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FINITE SYMMETRIES OF S^4

WEIMIN CHEN, SŁAWOMIR KWASIK AND REINHARD SCHULTZ

Abstract. This paper discusses topological and locally linear actions of finite groups on S^4. Local linearity of the orientation preserving actions on S^4 forces the group to be a subgroup of SO(5). On the other hand, orientation reversing topological actions of "exotic" groups G (*i.e.* $G \not\subset \mathrm{O}(5)$) on S^4 are constructed, and local linearity and stable smoothability of the actions are studied.

In the theory of transformation groups, one class of closed manifolds plays a special role; namely, standard spheres S^n. Classical results describe all finite group actions on S^1 and S^2; all such actions are conjugate to the linear ones.

The existence of wild topological actions on S^3 (*cf.* [6]) suggests restricting attention to group actions which are more regular. Consequently, one usually requires the action to be smooth or more generally locally linear (*cf.* [6]). If one restricts to locally linear, say orientation preserving, actions on S^3 (which are smoothable by [18]), then again all these actions are topologically conjugate to linear actions (*cf.* [10], [26]). In particular, if G is a finite group acting locally linearly, orientation-preservingly (and effectively) on S^3, then G is isomorphic to a subgroup of SO(4). This is a highly nontrivial result involving Perelman's work on the Thurston's Geometrization Conjecture (*cf.* [5], [26]). Strangely enough the fact that $G \subset \mathrm{SO}(4)$, as far as we know, does not have a direct proof (i.e. without using the linearity of the action). It should be pointed out that a locally linear action of G on S^3 is also topologically conjugate to a linear action in the orientation reversing case (*cf.* [10]). In this case $G \subset \mathrm{O}(4)$, but the main difficulties are related to the orientation preserving case which has attracted most of the attention (*cf.* [2]).

For S^4 there are smooth, orientation preserving actions of $G = \mathbb{Z}_p$ which are not conjugate to linear ones (not even topologically). The existence of nonlinear finite group actions on S^4 still left open possibility that these groups are subgroups of the corresponding orthogonal groups. In particular, there are two appealing conjectures (*cf.* [23], [37], [22]).

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Conjecture 1. *If a finite group G acts smoothly (or locally linearly) and orientation preservingly on S^4, then G is isomorphic to a subgroup of* $\mathrm{SO}(5)$.

Conjecture 2. *If G acts orientation reversingly on S^4, then G is isomorphic to a subgroup of* $\mathrm{O}(5)$.

Significant progress toward the proof of Conjecture 1 was made in [23]. Our first main result is the completion of the proof of Conjecture 1. In particular, the following holds.

Theorem 1. *If a finite group G acts locally linearly and orientation preservingly on a homology 4-sphere Σ, then G is isomorphic to a subgroup of* $\mathrm{SO}(5)$.

It is however worthwhile to note the following:

Observation. *There is a finite group π acting topologically and orientation-preservingly on S^4 (semifreely with two fixed points) such that π is not isomorphic to a subgroup of* $\mathrm{SO}(5)$.

Our remaining results concern Conjecture 2. In this direction we have the following:

Fact. *There is a finite group π acting topologically and orientation reversingly on S^4 without fixed points such that π is not isomorphic to a subgroup of* $\mathrm{O}(5)$.

The above fact gives a topological solution (in negative) to Conjecture 2. The locally linear (or smooth) case of Conjecture 2 is still open. To be more precise, we start with a construction of topological, orientation reversing, pseudo-free actions of groups $Q(8p,q)$, for certain pairs of (p,q), on S^4. Note that $Q(8p,q) \not\subset \mathrm{O}(5)$. Then we describe an approach for constructing potential counterexamples to the locally linear version of Conjecture 2 and then discuss stable smoothability of these potential locally linear actions. This approach uses computational techniques involving surgery groups (*cf.* [35], [27], [21]) and algebraic number theory. It also leads to questions concerning free finite group actions on homology 3-spheres which are of independent interest. Our results in this direction are summarized in the following:

Theorem 2. *There exist orientation reversing (pseudo-free) actions of groups $Q(8p,q)$, for some pairs (p,q), on S^4. Suppose such an action Φ is locally linear. If $\Phi \times \mathrm{id}_{\mathbb{R}}$ is not equivariantly smoothable, then there is also an action Φ' of the same type such that $\Phi' \times \mathrm{id}_{\mathbb{R}}$ is equivariantly smoothable, and likewise if $\Phi \times \mathrm{id}_{\mathbb{R}}$* is *equivariantly smoothable, then there is a locally linear action Φ' of the same type such that for all nonnegative integers k, the product action $\Phi' \times \mathrm{id}_{\mathbb{R}^k}$ is* not *equivariantly smoothable.*

In other words, the potential locally linear actions of $Q(8p,q)$ on S^4 come in twin pairs, such that one is stably equivariantly smoothable but the other is not.

In view of Theorem 2, we would like to advertise the smooth version of Conjecture 2, stated here as

Problem 1. *Let G be a finite group acting smoothly and orientation reversingly on S^4. Is G isomorphic to a subgroup of* $\mathrm{O}(5)$*?*

Our interest in the questions of this paper was also partly motivated by the paper of C. B. Thomas [33]. Namely, in [33] the existence of exotic smooth actions of finite groups was used to show that the natural inclusions of $\mathrm{O}(6)$ in the automorphism groups $Top(S^5)$ and $\mathrm{Diff}(S^5)$ are not homotopy equivalences. The exotic actions used by Thomas are actually by subgroups of $\mathrm{SO}(6)$. In particular these actions include the action first constructed by T. Petrie for the metacyclic group of order 21, *cf.* [33]. The constructions of this paper would then suggest that the inclusion of $\mathrm{O}(5)$ in $Top(S^4)$ is also not a homotopy equivalence, for the group actions constructed in this paper are by "exotic" groups which act on S^4 but obviously can not act linearly on S^4 because they are not subgroups of $\mathrm{O}(5)$.

In fact, one can use homotopy theoretic methods as in [29] to show that the group $\pi_3(\,Top(S^n)\,)$ contains a copy of $\mathbb{Z}_2$ if $n \geq 4$, which immediately implies that the natural inclusion of $\mathrm{O}(5)$ in $Top(S^4)$ is not a homotopy equivalence. On the other hand, if $m \leq 3$ the inclusions $\mathrm{O}(m+1) \subset \mathrm{Diff}(S^m) \subset Top(S^m)$ are homotopy equivalences; the case $m = 3$ is the celebrated result of A. Hatcher in [15]. This means that only the following question is open:

Four-dimensional Smale Conjecture. *The natural inclusion of* $\mathrm{O}(5)$ *in* $\mathrm{Diff}(S^4)$ *is a homotopy equivalence.*

This question provides additional motivation for asking whether the actions in Theorem 2 are in fact equivariantly smoothable (*cf.* Problem 1). One possible approach to study the smoothability question could involve the use of gauge theory, possibly combined with some other techniques; we hope to return to such problems in a future paper.

It would also be interesting to know if the group actions constructed in this paper can be used to provide a different proof that $\mathrm{O}(5) \not\simeq Top(S^4)$ in the spirit of [33].

The homotopy equivalence $\mathrm{O}(4) \subset Top(S^3)$ makes the following problem quite interesting.

Problem 2. *If a finite group acts effectively and topologically on S^3, must it be isomorphic to a subgroup of* $\mathrm{O}(4)$*?*

Finally, in view of Theorem 1, the following problem becomes of considerable interest.

Problem 3. *What is the smallest dimension n of the sphere S^n such that S^n admits an orientation preserving smooth (locally linear) action of a finite group G with G not being isomorphic to a subgroup of* $\mathrm{SO}(n+1)$*?*

The paper is divided into two parts. The first contains the proof of Theorem 1. The second contains the proof of Theorem 2. Our proofs combine quite intricate considerations involving the theory of finite groups and surgery theoretic considerations based on some computational results for Wall's surgery obstruction groups [36], [8].

1. Orientation preserving actions on S^4

Proof of Theorem 1. We start our proof by recalling some arguments from [22, 23].

Theorem 1 was verified by Mecchia and Zimmermann (*cf.* [22, 23]) in many special cases; *e.g.*, when G is nonsolvable. In fact, they showed that G must be isomorphic to a subgroup of $\mathrm{SO}(5)$, except for the following indeterminacy of index two in the case where G is solvable.

