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Abstract

In this paper we study approximation of rectangular partial sums of double Fourier series with respect to the Walsh-Kaczmarz system in the spaces $C$ and $L$. From this results we obtain different criterion of the uniform convergence and $L$-convergence of double Fourier-Kaczmarz series.

1 Introduction

L. Zhizhiashvili ([13], part. 2, Chap. 3) has established certain approximation properties of rectangular partial sums of the double trigonometric Fourier series in spaces $C$ and $L$. The analogous question for double Fourier series with respect to the Walsh-Paley system were treated in the works [2,8].

We will study approximation by rectangular partial sums of double Fourier series with respect to Walsh-Kaczmarz system to function $f \in L^p (I^2)$ in the

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norm of $L^p$ for $p = 1$ or $p = \infty$ (see Theorems 1, 2 and 3). From this theo-
rem one can obtain different criteria for the uniform convergence and $L^1$
-convergence of double Fourier series with respect to the Walsh-Kaczmarz
system, in particular, we established two-dimensional version of the Dini-
Lipschitz condition (see Corollary 1 and Corollary 2).

Results of somewhat different type can be obtained by using the variation
of a function.

Jordan [7] introduced a class of functions of bounded variation and, ap-
plying it to the theory of the Fourier series, he proved that if a continuous
function has bounded variation, then its Fourier series converges uniformly.
In 1906 Hardy [6] generalized the Jordan criterion to the double Fourier series
and introduced for the function of two variables the notion of bounded vari-
ation. He proved that if the continuous function of two variables has bounded
variation (in the sense of Hardy), then its Fourier series converges uniformly
in the sense of Pringsheim*. The analogous result for double Walsh-Fourier
series is verified by Moricz [8]. The author [3] has proved that in Hardy’s
theorem there is no need to require the boundedness of $V_{1,2}(f)$, in partic-
ular, it is proved that if $f$ is continuous function and has bounded partial
variation ($f \in PBV$) then its double trigonometric Fourier series converges
uniformly on $[0, 2\pi]^2$ in the sense of Pringsheim. The analogous result for
double Walsh-Fourier series is established in [4]. In this paper we study
analogical question in case of the Walsh-Kaczmarz system (see Theorem 4).

*A double series is said to converge in the sense of Pringsheim if its partial rectangular
sums converge.
2 Definitions and Notations

We shall denote the sets of all non-negative integer by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$ and the set of dyadic rational numbers in the unit interval $[0,1]$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p,n \in \mathbb{N}$, $0 \leq p \leq 2^n$.

Let $r_0(x)$ be a function defined on $[0,1)$ by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0,1/2) \\ -1, & \text{if } x \in [1/2,1) \end{cases}, \quad r_0(x + 1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \geq 1 \quad \text{and} \quad x \in [0,1).$$

Each $n \in \mathbb{N}$ it is possible write uniquely as

$$n = \sum_{k=0}^{m} n_k 2^k, \quad \text{(1)}$$

where $n_k = 0$ or 1. The numbers $n_k$ called the binary coefficients of $n$.

Given $x \in [0,1)$, the expansion

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \quad \text{(2)}$$

where each $x_k = 0$ or 1, will be called a dyadic expansion of $x$. If $x \in [0,1]\setminus\mathbb{Q}$, then (2) is uniquely determined. By the dyadic expansion $x \in \mathbb{Q}$ we choose the one for which $\lim_{k \to \infty} x_k = 0$.

The dyadic sum of $x,y \in [0,1)$ in terms of the dyadic expansion of $x$ and $y$ is defined by

$$x \oplus y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}.$$
The Walsh system in the Paley enumeration \( \{ w_n(x) : n \in \mathbb{N} \} \) is defined by

\[
w_n(x) = \prod_{j=0}^{m} (r_j(x))^{n_j}.
\]

Now we recall the definition of the Walsh-Kaczmarz system \( \{ \psi_n(x) : n \in \mathbb{N} \} \).

