which, with $\Sigma\cdot\Sigma=2+2g$, implies $b=-a(1+g)$. Now $E^2=-2$ means

$$(2+2g)a^2+4ab=-2,$$

which gives $a^2(1+g)=1$. Finally, with $F\cdot E=0 \pmod 2$, we get $2a=F\cdot E=2k$ for some $k\in\mathbb{Z}$, so that $a\in\mathbb{Z}$. But this contradicts $a^2(1+g)=1$ as $g\geq 1$. The proof of Proposition 6.2 is completed.

□

It is clear that Theorem 1.5 follows from Proposition 6.2 and Theorem 1.2.

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