There is an index two subgroup H of G which is isomorphic to a subgroup of $\mathrm{SO}(4)$ *such that $\Sigma^H = \{p_+, p_-\}$, and $\Sigma^G = \emptyset$.*

We will prove Theorem 1 by removing the above indeterminacy of G, showing that G is isomorphic to a subgroup of $\mathrm{O}(4)$. For reader's convenience, we shall give an outline of Mecchia and Zimmermann's arguments in [23] first. Let E be the maximal semisimple normal subgroup of G (recall that a semisimple group is a central product of quasisimple groups, where a quasisimple group is a perfect central extension of a simple group, *cf.* [31], Chapter 6.6.) Then the arguments in [23] are divided into two cases: (1) E is nontrivial, (2) E is trivial.

In an earlier paper [22], Mecchia and Zimmermann showed that if E is nontrivial, then E must be isomorphic to one of the following groups: $A_5, A_6, A_5^*, A_5^* \times_{\mathbb{Z}_2} A_5^*$, where A_5^* denotes the binary dodecahedral group. With this understood, the arguments for case (1) proceed by considering the subgroup $\tilde{E}$ which is generated by E and its centralizer in G, exploiting the fact that $G/\tilde{E}$ is isomorphic to a subgroup of the outer automorphism group $\mathrm{Out}\ (E)$ which is small for the possible groups E.

The index two indeterminacy occurs in case (2) where E is trivial. In this case, one considers instead the Fitting subgroup F of G (recall that the Fitting subgroup is the maximal nilpotent normal subgroup). The subgroup F^* generated by F and E is called the generalized Fitting subgroup of G, and as its main property, G/F^* is isomorphic to $\mathrm{Out}\ (F^*)$. Since E is trivial, one has in this case the crucial fact that G/F is isomorphic to a subgroup of $\mathrm{Out}\ (F)$. The Fitting subgroup F is a direct product of its Sylow p-subgroups, for different primes p, and each of the Sylow p-subgroups of F is normal in G. With this understood, the aforementioned index two indeterminacy in [23] occurs precisely in the following situation:

Let S be any Sylow p-subgroup of F, and let Z be the maximal elementary Abelian subgroup in the center of S (note that Z is normal in G). Then one of the following is true (*cf.* [23], pp. 745-746).

(i) Z has rank one and $\Sigma^Z = \{p_+, p_-\}$.
(ii) Z has rank two and $\Sigma^Z = \{p_+, p_-\}$.
(iii) Z has rank three and $\Sigma^Z = \{p_+, p_-\}$, with $p = 2$.

With the preceding understood, our arguments for removing the index two indeterminacy in [23] are based on the following Atiyah-Bott Theorem (*cf.* [36]).

Atiyah-Bott Theorem: *Let a finite group Γ act locally linearly, semifreely, and orientation preservingly on Σ with two isolated fixed points $\{p_+, p_-\}$. Then there is an orientation reversing isomorphism between $T_{p_+}\Sigma$ and $T_{p_-}\Sigma$, under which the corresponding representations of Γ become equivalent.*

The Atiyah-Bott Theorem is a crucial ingredient for proving that in the case of orientation–preserving actions of G on Σ, the aforementioned index two subgroup H of G has certain special properties. Namely, we have the following key lemma.

Lemma 1.1. *Let $h \in H$ be any element such that $h^2 \neq 1$ and for any $k \in \mathbb{Z}$ with $h^k \neq 1$, $\Sigma^{h^k} = \{p_+, p_-\}$. Then for any $\lambda \in G \setminus H$, λ does not lie in the set*

$$C_G^*(h) = \{x \in G | xhx^{-1} = h \text{ or } h^{-1}\}.$$

Proof. Assume $\lambda \in C_G^*(h)$. We shall derive a contradiction.

Consider first the case where $\lambda \in C_G(h)$, the centralizer of h. In this case, the fixed-point set Σ^λ, which is either a 0-sphere or 2-sphere, is invariant under h. Furthermore, it is disjoint from $\Sigma^h = \{p_+, p_-\}$ because λ switches the two points. But this is a contradiction, for either $h^2 \neq 1$ fixes Σ^λ or h has a fixed point on it.

Consider next the case where $\lambda \in C_G^*(h) \setminus C_G(h)$. By the Atiyah-Bott Theorem, there is an orientation reversing isomorphism between $T_{p_+}\Sigma$ and $T_{p_-}\Sigma$, under which the corresponding action of h becomes equivalent. We will denote this real 4-dimensional $\langle h \rangle$-representation by V. Note that $\lambda : V \to V$ is orientation reversing, and for any $v \in V$, $h^{-1}\lambda(v) = \lambda h(v)$. We will show these two conditions contradict each other.

First, a digression. Let A be a 2×2 matrix of real coefficients and let

$$H_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $\sin\theta \neq 0$, such that $H_\theta^{-1} A = A H_\theta$. Then it's an elementary exercise to see that there are $a, a' \in \mathbb{R}$, such that

$$A = \begin{pmatrix} a & a' \\ a' & -a \end{pmatrix}$$

In particular, $\det A = -a^2 - (a')^2 \leq 0$ for any such a matrix A, and $\det A = 0$ if and only if $A = 0$.

Now suppose $V = V_1 \oplus V_2$ where V_1, V_2 are 2-dimensional, non-equivalent $\langle h \rangle$-representations. Then $\lambda(V_i) = V_i$, $i = 1, 2$, because $\lambda(V_i)$ is equivalent to V_i for each i, as $\lambda \in C^*_G(h)$ by assumption. Then $h^{-1}\lambda(v) = \lambda h(v)$ implies that the restriction of λ on each V_i may be represented by a 2×2 matrix A_i satisfying $H_{\theta_i}^{-1} A_i = A_i H_{\theta_i}$ for some θ_i. Note that $\Sigma^{h^2} = \{p_+, p_-\}$ implies that $\sin\theta_i \neq 0$ for each i, and consequently, $\det A_i \leq 0$. But this implies that $\lambda : V \to V$ is orientation preserving, which is a contradiction.

Suppose $V = V_1 \oplus V_2$ where V_1, V_2 are 2-dimensional $\langle h \rangle$-representations which are equivalent. Without loss of generality, we assume, after giving proper orientation to each V_i, that the action of h on each V_i is given by a rotation by an angle θ, where $0 < \theta < \pi$ (in particular, $\sin\theta \neq 0$). If we represent λ by a matrix $A = (A_{ij})$, where each A_{ij} is a 2×2 matrix, then $h^{-1}\lambda(v) = \lambda h(v)$ implies that $H_\theta^{-1} A_{ij} = A_{ij} H_\theta$ for each i, j. Without loss of generality, assume A_{11} is non-singular. Then

$$\det A = \det A_{11} \cdot \det(A_{22} - A_{21} A_{11}^{-1} A_{12}) > 0,$$

because $A_{22} - A_{21} A_{11}^{-1} A_{12}$ satisfies the equation $H_\theta^{-1} A = A H_\theta$ also, and hence has non-positive determinant. But this contradicts the fact that $\lambda : V \to V$ is orientation reversing. Hence Lemma 1.1 follows.

□

Given the above, we shall divide the proof of Theorem 1 into two stages. In Stage 1, we assume that there is a prime p such that either (ii) or (iii) is true.

Proposition 1.2. *Suppose there is a prime p such that either (ii) or (iii) is true. Then G is isomorphic to a subgroup of* $\mathrm{SO}(4)$.

The proof of Proposition 1.2 is based on the following observation.

Observation 1.3. *Suppose $\lambda^2 = 1$ for any $\lambda \in G \setminus H$. Then G is isomorphic to a subgroup of* $\mathrm{SO}(4)$.