Set \( \psi_0(x) = 1 \), while for \( n \geq 1 \), which is given by expression (1) with \( n_m = 1 \), set

\[
\psi_n(x) = r_m(x)^{m-1} \prod_{j=0}^{m-1} (r_{m-j-1}(x))^{n_j}.
\]

Let us consider the kernel of Dirichlet for the Walsh-Paley system:

\[
D_n(x) = \sum_{j=0}^{n-1} \omega_j(x)
\]

and for the Walsh-Kaczmarz system

\[
K_n(x) = \sum_{j=0}^{n-1} \psi_j(x).
\]

We will need the well-known equality for the Dirichlet kernel of the Walsh-Paley system [see 5, p. 27]

\[
D_{2^n}(x) = \begin{cases} 
2^n, & \text{if } x \in [0, 1/2^n) ; \\
0, & \text{if } x \in [1/2^n, 1). 
\end{cases}
\]

The transformation \( \tau_n \) for \( x \in [0, 1) \) defined by

\[
\tau_n(x) = \sum_{k=0}^{n-1} x_{n-k-1} 2^{-(k+1)} + \sum_{j=n}^{\infty} x_j 2^{-(j+1)}.
\]

We will apply the transformation \( \tau_n \) also for integers \( p \geq 0 \), given by

\[
\tau_n(p) = \sum_{j=0}^{n-1} x_{n-j-1} 2^j,
\]
where
\[ p = \sum_{j=0}^{n-1} x_j 2^j. \]

Given \( n \geq 0 \) and \( 0 \leq p < 2^n \), we set
\[ I_n(p) = [p2^{-n}, (p + 1)2^{-n}) \]

It is evident that the transformation \( \tau_n(x) \) maps the segment \( I_n(p) \) on the segment \( I_n(\tau_n(p)) \).

It is known [11] that
\[ K_n(x) = D_{2^k}(x) + r_k(x)D_m(\tau_k(x)) \quad \text{for} \quad n = 2^k + m, \ 0 \leq m < 2^k \quad (4) \]

and
\[ |D_m(\tau_k(x))| \leq \frac{2^k}{\tau_k(p)} \quad \text{for} \quad x \in I_k(p), \ p = 1, 2, ..., 2^k - 1. \quad (5) \]

We consider the double system \( \{\psi_n(x) \times \psi_m(y) : n, m \in \mathbb{N}\} \) on the unit square \( I^2 = [0, 1] \times [0, 1] \).

If \( f \in L(I^2) \), then
\[ \hat{f}(n,m) = \int_0^1 \int_0^1 f(x,y) \psi_n(x)\psi_m(y) dx dy \]
is the \((n,m)\)-th Fourier coefficient of \( f \).

The rectangular partial sums of double Fourier series with respect to the Walsh-Kaczmarz system are defined by
\[ S_{M,N}(f;x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m,n) \psi_m(x)\psi_n(y). \]

As usual, denote by \( L(I^2), (1 \leq p \leq \infty) \) the set of all measurable functions defined on \( I^2 \), for which
\[ \|f\|_1 = \int_0^1 \int_0^1 |f(x,y)| dx dy \]
is finite. Furthermore, let \( C(I^2) \) be the set of all functions \( f : I^2 \to R \) that are uniformly continuous from the dyadic topology of \( I^2 \) to the usual topology of \( R \) with the norm (see [9], pp. 9-11)

\[
\|f\|_C = \sup_{x,y \in I^2} |f(x,y)| \quad (f \in C(I^2)).
\]

The total modulus of continuity, and the total integrated modulus of continuity are defined by

\[
\omega(f; \delta_1, \delta_2)_C = \sup \{ \|f(x \oplus u, y \oplus v) - f(x, y)\|_C : 0 \leq u < \delta_1, 0 \leq v < \delta_2 \},
\]

\[
\omega(f; \delta_1, \delta_2)_1 = \sup \{ \|f(x \oplus u, y \oplus v) - f(x, y)\|_1 : 0 \leq u < \delta_1, 0 \leq v < \delta_2 \},
\]

while the partial moduli of continuity, and the partial integrated moduli of continuity are defined by

\[
\omega_1(f; \delta_1)_C = \omega(f; \delta_1, 0)_C \quad \text{and} \quad \omega_2(f; \delta_2)_C = \omega(f; 0, \delta_2)_C \quad f \in C(I^2),
\]

\[
\omega_1(f; \delta_1)_1 = \omega(f; \delta_1, 0)_1 \quad \text{and} \quad \omega_2(f; \delta_2)_1 = \omega(f; 0, \delta_2)_1 \quad f \in L(I^2).
\]

We also use the notion of the mixed modulus of continuity, and the mixed integrated modulus of continuity are defined as follows

\[
\omega_{1,2}(f; \delta_1, \delta_2)_C = \sup \{ \|f(x \oplus u, y \oplus v) - f(x \oplus u, y)