Proof. We pick a $t \in G \setminus H$, and observe that $G \setminus H = \{ht | h \in H\}$. Then $\lambda^2 = 1$ for any $\lambda \in G \setminus H$ gives $t^2 = 1$, $htht = 1$, which implies that

$$h^{-1} = tht^{-1}, \ \ \forall h \in H.$$

Furthermore, for any $h_1, h_2 \in H$,

$$h_1 h_2 = (th_1^{-1}t^{-1}) \cdot (th_2^{-1}t^{-1}) = t(h_2 h_1)^{-1} t^{-1} = h_2 h_1.$$

Consequently, H is Abelian, and G has the following presentation

$$G = \{h \in H, t | t^2 = 1, tht^{-1} = h^{-1}, \forall h \in H\}.$$

To see that G is isomorphic to a subgroup of SO(4), note that since $H \subset \mathrm{SO}(4)$ is Abelian, there is an identification $\mathbb{R}^4 = \mathbb{C}^2$ such that H can be represented by complex 2×2 matrices which are unitary and symmetric. With this understood, we extend the faithful representation of H on $\mathbb{C}^2$ to a faithful representation of G on $\mathbb{R}^4$ by defining the action of t as follows:

$$t \cdot (z_1, z_2) = (\bar{z}_1, \bar{z}_2), \quad (z_1, z_2) \in \mathbb{C}^2.$$

This indeed is compatible with the action of H, as for any $h \in H$,

$$tht^{-1}(z_1, z_2) = th(\bar{z}_1, \bar{z}_2) = \bar{h}(z_1, z_2) = h^{-1}(z_1, z_2).$$

□

Given the above observation, Proposition 1.2 follows from Lemma 1.4 and Lemma 1.5 below, which deal with (ii) and (iii) respectively. But first of all, the following fact, which follows from a simple argument in Smith theory, will be frequently used:

Let $\lambda \in G \setminus H$ such that $\lambda^2 \neq 1$. Then Σ^λ is a 0-sphere disjoint from $\{p_+, p_-\}$ and Σ^{λ^2} is a 2-sphere containing $\{p_+, p_-\}$.

Lemma 1.4. *Suppose there is a Sylow p-subgroup S of the Fitting subgroup F such that the maximal elementary Abelian subgroup Z of S has rank two with $\Sigma^Z = \{p_+, p_-\}$, then $\lambda^2 = 1$ for any $\lambda \in G \setminus H$.*

Proof. First, as shown in [22], Lemma 4.1, there are exactly two index p subgroups Z_1, Z_2 of Z whose fixed-point set is a 2-sphere, and for all other index p subgroups of Z the fixed-point set is a 0-sphere. For convenience we sketch its proof here. The action of Z on Σ obeys the following Borel Formula (*cf.* [6])

$$4 - r = \Sigma_i (n(Z_i) - r)$$

where Z_i is running over all index p subgroups of Z, $n(Z_i)$ denotes the dimension of Σ^{Z_i}, and r is the dimension of Σ^Z. Since each Z_i is cyclic, $n(Z_i)$ is either 0 or 2 by Smith theory. The claim follows easily from Borel Formula with $r = 0$.

Let $\lambda \in G \setminus H$ and assume $\lambda^2 \neq 1$. The action of λ on Z by conjugation will leave the set $\{Z_1, Z_2\}$ invariant; in particular, λ^2 leaves each of Z_1 and Z_2 invariant under conjugation. Consequently, Σ^{Z_i} $(i = 1, 2)$ is invariant under λ^2. Now observe that the three 2-spheres Σ^{Z_1}, Σ^{Z_2} and Σ^{λ^2} all contain $\{p_+, p_-\}$. The tangent plane of Σ^{λ^2} at p_+ is fixed under λ^2, and the tangent planes of Σ^{Z_1} and Σ^{Z_2} at p_+ are distinct. It follows that Σ^{λ^2} must coincide with one of Σ^{Z_1}, Σ^{Z_2}, say Σ^{Z_1}. This implies that $\Sigma^\lambda \subset \Sigma^{\lambda^2} = \Sigma^{Z_1}$, and Σ^{Z_1} is invariant under λ. It follows then that each of Z_1 and Z_2 is invariant under the action of λ. Particularly, Σ^{Z_2} is invariant under λ. Now observe that $\Sigma^{Z_1} \cap \Sigma^{Z_2} = \Sigma^Z = \{p_+, p_-\}$, so that the action of λ on Σ^{Z_2} has no fixed points. This implies that λ^2 fixes Σ^{Z_2} also, which is a contradiction. Hence $\lambda^2 = 1$. □

Lemma 1.5. *Suppose the maximal elementary Abelian subgroup Z of the Sylow 2-subgroup S of F has rank three with $\Sigma^Z = \{p_+, p_-\}$. Then $\lambda^2 = 1$ for any $\lambda \in G \setminus H$.*

Proof. There are seven subgroups of index 2 in Z. As shown in [22], Lemma 4.1, the Borel Formula implies that four of the subgroups have 1-dimensional fixed-point set, which is a 1-sphere, and three subgroups have 0-dimensional fixed-point set. Equivalently, of the seven involutions of Z, six of them has fixed-point set a 2-sphere, and exactly one involution has fixed point set a 0-sphere, which is given by $\Sigma^Z = \{p_+, p_-\}$.

Denote by $h \in Z$ the involution with $\Sigma^h = \Sigma^Z = \{p_+, p_-\}$. We denote the action of any $g \in G$ on Z given by conjugation by $g(x) = gxg^{-1}$, $\forall x \in Z$. Then clearly $g(h) = h$, $\forall g \in G$.

We first prove that either $\lambda^2 = 1$ or $\lambda^6 = 1$ for any $\lambda \in G \setminus H$. Let $Z_0 = Z \setminus \{1, h\}$ be the set of the six involutions whose fixed-point set is a 2-sphere. Let $\lambda \in G \setminus H$ such that $\lambda^2 \neq 1$. We shall examine the action of λ on Z_0.

Two observations first: (a) if $x \in Z_0$ is fixed under λ, i.e., $\lambda(x) = x$, so is $xh \in Z_0$, as $\lambda(xh) = \lambda(x)\lambda(h) = xh$; (b) for any $x \in Z_0$, if $xh = \lambda^k(x)$ is in the orbit of x, then $\lambda^{2k}(x) = \lambda^k(xh) = \lambda^k(x)\lambda^k(h) = xhh = x$.

With this understood, we first show that λ can not act transitively on Z_0. This is because for any $x \in Z_0$, (b) implies that $xh = \lambda^3(x)$. It follows easily that the orbit $x, \lambda(x), \cdots, \lambda^5(x)$ has the following form

$$x, y, z, xh, yh, zh,$$

where $z = xy$ or xyh. Assume first that $z = xy$. Then the subgroup generated by x, y, z is transformed under λ to the subgroup generated by y, z, xh. However, this is a contradiction because the two subgroups have different ranks. In the case where $z = xyh$, the subgroup generated by zh, x, y is transformed under λ to the subgroup generated by x, y, z. The former has rank two but the latter has rank three, which is also a contradiction. Hence the claim follows.

Next, it follows easily from (a) that the action of λ on Z_0 can not have an orbit consisting of five elements. Now consider the next case where the action of λ has an orbit consisting of four elements, say x_1, x_2, x_3, x_4. Let $\{x, y\} = Z_0 \setminus \{x_i\}$ be the complement. We claim $y = xh$. To see this, note that if x is fixed under λ, then so is xh, which implies $xh \in Z_0 \setminus \{x_i\}$. Hence $y = xh$. If x is not fixed under λ, then $y = \lambda(x)$, $x = \lambda(y)$, so that xy is fixed under λ. This implies that $xy = h$, which is equivalent to $y = xh$. With this understood, note that the subgroup generated by x, y is invariant under λ, which has rank two, and furthermore, note that its fixed-point set is $\{p_+, p_-\}$, and that exactly two subgroups of index 2 has fixed-point set a 2-sphere. The argument of the previous lemma then shows that $\lambda^2 = 1$ must be true, which is a contradiction. This proves the claim that the action of λ can not have an orbit consisting of four elements.