- f(x, y \oplus v) + f(x, y)\|_C : 0 \leq u < \delta_1, 0 \leq v < \delta_2 \}, \quad f \in C(I^2),
\]

\[
\omega_{1,2}(f; \delta_1, \delta_2)_1 = \sup \{ \|f(x \oplus u, y \oplus v) - f(x \oplus u, y)

- f(x, y \oplus v) + f(x, y)\|_1 : 0 \leq u < \delta_1, 0 \leq v < \delta_2 \}, \quad f \in L(I^2).
\]

A function \( f : I^2 \to R \) is said to be of bounded variation in the sense of Hardy (\( f \in HBV(I^2) \)) if there exists a constant \( K \) such that for any partition

\[
\Delta_1 : 0 \leq x_0 < x_1 < x_2 < \cdots < x_n \leq 1,
\]
\[ \Delta_2 : 0 \leq y_0 < y_1 < y_2 < \cdots < y_m \leq 1, \]

we have

\[
V_{1,2} (f) = \sup_{\Delta_1 \times \Delta_2} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |f(x_i, y_j) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_{i+1}, y_{j+1})| \leq K,
\]

\[
V_1 (f) = \sup_y \sup_{\Delta_1} \sum_{i=0}^{n-1} |f(x_i, y) - f(x_{i+1}, y)| \leq K,
\]

\[
V_2 (f) = \sup_x \sup_{\Delta_2} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |f(x, y_j) - f(x, y_{j+1})| \leq K.
\]

**Definition 1** ([4]) We say that the function \( f : I^2 \to R \) is bounded partial variation (\( f \in \text{PBV} (I^2) \)) if \( V_1 (f) \) and \( V_2 (f) \) are finite.

Given a function \( f(x, y) \), periodic in both variables with period 1, for \( 0 \leq j < 2^m \) and \( 0 \leq i < 2^n \) and integers \( m, n \geq 0 \) we set

\[
\Delta^m_j f(x, y)_1 = f\left(x \oplus 2j2^{-m-1}, y\right) - f\left(x \oplus (2j + 1)2^{-m-1}, y\right),
\]

\[
\Delta^n_i f(x, y)_2 = f\left(x, y \oplus 2i2^{-n-1}\right) - f\left(x, y \oplus (2i + 1)2^{-n-1}\right),
\]

\[
\Delta^{mn}_{ji} f(x, y) = \Delta^n_i \left( \Delta^m_j f(x, y) \right)_1 = \Delta^m_j \left( \Delta^n_i f(x, y) \right)_2
\]

\[
= f\left(x \oplus 2j2^{-m-1}, y \oplus 2i2^{-n-1}\right) - f\left(x \oplus (2j + 1)2^{-m-1}, y \oplus 2i2^{-n-1}\right)
\]

\[
- f\left(x \oplus 2j2^{-m-1}, y \oplus (2i + 1)2^{-n-1}\right) + f\left(x \oplus (2j + 1)2^{-m-1}, y \oplus (2i + 1)2^{-n-1}\right).
\]

Furthermore, set \( \lambda^m_0 = 1 \) and \( \lambda^n_j = (\tau_m (j))^{-1} \) for \( 1 \leq j < 2^m \) and

\[
W^{(1)}_m (f; x, y) = \sum_{j=0}^{2^m-1} \lambda^m_j \left| \Delta^m_j f(x, y) \right|_1,
\]

\[
W^{(2)}_n (f; x, y) = \sum_{i=0}^{2^n-1} \lambda^n_i \left| \Delta^n_i f(x, y) \right|_2,
\]

\[
W_{mn} (f; x, y) = \sum_{j=0}^{2^m-1} \sum_{i=0}^{2^n-1} \lambda^m_j \lambda^n_i \left| \Delta^{mn}_{ji} f(x, y) \right|.
\]
2.1 Main Results

**Theorem 1** Let $M, N$ be positive integers such that $M = 2^m + j$, $0 \leq j < 2^m$ and $N = 2^n + i$, $0 \leq i < 2^n$, for some integers $m, n \geq 0$. If $f \in C(I^2)$, then

$$\|S_{M,N}(f) - f\|_C \leq \omega \left( f; \frac{1}{2^m}, \frac{1}{2^n} \right)_C + \frac{1}{2} \left\| W_m^{(1)}(f) \right\|_C + \frac{1}{2} \left\| W_n^{(2)}(f) \right\|_C + \frac{1}{4} \| W_{mn}(f) \|_C.$$