Suppose the action of λ has an orbit consisting of three elements x_1, x_2, x_3. We will show that $\lambda^6 = 1$. To see this, note first that (b) implies that for any $1 \leq i, j \leq 3$, $x_i h \neq x_j$. It follows easily that $x_1 h, x_2 h, x_3 h$ form another orbit. Moreover, $x_3 = x_1 x_2$ or $x_3 = x_1 x_2 h$. In the former case, x_1, x_2, x_3 generate a rank two subgroup Z' of Z invariant under λ, whose fixed-point set $\Sigma^{Z'}$ is a 1-sphere containing $\{p_+, p_-\}$ because each Σ^{x_i} is a 2-sphere (*cf.* [22], Lemma 4.1). Moreover, λ permutes the 2-spheres Σ^{x_1},Σ^{x_2}, Σ^{x_3}. The 1-sphere $\Sigma^{Z'}$ is the intersection of Σ^{x_1}, Σ^{x_2}, Σ^{x_3}, which is invariant under λ. Since λ switches p_+ and p_-, it follows that λ^2 fixes $\Sigma^{Z'}$. Now there are four 2-spheres Σ^{x_1},Σ^{x_2}, Σ^{x_3}, and Σ^{λ^2} intersecting at the 1-sphere $\Sigma^{Z'}$. Examining the action of λ on the normal bundle of $\Sigma^{Z'}$, we see that λ^6 acts as identity in a neighborhood of $\Sigma^{Z'}$, which implies $\lambda^6 = 1$. In the latter case, apply the above argument to the orbit $x_1 h, x_2 h, x_3 h$, which generates a λ-invariant subgroup of rank two. Hence the claim $\lambda^6 = 1$ follows.

If the action of λ has an orbit consisting of two elements x_1, x_2, then we claim that there is a $z \in Z_0$ such that the set $\{z, zh\}$ is invariant under λ, which we have seen is a contradiction to the assumption $\lambda^2 \neq 1$ as we argued in the previous lemma. To see the claim, note that if $x_2 = x_1 h$, then we are done. If $x_2 \neq x_1$, then $x_1 h, x_2 h$ form another orbit. The complement $Z_0 \setminus \{x_i, x_i h\}$ is a set consisting of z, zh which is invariant under λ. Hence the claim follows.

Finally, if λ fixes every element of Z_0, then every subset $\{x, xh\}$ is invariant under λ, which is a contradiction. This shows that either $\lambda^2 = 1$ or $\lambda^6 = 1$ for any $\lambda \in G \setminus H$.

We shall further rule out the possibility that λ has order 6. To this end, we consider the centralizer $C_G(Z)$ of Z. We first show that for any $g \in C_G(Z)$, $g^2 = 1$. To see this, note that the action of g by conjugation leaves each 2-sphere Σ^z, where $z \in Z_0$, invariant, which all contains $\{p_+, p_-\}$. In particular, consider 2-spheres Σ^x, Σ^y, Σ^{xy}, for some choices of $x, y \in Z_0$ with $xy \in Z_0$. The action of g leaves each of them invariant, and so also leaves their intersection, which is a 1-sphere, invariant. If $g \in H$, then g fixes $\{p_+, p_-\}$, and g is either identity or a reflection on the 1-sphere. It follows easily that g^2 fixes each of the 2-spheres Σ^x, Σ^y, Σ^{xy}, hence $g^2 = 1$. If $g \in G \setminus H$, then g switches p_+ and p_-, and the action of g on the 1-sphere is by a rotation of order 2. It follows also that g^2 fixes each of the 2-spheres Σ^x, Σ^y, Σ^{xy}, and hence $g^2 = 1$. With this understood, we see that $C_G(Z)$ is an elementary Abelian 2-group. Now note that $S \subseteq C_G(Z)$ because Z lies in the center of S, so that $S = Z$ by the maximality of Z. On the other hand, since each Sylow p-subgroup for odd p lies in $C_G(Z)$, we see that $F = S = Z$ immediately.

With the above preparation, we now rule out the possibility that λ has order 6. The key observation is that in the proof of $\lambda^6 = 1$, we also showed that in that case λ^3 acts on $Z = F$ trivially. But we know that G/F is a subgroup of Out (F).

This implies that $\lambda^3 \in F = Z \subset H$. Consequently, $\lambda \in H$ because $\lambda^2 \in H$, which is a contradiction. This rules out the possibility that λ has order 6, and consequently, $\lambda^2 = 1$ for any $\lambda \in G \setminus H$. The proof of Lemma 1.5 is complete.

□

Now we enter Stage 2 of the proof, where we assume that for every Sylow p-subgroup of F, (i) is true. Furthermore, without loss of generality, we assume that there is a $\lambda \in G \setminus H$ such that $\lambda^2 \neq 1$. We remark that the Atiyah-Bott Theorem, through Lemma 1.1, played a crucial role in this stage.

Proposition 1.6. *Suppose (i) is true for every Sylow p-subgroup of F. Then G is isomorphic to a subgroup of* $\mathrm{O}(4)$.

It turns out that the Fitting subgroup F must be cyclic, and there is a $\lambda \in G \setminus H$ of order 4, such that H is the dihedral group generated by F and λ^2.

We begin with a group-theoretic observation.

Observation 1.7. *Let ξ be an element of order p^n where p is prime. Suppose λ is an element such that*

$$\lambda\xi\lambda^{-1} = \xi^k, \text{ where } k \neq 1 \pmod{p^n}$$

and λ^2 commutes with ξ. Then

- *if p is odd, then $k = -1 \pmod{p^n}$;*
- *if $p = 2$, then either $k = -1 \pmod{p^n}$ or $k = \pm 1 \pmod{p^{n-1}}$. Moreover, in the latter case, $n \geq 3$, and $\lambda\xi^2\lambda^{-1} = \xi^{\pm 2}$.*

Proof. First, the assumptions $\lambda\xi\lambda^{-1} = \xi^k$ and λ^2 commutes with ξ imply that $k^2 = 1 \pmod{p^n}$, or equivalently, $(k-1)(k+1) = 0 \pmod{p^n}$. Since $k \neq 1 \pmod{p^n}$, one must have either $k = -1 \pmod{p^n}$, or else there exist m_1, m_2 satisfying $1 \leq m_1, m_2 \leq n-1$ and $m_1 + m_2 \geq n$, such that

$$k - 1 = p^{m_1}u_1, k + 1 = p^{m_2}u_2$$

for some u_1, u_2 which are relatively prime to p.

We continue the discussion assuming that $k \neq -1 \pmod{p^n}$. Consider first the case where $m_1 \leq m_2$. We write

$$2k = p^{m_1}u_1 + p^{m_2}u_2 = p^{m_1}(u_1 + p^{m_2-m_1}u_2),$$

which implies $p^{m_1} = 2$. Consequently, $p = 2$, $m_1 = 1$, and $m_2 \geq n - m_1 = n - 1$ which implies $m_2 = n - 1$ and $k = -1 \pmod{p^{n-1}}$. Finally, observe that $m_2 - m_1 > 0$ because k is odd, which implies that $n = m_1 + m_2 \geq 3$. For the case where $m_2 \leq m_1$, a similar argument shows that $k = 1 \pmod{p^{n-1}}$, with $n \geq 3$.

□

The following observation plays a key role in determining the Sylow 2-subgroup of F.

Observation 1.8. *Let $h \in H$ be an element of order 2^n such that for any k with $h^k \neq 1$, $\Sigma^{h^k} = \{p_+, p_-\}$. If the normalizer of the subgroup generated by h contains a $\lambda \in G \setminus H$, then $n = 1$, i.e., $h^2 = 1$.*

Proof. We assume $n \geq 2$, and derive a contradiction.

Note that $\lambda \in G \setminus H$ has an even order, say $|\lambda| = k2^{m+1}$ where k is odd and $m \geq 0$. Then $\lambda^k \in G \setminus H$ and is also contained in the normalizer of the subgroup generated by h. After replacing λ by λ^k, we may assume λ has order 2^{m+1}, $m \geq 0$.

Set $\mu = \lambda^{2^m}$. Then $\mu^2 = 1$, and by Observation 1.7, we have $\mu h \mu^{-1} = h^k$, where k satisfies either $k = \pm 1 \pmod{2^n}$ or $k = \pm 1 \pmod{2^{n-1}}$ with $n \geq 3$. It follows that $\mu \xi \mu^{-1} = \xi^{\pm 1}$ for $\xi = h$, or h^2, with the order of ξ being a power of 2 which is greater than 2.

If $m = 0$, then $\mu = \lambda \in G \setminus H$, which contradicts Lemma 1.1. Suppose $m > 1$. We claim $\mu \xi \mu^{-1} = \xi$ must be true. Suppose to the contrary that $\mu \xi \mu^{-1} = \xi^{-1}$. Then if we set $\tau = \lambda^{2^{m-1}}$, then $\tau \xi \tau^{-1} = \xi^k$ for some k, where k satisfies $k^2 = -1 \pmod{|\xi|}$. But there is no solution for k if $|\xi|$ is a power of 2 greater than 2. Hence the claim follows. Now applying Observation 1.7 to τ, we get $\tau \xi \tau^{-1} = \xi^{\pm 1}$ for some ξ, with the order of ξ being a power of 2 which is greater than 2. By induction, we get a contradiction.

□

As an immediate corollary, we obtain

Lemma 1.9. *The Sylow 2-subgroup S of the Fitting subgroup F is of order 2.*

Proof. Since the maximal elementary Abelian subgroup Z of the center of S has rank one by assumption, S is either cyclic, dihedral, quaternion, or generalized quaternion (*cf.* Suzuki [31], pp. 58-59). Now observe that the number of cyclic subgroups of maximal order of S is one except for the case when S is quaternion, and when S is quaternion the number is 3. It follows that for a given $\lambda \in G \setminus H$, there exists a cyclic subgroup of maximal order, denoted by C, whose normalizer contains λ. On the other hand, since $Z \subseteq C$ has fixed points $\{p_+, p_-\}$, it follows that every element of C has fixed points $\{p_+, p_-\}$. By Observation 1.8, $C \cong \mathbb{Z}_2 \cong Z$, from which it follows that $S = Z \cong \mathbb{Z}_2$.

□

Since we assume that (i) is true for every Sylow p-subgroup of F, in particular, the maximal elementary Abelian subgroup Z of the center of Sylow p-subgroup for odd p has rank one, the Sylow p-subgroups with p odd are all cyclic (*cf.* Suzuki [31], pp. 58-59). Consequently, with Lemma 1.9, we see that F is cyclic. Moreover, note that $F \subseteq H$. Since G/F is a subgroup of Out F = Aut F which is cyclic, we see that H/F is cyclic, and we conclude that H is metacyclic.

Lemma 1.10. *Suppose there is a $\lambda \in G \setminus H$ such that $\mu = \lambda^2 \neq 1$, and F contains a Sylow p-subgroup of odd p. Then H is a dihedral group generated by F and μ, with $\mu^2 = 1$.*

Proof. We will exploit the double cover $\rho : \mathrm{SO}(4) \to \mathrm{SO}(3) \times \mathrm{SO}(3)$, whose kernel is $\pm I$. Note that $\ker \rho \cap H$ coincides with the Sylow 2-subgroup of F. We denote by K the subgroup of H generated by F and μ, which is also metacyclic. We denote by K', H', F', μ' the image of K, H, F, μ under ρ respectively. Note that the product of Sylow p-subgroups of F with p odd is mapped injectively onto F'.

We will first show that K' is dihedral, which is generated by F' and μ'. To this end, we let π_i be the projection from $\mathrm{SO}(3) \times \mathrm{SO}(3)$ onto its i-th factor, where $i = 1, 2$, and let $K_i = \pi_i(K')$, $F_i = \pi_i(F')$, and $\mu_i = \pi_i(\mu')$. Since K' is metacyclic, so are both K_1, K_2. Since the only metacyclic subgroups of $\mathrm{SO}(3)$ are either cyclic or dihedral, we see that μ_i^2 lies in the centralizer of F_i. Consequently, $(\mu')^2$ lies in the centralizer of F'. It follows easily that μ^2 lies in the centralizer of each Sylow p-subgroup of odd order. Now observe that each Sylow p-subgroup of odd order acts on Σ semifreely, it follows from Observation 1.7 and Lemma 1.1 (applied to λ) that $\mu\xi\mu^{-1} \neq \xi$ for each generator ξ in the Sylow p-subgroups of odd order. Applying Observation 1.7 again (this time to μ), with the fact that μ^2 lies in the centralizer of each Sylow p-subgroup of odd order, we see that $\mu\xi\mu^{-1} = \xi^{-1}$ for each ξ in the Sylow p-subgroups of odd order, which implies that each K_i is either trivial, $\mathbb{Z}_2$, or dihedral. Consequently, K' is dihedral and $(\mu')^2 = 1$. Since μ has a 2-dimensional fixed-point set, ρ is injective on the subgroup generated by μ. This implies that $\mu^2 = 1$.

Next we show that H is generated by F and μ. To this end, let $H_i = \pi_i(H')$, $i = 1, 2$. Then H_i is either cyclic or dihedral, because H' is metacyclic. This implies that for any $g' \in H'$, which is the image of $g \in H$ under ρ, if we let $g_i = \pi_i(g')$, then g_i^2 lies in the centralizer of F_i, which implies as earlier that g^2 lies in the centralizer of each Sylow p-subgroup of odd p. By Observation 1.7, $g\xi g^{-1} = \xi^{\pm 1}$ on each Sylow p-subgroup of odd p. We claim that either $g\xi g^{-1} = \xi^{-1}$ or $g\xi g^{-1} = \xi$ on F, which would imply in turn that either μg or g lies in F, and we are done.

The key point is the following: the real representation of each Sylow p-subgroup of odd p on the tangent spaces of p_+, p_- splits into a direct sum of two non–equivalent real 2-dimensional representations. This would imply that each of the two 2-dimensional representations is preserved by g, and furthermore, $g\xi g^{-1} = \xi^{-1}$ or $g\xi g^{-1} = \xi$ depends on whether g is a reflection or a rotation on each of them. This would then imply that if $g\xi g^{-1} = \xi^{-1}$, respectively, $g\xi g^{-1} = \xi$ for one Sylow p-subgroup of odd p, then $g\xi g^{-1} = \xi^{-1}$, respectively, $g\xi g^{-1} = \xi$ for all other Sylow p-subgroups of odd p. Hence the claim follows.

To see that the two 2-dimensional representations are non-equivalent, suppose to the contrary that they are equivalent. Then as argued in the proof of Lemma 1.1 (recall that each Sylow p-subgroup of odd order acts on Σ semifreely), the fact that $\lambda\xi\lambda^{-1} = \xi^k$ for some k satisfying $k^2 = -1 \pmod{p^n}$ implies that λ is represented by a matrix $A = (A_{ij})$, such that A_{ij} satisfies $A_{ij}H_\theta = H_{k\theta}A_{ij}$ for some θ where $\sin\theta \neq 0$. It follows that A_{ij}^2 satisfies

$$A_{ij}^2 H_\theta = H_\theta^{-1} A_{ij}^2,$$

so that A_{ij}^2 has the form

$$\begin{pmatrix} a & a' \\ a' & -a \end{pmatrix}.$$

But this is a contradiction, as on the one hand, $\det A_{ij}^2 = (\det A_{ij})^2 \geq 0$, and on the other hand, $\det A_{ij}^2 = -a^2 - (a')^2 \leq 0$, which implies $A_{ij} = 0$ for all i, j.

□

Proof of Proposition 1.6. If F contains no Sylow p-subgroups of odd p, then $F \cong \mathbb{Z}_2$, and it follows easily that $G \cong \mathbb{Z}_2$, which contradicts the assumption that there is a $\lambda \in G \setminus H$ such that $\lambda^2 \neq 1$. Hence, F must contain a Sylow p-subgroup of odd p. By Lemma 1.10, G is isomorphic to a subgroup of $\mathrm{O}(4)$ by the following action: if (z_1, z_2) is the complex representation of F given by the action on the tangent space at p_+ or p_-, then the action of G is the one where the action of λ is given by $\lambda \cdot (z_1, z_2) = (\bar{z}_2, z_1)$.

□

The proof of Theorem 1 is now complete.

2. Orientation reversing actions on S^4: local linearity and stable smoothability

This section uses methods of surgery theory to study Conjecture 2. Since these methods are of considerable complexity we start with general comments which we hope will shed some light onto our considerations.

Let G be a finite group acting freely on some sphere S^{n-1}, where $n \geq 2$ and $(n-1)$ is odd. Then it follows that G has a free resolution of period n, and its Tate cohomology $\widehat{H}^*(G; \mathbb{Z})$ is periodic of period n (*cf.* [25]).

Now given a finite group G of period n one can ask if G can act freely on a finite CW-complex X with $X \simeq S^{n-1}$.