**Theorem 2** Let $M, N$ be positive integers such that $M = 2^m + j$, $0 \leq j < 2^m$ and $N = 2^n + i$, $0 \leq i < 2^n$, for some integers $m, n \geq 0$. If $f \in C(I^2)$, then

$$\|S_{M,N}(f) - f\|_C \leq c^* \left\{ \omega_1 \left( f; \frac{1}{2^m} \right)_C m + \omega_2 \left( f; \frac{1}{2^n} \right)_C n + \omega_{1,2} \left( f; \frac{1}{2^m}, \frac{1}{2^n} \right)_C mn \right\}.$$

**Corollary 1** Let $f \in C(I^2)$ and

$$\omega_1 \left( f; \frac{1}{2^m} \right)_C m \to 0 \quad \text{as} \quad m \to \infty,$$

$$\omega_2 \left( f; \frac{1}{2^n} \right)_C n \to 0 \quad \text{as} \quad n \to \infty,$$

$$\omega_{1,2} \left( f; \frac{1}{2^m}, \frac{1}{2^n} \right)_C mn \to 0 \quad \text{as} \quad m, n \to \infty,$$

then the double Fourier series with respect to Walsh-Kaczmarz system converges uniformly on $I^2$.

**Theorem 3** Let $M, N$ be positive integers such that $M = 2^m + j$, $0 \leq j < 2^m$ and $N = 2^n + i$, $0 \leq i < 2^n$, for some integers $m, n \geq 0$. If $f \in L(I^2)$, then

$$\|S_{M,N}(f) - f\|_1 \leq c \left\{ \omega_1 \left( f; \frac{1}{2^m} \right)_1 m + \omega_2 \left( f; \frac{1}{2^n} \right)_1 n + \omega_{1,2} \left( f; \frac{1}{2^m}, \frac{1}{2^n} \right)_1 mn \right\}.$$

*In this paper the constant $c$ is absolute constant and may different in different contexts.*
Corollary 2 Let $f \in L(I^2)$ and

$$\omega_1 \left( f; \frac{1}{2^m} \right) m \to 0 \quad \text{as} \quad m \to \infty,$$

$$\omega_2 \left( f; \frac{1}{2^n} \right) n \to 0 \quad \text{as} \quad n \to \infty,$$

$$\omega_{1,2} \left( f; \frac{1}{2^m}, \frac{1}{2^n} \right) mn \to 0 \quad \text{as} \quad n,m \to \infty,$$

then the double Fourier series with respect to Walsh-Kaczmarz system converges to $f$ in $L$-norm.

Theorem 4 Let $f \in C(I^2) \cap PBV(I^2)$. Then the double Fourier series with respect to Walsh-Kaczmarz system converges uniformly on $I^2$.

2.2 Proof of Main Results

Proof of Theorem 1. From (4), we write

$$S_{M,N}(f; x, y) - f(x, y) = \int_0^1 \int_0^1 \left[ f(x \oplus u, y \oplus v) - f(x, y) \right] K_M(u) K_N(v) \, dudu$$

$$= \int_0^1 \int_0^1 \left[ f(x \oplus u, y \oplus v) - f(x, y) \right] D_{2m}(u) D_{2n}(v) \, dudu$$

$$+ \int_0^1 \int_0^1 \left[ f(x \oplus u, y \oplus v) - f(x, y) \right] D_{2m}(u) r_n(v) D_i(\tau_n(v)) \, dudu$$

$$+ \int_0^1 \int_0^1 \left[ f(x \oplus u, y \oplus v) - f(x, y) \right] D_{2n}(v) r_m(u) D_j(\tau_m(u)) \, dudv$$

$$+ \int_0^1 \int_0^1 \left[ f(x \oplus u, y \oplus v) - f(x, y) \right] r_n(v) r_m(u) D_{ij}(\tau_m(u)) D_i(\tau_n(v)) \, dudv$$
= I + II + III + IV.  \tag{6}

We obtain by (3) that

\[
\|I\|_C = 2^{n+m} \left\| \int_0^{1/2m} \int_0^{1/2n} [f(x \oplus u, y \oplus v) - f(x, y)] \, du \, dv \right\|_C
\leq \omega \left( f; \frac{1}{2m}, \frac{1}{2n} \right)_C. \tag{7}
\]