It turns out that there is a finiteness obstruction $\sigma_n(G)$ (introduced by R. Swan in [32]) for existence of such an action. This obstruction takes value in a certain quotient of $\widetilde{K_0}(\mathbb{Z}[G])$; *i.e.*, in $\widetilde{K_0}(\mathbb{Z}[G])/T_G$ (*cf.* [9]).

Computation of these finiteness obstructions is in general a very complicated and quite technical task. For various groups it was carried successfully however by R. J. Milgram and I. Madsen (*cf.* [24], [21]). Quite extensive calculations were

done for the class of groups $Q(2^k a, b, c)$, where a, b, c are coprime integers, $k \geq 2$ and $Q(2^k a, b, c)$ is given by the semi-direct product

$$1 \to \mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c \to Q(2^k a, b, c) \to Q(2^k) \to 1$$

in which $Q(2^k)$ is the quaternionic 2-group (*cf.* [9]). In particular for $\pi = Q(8p, q, 1)$, there are conditions on p, q (*cf.* [24], [21]) which imply $\sigma_4(\pi) = 0$.

Let $\widetilde{X} \simeq S^3$ be a finite complex with a free action of π on $\widetilde{X}$. Then $\widetilde{X}/\pi = X$ is a finite 3-dimensional Poincaré complex and hence it is equipped with a Spivak normal bundle (*i.e.*, a homotopy spherical fibration; *cf.* [36], [9]). Let $f : X \to BSG$ be the classifying map for this fibration (*cf.* [9]).

By considering Sylow p-subgroups π_p of π and using the fact that there are manifold models (linear space forms) for each lifting $\widetilde{X}_{(p)} \simeq S^3/\pi_p$ one can conclude (*cf.* [9], [21]) that $f : X \to BSG$ lifts to $f : X \to BSO$ where BSO is the classifying space for oriented bundles. The existence of such a lifting leads to the existence of a normal map

$$f : (M^3, \nu_{M^3}) \to (X, \xi_X)$$

where ν is the stable normal bundle, and ξ_X is the Spivak normal bundle. Rather intricate and quite lengthy computations (see [3], [9], [21]) show that the surgery obstruction $\lambda(f) \in L^h_3(\pi)$ is trivial for each of the pairs

$$(p, q) = (3, 313), (3, 433), (3, 601), (7, 113), (5, 461), (7, 809), (11, 1321), (17, 103)\ .$$

In dimension 3 this means (*cf.* [16]) that there is a manifold M^3 and a map $k : M^3 \to X$ which is a $\mathbb{Z}[\pi]$-homology equivalence (in other words: there is a free action of π on some integral homology 3-sphere $\widetilde{M^3}$).

Note. The functorial splitting of the rational group algebra $Q[Q(8p, q)]$ in [4], pp. 452, leads to a corresponding splitting of the real representation ring $R[Q(8p, q)]$. It follows that the only irreducible faithful real representation has dimension 8. Consequently, $Q(8p, q)$ is not isomorphic to a subgroup of $\mathrm{O}(n)$ for $n \leq 7$.

Construction of actions in Theorem 2. Let π be the group $Q(8p, q)$, where (p, q) is any of the pairs mentioned earlier. Let E_X be the total space of a twisted I-bundle over X (= the unit disk bundle of a real line bundle).

Now if $Q(8)$ is the quaternionic group given by

$$Q(8) = \{x, y | x^4 = 1, x^2 = y^2, yxy^{-1} = x^{-1}\}$$

let $w : Q(8) \to \mathbb{Z}_2$ be a nontrivial orientation homomorphism given by

$$w(x) = +1,\ w(y) = -1.$$

It induces a corresponding homomorphism

$$w : Q(8p, q) \to \mathbb{Z}_2$$

which in turn yields a specific line bundle, and hence a specific choice of E_X.

Next, let E_{M^3} be the pull back of E_X by k. Then there is a degree one map

$$h : (E_{M^3}, \partial) \to (E_X, \partial)$$

and h is a $\mathbb{Z}[\pi]$-homology equivalence between manifold with boundary E_{M^3} and E_X. It turns out that $\lambda(h) \in L^h_0(\pi, w)$ is trivial (*i.e.*, $L^h_0(\pi, w) \cong \Gamma_0(\mathcal{F}, w)$), where $\Gamma_0(\mathcal{F}, w)$ are the homological surgery obstruction groups as in [7]. Here $\mathcal{F} : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]$ is the identity homomorphism.

Consequently we can replace the manifold E_{M^3} by a manifold (W^4, ∂) homotopy equivalent (rel ∂) to (E_X, ∂).

In particular, $h_0 = h|_{\partial W^4} : \partial W^4 \to \partial E_X$ is a $\mathbb{Z}[\tau]$-homology equivalence where $\tau \subset \pi$ is a subgroup of index two; in fact, $\tau \cong Q(4pq)$.

Now applying the above argument to

$$h_0 \times \mathrm{id} : \partial W^4 \times I \to \partial E_X \times I$$

one obtains a manifold (N^4, ∂) homotopy equivalent to $\partial E_X \times I$. Let

$$\overline{N} = N^4 \cup_\partial N^4 \cup_\partial \dots .$$

and form $M^4 = W^4 \cup_\partial \overline{N}$. Then M^4 is a one-ended manifold with the universal covering $\widetilde{M^4} \approx_{top} S^3 \times \mathbb{R}$ (*cf.* [11]).

As a consequence, the two-point compactification of $\widetilde{M^4}$ gives an action of π on S^4; the induced group action on the two "points at infinity" is given by the homomorphism w described above. This action is fixed-point free (it is pseudo-free with two singular points having isotropy group τ).

Consequently we have

Fact. *There is a finite group π acting topologically and orientation reversingly on S^4 without fixed points such that π is not isomorphic to a subgroup of* $\mathrm{O}(5)$.

Remarks. Starting with the trivial I-bundle over X (i.e., with $X \times I$) and repeating the above construction with the trivial orientation homomorphism $w : Q(8p, q) \to \mathbb{Z}_2$ one gets

Observation. *There is a finite group π acting topologically and orientation-preservingly on S^4 semifreely with two fixed points such that π is not isomorphic to a subgroup of* $\mathrm{SO}(5)$.

Note. Constructions like the ones above were known to the experts right after Freedman's work on 4-dimensional topological surgery.

The above *Fact* gives a topological solution to Conjecture 2. It would be interesting to see if one can transform the topological action constructed above into a locally linear one (and therefore provide counterexamples to Conjecture 2 in [23]). It is not clear at this moment if this is the case, but we describe below a possible strategy for achieving this, namely:

Let (p,q) be a pair of primes such that $Q(8p,q)$ acts freely on a homology 3-sphere Σ^3. Write $\pi = Q(8p,q)$ and $\tau = Q(4pq)$ where $\tau \subset \pi$ is a subgroup of index two.

Let $h : \Sigma^3/\pi \to X^3/\pi$ be a $\mathbb{Z}[\pi]$-homology equivalence, where X^3/π is a finite Poincaré complex with $\widetilde{X^3} \simeq S^3$.

Consider the exact homology surgery sequence in dimension 3 (*e.g.*, see [16]):

$$L^h_0(\pi) \xrightarrow{\gamma} \mathcal{S}_H(X^3/\pi) \xrightarrow{\eta} [X^3/\pi, G/Top] \xrightarrow{\Theta^h_3} L^h_3(\pi)$$

The transfer to the 2-fold cover $X/\tau \to X/\pi$ gives a commutative diagram

$$\begin{array}{ccccccc} L^h_0(\tau) & \xrightarrow{\gamma} & \mathcal{S}_H(X^3/\tau) & \xrightarrow{\eta} & [X^3/\tau, G/Top] & \xrightarrow{\Theta^h_3} & L^h_3(\tau) \\ \uparrow{\scriptstyle tr_*} & & \uparrow{\scriptstyle tr_*} & & \uparrow{\scriptstyle tr_*} & & \uparrow{\scriptstyle tr_*} \\ L^h_0(\pi) & \xrightarrow{\gamma} & \mathcal{S}_H(X^3/\pi) & \xrightarrow{\eta} & [X^3/\pi, G/Top] & \xrightarrow{\Theta^h_3} & L^h_3(\pi) \end{array}$$

Since $X^3/\tau \simeq S^3/\tau$ and S^3/τ is a manifold we have a base point in $\mathcal{S}_H(X^3/\tau)$.

Let $tr_*(h) = \widetilde{h} : \Sigma^3/\tau \to S^3/\tau$ be a $\mathbb{Z}[\pi]$-homology equivalence and let $[\widetilde{h}] \in \mathcal{S}_H(X^3/\tau)$. We claim that $\eta[\widetilde{h}] = 0$ in $[X^3/\tau, G/Top]$. Indeed, $[X^3/\tau, G/Top] \cong H_1(\tau;\mathbb{Z}_2) \cong \mathbb{Z}_2$ and since Θ^h_3 is a monomorphism (see [16]), our claim follows.

This means that there is an element $\widetilde{x} \in L^h_0(\tau)$ with $[\widetilde{h}] = \gamma\widetilde{x}$.

Hypothesis 2.1. *Suppose that there is an element $x \in L^h_0(\pi)$ such that $tr_*(x) = \widetilde{x}$.*

Assuming the above, we act on $[h] \in \mathcal{S}_H(X^3/\pi)$ by $(-x)$. Then we get a new $\mathbb{Z}[\pi]$-homology equivalence:

$$\bar{h} : \Sigma'/\pi \to X^3/\pi$$

with the lifting $\widetilde{\bar{h}} : \Sigma'/\tau \to S^3/\tau$ such that $\widetilde{\bar{h}}$ is H-cobordant to $\mathrm{id}_{S^3/\tau}$.

Let $\overline{W^4}$ be such an H-cobordism

$$(\overline{W^4}; \Sigma'/\tau, S^3/\tau) = (\overline{W^4}; \partial_0, \partial_1).$$

Now in our construction of the one-ended manifold M^4 we simply take

$$M^4 = W^4 \cup_{\partial_0} \overline{W^4} \cup_{\partial_1} (S^3/\tau) \times [0,\infty).$$

Obviously, $\widetilde{M^4} \approx_{top} S^3 \times \mathbb{R}$ and the action of π on the two-point compactification is locally linear.

To be even more specific, we should point out that local linearity of the above actions is related to the following question, which may be interesting in its own right:

Suppose that G is a finite group which acts freely on some integral homology 3-sphere Σ^3, let $H \subset G$ be a subgroup which acts freely on S^3, and assume that there is a twisted $\mathbb{Z}[H]$-homology equivalence from Σ^3/H to some spherical space

form $L(\Sigma, H)$. Can one find a free G-action on some homology 3-sphere Σ' such that the Atiyah-Singer ρ-invariants (cf. [1]*) of (Σ', H) and $L(\Sigma, H)$ are equal?*

If G acts freely and smoothly on some closed oriented 3-manifold M^3, then the invariant $\rho(M^3, G)$ of [1] is a complex valued function on $G \setminus \{1\}$ which is a rational number times the character of a virtual real representation of G.

Analogous questions in higher dimensions are studied in [12].

Assume now that $G = Q(8p, q)$ where p and q are distinct odd primes, and take H to be the index 2 subgroups $Q(4pq)$, which is a generalized quaternion group, as before, choose (p, q) so that G acts freely on some integral homology 3-sphere. For some choices of p and q — including $(17, 103)$, $(3, 313)$ and $(3, 433)$ — an affirmative answer to the ρ-invariant question would imply that one can choose the group actions constructed above to be locally linear; this uses computational techniques from [35], [27] and [21] together with class number computations for certain rings of algebraic integers.

Proof of Theorem 2. We begin with the following commutative diagram, in which all vertical and horizontal lines are exact fibration sequences:

$$\begin{array}{ccccccc}
Top/PL & \longrightarrow & G/PL & \xrightarrow{\varphi'} & G/Top & \xrightarrow{\overline{k}} & K(\mathbb{Z}_2, 4) \\
\Big\downarrow{=} & & \Big\downarrow{i_1} & & \Big\downarrow{i_2} & & \Big\downarrow{=} \\
Top/PL & \longrightarrow & BPL & \xrightarrow{\varphi} & BTop & \xrightarrow{k} & K(\mathbb{Z}_2, 4) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\cdot & \longrightarrow & BG & \xrightarrow{=} & BG & \longrightarrow & \cdot
\end{array}$$

In this diagram k denotes the universal triangulation obstruction of [17]. Since all spaces in the diagram are H-spaces and all morphisms are H-maps (*cf.* [17]), it follows that we have the following exact sequence of abelian groups, in which M/∂ is an abbreviation for $M/\partial M$:

$$[M/\partial, Top/PL] \longrightarrow [M/\partial, G/PL] \xrightarrow{\varphi'_*} [M/\partial, G/Top] \xrightarrow{\overline{k}_*} H^4(M/\partial; \mathbb{Z}_2)$$

Observe that the group at the extreme left is isomorphic to $H^3(M/\partial; \mathbb{Z}_2)$ and the group at the extreme right is isomorphic to $\mathbb{Z}_2$.

Now the 4-stage Postnikov approximation to G/Top is $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4)$, and the corresponding approximation to G/PL has classification invariant $\beta \circ \mathrm{Sq}^2 \in H^5(K(\mathbb{Z}_2, 2); \mathbb{Z})$, where $\beta : H^4(-; \mathbb{Z}_2) \to H^5(-; \mathbb{Z})$ is the Bockstein operation for the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0 \, .$$

The results of [17] then imply that the group

$$\operatorname{coker} \varphi'_* : [M/\partial, G/PL] \to [M/\partial, G/Top] \cong$$

$$\text{image } \overline{k}_* : [M/\partial, G/Top] \to H^4(M/\partial; \mathbb{Z}_2) \;\cong\; \mathbb{Z}_2$$

is equal to the subgroup of $H^4(M/\partial; \mathbb{Z}_2)$ given by

$$\text{red}_2\, H^4(M/\partial; \mathbb{Z}) \;+\; \text{Sq}^2 H^2(M/\partial; \mathbb{Z}_2)$$

where red_2 denotes the mod 2 reduction operation from $\mathbb{Z}$ to $\mathbb{Z}_2$ coefficients. It follows that the cokernel of φ'_* is nontrivial and the map $\overline{k}_*$ is onto. Furthermore, if $c : M/\partial \to S^4$ is a map with mod 2 degree equal to 1 which collapses the closed complement of a coordinate disk $D^4 \cong D \subset \text{Int}(M)$ to a point, then it follows that the composite

$$\mathbb{Z} \cong \pi_4(G/Top) \xrightarrow{c^*} [M/\partial, G/Top] \xrightarrow{\overline{k}_*} H^4(M/\partial; \mathbb{Z}_2) \cong \mathbb{Z}_2$$

is onto.

It is worthwhile to point out that $c^*(1)$ is the normal invariant of $(M\#|E_8|, \partial) \to (M, \partial)$ and this map is normally bordant to a homotopy equivalence. Here $|E_8|$ is the manifold constructed by M. Freedman in [11].

Claim. *If $w : Q(8p, q) \to \mathbb{Z}_2$ is the previously described homomorphism, then the surgery obstruction map $\sigma : [M/\partial, G/Top] \to L^h_0(Q(8p, q), w)$ is trivial on the image of c^*.*

Before proving the claim, we shall explain how it implies Theorem 2. There are two cases depending upon whether or not the relative Kirby-Siebenmann obstruction for the pair $(M, \partial M)$ is trivial or nontrivial (note that the boundary is uniquely smoothable and in fact is a spherical space form). In both cases, one crucial point is the fact that if $h : (N, \partial N) \to (M, \partial M)$ is a homotopy equivalence which is a diffeomorphism on the boundary and the normal invariant of h is the nontrivial element x in the image of c^*, then the relative Kirby-Siebenmann invariants of $(N, \partial N)$ and $(M, \partial M)$ are opposites because their difference is the nontrivial class $\overline{k}_* \circ c^*(x)$.