It is well-known that

\begin{align*}
\text{a) } I_m(j) &= I_{m+1}(2j) \cup I_{m+1}(2j+1); \\
\text{b) } w_{2^m}(u) &= \begin{cases} 1, & \text{if } u \in I_{m+1}(2j) \\
-1, & \text{if } u \in I_{m+1}(2j+1); \end{cases} \\
\text{c) } t &= u \oplus 2^{-m-1} \text{ is a one-to-one mapping of } I_{m+1}(2j) \text{ onto } I_{m+1}(2j+1). 
\end{align*}

Thus, by (3), and (a)-(d)

\[
II = 2^m \int_{I_m(0)} \int_0^{1} [f(x \oplus u, y \oplus v) - f(x, y)] R_n(v) D_i(\tau_n(v)) \, du \, dv
\]

\[
= 2^m \sum_{r=0}^{2^n-1} \int_{I_m(0)} \left( \int_{I_{n+1}(2r)} [f(x \oplus u, y \oplus v) - f(x, y)] R_n(v) D_i(\tau_n(v)) \, dv \right) \, du
\]

\[
= 2^m \sum_{r=0}^{2^n-1} \int_{I_m(0)} \left( \int_{I_{n+1}(2r)} [f(x \oplus u, y \oplus v) - f(x, y)] D_i(\tau_n(v)) \, dv \\
- \int_{I_{n+1}(2r+1)} [f(x \oplus u, y \oplus v) - f(x, y)] D_i(\tau_n(v)) \, dv \right) \, du
\]

\[
= 2^m \sum_{r=0}^{2^n-1} \int_{I_m(0)} \int_{I_{n+1}(2r)} \left[ f(x \oplus u, y \oplus v) - f(x \oplus u, y \oplus v \oplus 2^{-n-1}) \right] \\
\times D_i(\tau_n(v)) \, du \, dv
\]
\begin{align*}
&= 2^m \int_{I_m(0)} \int_{I_{n+1}(0)} \left[ f(x \oplus u, y \oplus v) - f(x \oplus u, y \oplus v \oplus 2^{-n-1}) \right] \\
&\quad \times D_i(\tau_n(v)) \, du \, dv \\
+ &2^m \sum_{r=1}^{2^n-1} D_i(\tau_n(r)) \int_{I_m(0)} \int_{I_{n+1}(0)} \left[ f(x \oplus u, y \oplus v \oplus 2r2^{-n-1}) - f(x \oplus u, y \oplus v \oplus (2r + 1)2^{-n-1}) \right] \, du \, dv.
\end{align*}

From (5), we have

\[|II| \leq 2^m i \int_{I_m(0)} \int_{I_{n+1}(0)} |\Delta^n f(x \oplus u, y \oplus v)_2| \, du \, dv\]

\[+ 2^{m+n} \int_{I_m(0)} \int_{I_{n+1}(0)} \sum_{r=1}^{2^n-1} \frac{1}{\tau_n(r)} |\Delta^n f(x \oplus u, y \oplus v)_2| \, du \, dv,\]

consequently,

\[\|II\|_C \leq \frac{1}{2} \left\|\mathcal{W}_n^{(2)}(f)\right\|_C. \tag{8}\]

The estimation of III is analogous to the estimation of II and we have

\[\|III\|_C \leq \frac{1}{2} \left\|\mathcal{W}_m^{(1)}(f)\right\|_C. \tag{9}\]

Following a similar pattern to the case of II, by (a)-(d) we obtain

\begin{align*}
IV &= \sum_{s=0}^{2^m-1} \sum_{r=0}^{2^n-1} D_i(\tau_n(r)) D_j(\tau_m(s)) \left( \int_{I_{m+1}(2s)} \int_{I_{n+1}(2r)} - \int_{I_{m+1}(2s+1)} \int_{I_{n+1}(2r)} \right) \\
&\quad + \int_{I_{m+1}(2s+1)} \int_{I_{n+1}(2r+1)} \int_{I_{m+1}(2s+1)} \int_{I_{n+1}(2r+1)} [f(x \oplus u, y \oplus v) - f(x, y)] \, du \, dv
\end{align*}
\[
\begin{align*}
&= \sum_{s=0}^{2^n-1} \sum_{r=0}^{2^m-1} D_i (\tau_n (r)) D_j (\tau_m (s)) \\
&\quad \times \int_{I_{m+1}(2s)} \int_{I_{n+1}(2r)} [f (x \oplus u \oplus 2^{-m-1}, y \oplus v \oplus 2^{-n-1}) \\
&\quad - f (x \oplus u \oplus 2^{-m-1}, y \oplus v) \\
&\quad - f (x \oplus u, y \oplus v \oplus 2^{-n-1}) + f (x \oplus u, y \oplus v)] dudv
\end{align*}
\]