Suppose first that the relative Kirby-Siebenmann invariant of $(M, \partial M)$ is nontrivial, and let h be as above. Then by the last sentence of the preceding paragraph the relative invariant of $(N, \partial N)$ is trivial. The boundary of N is a spherical space form for the group $K = \text{Kernel}(w)$; let V be the associated 4-dimensional linear representation of that group, and let $D(V)$ denote its unit disk. If Σ^4 is formed by equivariantly identifying the boundaries of $\widetilde{N}$ and $Q(8p, q) \times_K D(V)$, then it follows that Σ^4 is a locally linear pseudofree $Q(8p, q)$-manifold, and the direct sum of its topological equivariant tangent bundle in the sense of [19] with a trivial 1-dimensional bundle comes from an equivariant vector bundle. Therefore the results of [19] imply that $\Sigma^4 \times \mathbb{R}^k$ is equivariantly smoothable for all $k \geq 1$.

In contrast, if the relative Kirby-Siebenmann invariant of $(M, \partial M)$ is trivial and h is as above, then we may again construct Σ^4 as in the preceding paragraph; if the direct sum of its topological equivariant tangent bundle in the sense of [19]

with a trivial k-dimensional bundle does NOT come from an equivariant vector bundle for all $k \geq 1$, then the results of [19] and [20] will imply that no product of the form $\Sigma^4 \times \mathbb{R}^k$ can be equivariantly smoothable. We need to show this tangent bundle assertion holds if the Kirby-Siebenmann invariant of $(M, \partial M)$ is trivial, which means that the corresponding invariant for $(N, \partial N)$ is not trivial. We shall do this by assuming the latter holds but $\Sigma^4 \times \mathbb{R}^k$ is nevertheless equivariantly smoothable and deriving a contradiction.

Suppose that a product as above is smoothable; without loss of generality we may assume that $k \geq 5$. Since topological and linear equivalence are the same for semifree representations (*e.g.*, see [14], Corollary 3.10), it follows that an invariant tubular neighborhood E of the singular set must be equivariantly diffeomorphic to

$$\mathbb{R}^k \times (Q(8p, q) \times_K D(V))$$

where K and V are defined as above. Let $\widetilde{W}$ be the closure of the complement of such an invariant tubular neighborhood, and let W be its (smooth) orbit space. It follows immediately that the relative Kirby-Siebenmann obstruction for $(W, \partial W)$ is trivial.

On the other hand, we claim that this obstruction is equal to the relative Kirby-Siebenmann obstruction for $(N, \partial N) \times \mathbb{R}^k$. To see this, first observe that we can choose the invariant smooth tubular neighborhood E to be so small that it lies in the interior of

$$(Q(8p, q) \times_K D(V)) \times \mathbb{R}^k \subset \Sigma^4 \times \mathbb{R}^k$$

(viewing the left side as a subset of the right via the construction of $\Sigma^4 \times \mathbb{R}^k$). Now the region between ∂E and $Q(8p, q) \times_K S(V) \times \mathbb{R}^k$ is a proper equivariant s-cobordism (*e.g.*, this follows from [30]), and this fact allows us to extend the linear structure on the topological equivariant tangent bundle of E (given by smoothability) to a linear structure on the equivariant topological tangent bundle of $(Q(8p, q) \times_K D(V)) \times \mathbb{R}^k$. It is natural to ask whether or not this linear structure coincides with the standard one given by product constructions on the equivariant tangent bundle of $D(V)$, at least up to homotopy; fortunately, this follows directly from the K-equivariant contractibility of $D(V) \times \mathbb{R}^k$ and the fact that the local representation of K at its fixed points is $V \times \mathbb{R}^k$ in all cases. On the orbit space level this compatibility of linear equivariant tangent bundle structures implies that the relative Kirby-Siebenmann invariants for $(W, \partial W)$ and $(N, \partial N) \times \mathbb{R}^k$ must be equal, and as noted in an earlier paragraph this contradiction implies that no product of the form $\Sigma^4 \times \mathbb{R}^k$ can be equivariantly smoothable.

To summarize, regardless of whether or not the original action on S^4 is stably smoothable, we have constructed a second action which has the opposite property.

Proof of Claim. Let $D \subset \mathrm{Int}(M)$ be an embedded closed 4-disk which extends to a slightly larger open 4-disk, so that $M - \mathrm{Int}(D)$ is a manifold with boundary. One can construct a relative surgery problem (*i.e.*, a homeomorphism on the boundary) with target M and surgery obstruction $c^*(x) \in [M/\partial, G/Top]$ as follows: Start with a relative topological surgery problem (f, Φ) with target D^4 which corresponds to a generator $x \in \pi_4(G/Top) \cong \mathbb{Z}$. Now construct a relative surgery problem (f', Φ') with target M from (f, Φ) and the identity on $M - \mathrm{Int}(D)$ by identifying the homeomorphism to ∂D^4 in the first problem with the identity on ∂D in the second. A direct analysis of this construction shows that the resulting surgery problem represents $c^*(x)$ and that the surgery obstruction is given by $\sigma(f') = i_*\sigma(f)$, where $i_* : L_0(1) \longrightarrow L_0^s(Q(8p, q), w)$ is induced by the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}[Q(8p, q)]$ of rings with unit. Therefore the proof of the Claim reduces to showing that i_* is trivial. We can reduce things further as follows: The Sylow 2-subgroup of $Q(8p, q)$ is the quaternionic group Q_8, so it suffices to prove that $L_0(1) \to L_0^s(Q_8, w')$ is trivial, where w' is the restriction of w. Since the forgetful map from $L_0^s(Q_8, w')$ to $L_0^h(Q_8, w')$ is an isomorphism (because $Wh(Q_8) = 0$), it actually suffices to check that the map $j_* : L_0(1) \to L_0^h(Q_8, w')$ is trivial.

The results of [35] and [8] show that $L_0^h(Q_8, w')$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Furthermore, the results of [13], pp. 122, show that one summand is detected by the composite

$$L_0^h(Q_8, w') \xrightarrow{f} L_0^p(Q_8, w') \xrightarrow{w_*} L_0(\mathbb{Z}_2, -) \cong \mathbb{Z}_2$$

where f is the forgetful map from homotopy to projective Wall groups (see [28]) and w_* is induced by the group homomorphism $w' : Q_8 \to \mathbb{Z}_2$; we can suppress the superscipt on the third Wall group in the display because $Wh(\mathbb{Z}_2)$ and $\widetilde{K_0}(\mathbb{Z}[\mathbb{Z}_2])$ are both trivial. In fact, the results of [13] show that w_* is an isomorphism and the map f is onto; one of the $\mathbb{Z}_2$ summands in $L_0^h(Q_8, w')$ maps onto the generator of $L_0^p(Q_8, w')$, and the other is given by the kernel of f. By the Rothenberg exact sequence relating L^h and L^p (see [28]), this kernel is the image of the connecting homomorphism

$$\delta : \mathbb{Z}_2 \cong H^*\left(\mathbb{Z}_2; \widetilde{K_0}(\mathbb{Z}[Q_8])\right) \to L_0^h(Q_8, w')$$

which is injective. Furthermore, if $b : \mathbb{Z}_4 \to Q_8$ denotes the inclusion of the kernel of w', then the results described on page 633 of [8] imply that the image of δ maps nontrivially under the homomorphism

$$L_0^h(Q_8, w') \to L_0^h(Q_8, \mathbb{Z}_4; w')$$

in the long exact sequence of L^h groups associated to the inclusion homomorphism b. Therefore, the triviality of the mapping j_* is equivalent to the triviality of both

$L_0(1) \to L_0(\mathbb{Z}_2, -)$ and the composite

$$L_0(1) \longrightarrow L_0^h(Q_8, w') \longrightarrow L_0^h(Q_8, \mathbb{Z}_4; w') \ .$$

The first of these maps is trivial by results of Wall (see [36], pp. 173, and [34], (4.13, Complement), for details). To prove the triviality of the second map, observe that j_* factors as a composite

$$L_0(1) \longrightarrow L_0(\mathbb{Z}_4) \xrightarrow{b_*} L_0^h(Q_8, w')$$

and by [8] there is an exact sequence

$$L_0(\mathbb{Z}_4) \xrightarrow{b_*} L_0^h(Q_8, w') \longrightarrow L_0^h(Q_8, \mathbb{Z}_4; w') \longrightarrow 0 \ .$$

The combination of these statements imply that $L_0(1) \to L_0^h(Q_8, \mathbb{Z}_4; w')$ is trival, and as noted above this completes the proof of the claim. □

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