We obtain by (5) that

\[
|IV| \leq ij \int_{I_{m+1}(0)} \int_{I_{n+1}(0)} |\Delta_{00}^{mn} f (x \oplus u, y \oplus v)| dudv
\]

\[
+ j2^n \int_{I_{m+1}(0)} \int_{I_{n+1}(0)} 2^{n-1} \sum_{r=1}^{2^m-1} \frac{1}{\tau_n (r)} |\Delta_{0r}^{mn} f (x \oplus u, y \oplus v)| dudv
\]

\[
+ i2^m \int_{I_{m+1}(0)} \int_{I_{n+1}(0)} 2^{m-1} \sum_{s=1}^{2^n-1} \frac{1}{\tau_m (s)} |\Delta_{sr}^{mn} f (x \oplus u, y \oplus v)| dudv
\]

\[
+ 2^{n+m} \int_{I_{m+1}(0)} \int_{I_{n+1}(0)} 2^{m-1} 2^{n-1} \sum_{r=1}^{2^m-1} \sum_{s=1}^{2^n-1} \frac{1}{\tau_n (r)} \frac{1}{\tau_m (s)} |\Delta_{sr}^{mn} f (x \oplus u, y \oplus v)| dudv
\]

\[
\leq 2^{n+m} \int_{I_{m+1}(0)} \int_{I_{n+1}(0)} W_{mn} f (f; x \oplus u, y \oplus v) dudv,
\]

12
consequently,
\[ \|IV\|_c \leq \frac{1}{4} \|W_{mn}(f)\|_c. \]  
(10)

Combining (6)-(10) we complete the proof of Theorem 1.

Proof of Theorem 2. Since
\[ \sum_{r=0}^{2^n-1} \lambda^n_r = 1 + \sum_{r=1}^{2^n-1} \frac{1}{\tau_n(r)} = 1 + \sum_{r=1}^{2^n-1} \frac{1}{r} \leq cn \]
and
\[ W_m^{(1)}(f; x, y) \leq \omega_1 \left( f; \frac{1}{2m} \right)_C \sum_{r=0}^{2^m-1} \lambda^m_r \leq c \omega_1 \left( f; \frac{1}{2m} \right) m, \]
\[ W_n^{(2)}(f; x, y) \leq \omega_2 \left( f; \frac{1}{2n} \right)_C \sum_{i=0}^{2^n-1} \lambda^n_i \leq c \omega_2 \left( f; \frac{1}{2n} \right) n, \]
\[ W_{mn}(f; x, y) \leq \omega_{1,2} \left( f; \frac{1}{2m}, \frac{1}{2n} \right)_C \sum_{r=0}^{2^{m-1}2^n-1} \sum_{i=0}^{2^n-1} \lambda^m_r \lambda^n_i \leq c \omega_{1,2} \left( f; \frac{1}{2m}, \frac{1}{2n} \right) nm, \]
the validity of Theorem 2 follows from Theorem 1.

Calculations, similar to those that were performed in the proofs of Theorems 1, 2 and an application of the Minkowski inequality yield the validity of Theorem 3.

Proof of Theorem 4. On the basis of Theorem 1, it suffices to show that
\[ \|W_m^{(1)}(f)\|_C \to 0 \quad \text{as} \quad m \to \infty, \]
\[ \|W_n^{(2)}(f)\|_C \to 0 \quad \text{as} \quad n \to \infty, \]
\[ \|W_{mn}(f)\|_C \to 0 \quad \text{as} \quad n, m \to \infty. \]

Let
\[ A_r^m = \left\{ j : j = 1, 2^m - 1, 2^r \leq \tau_m(j) < 2^{r+1} \right\}, \quad r = 0, 1, ..., m - 1. \]
Then it is evident that
\[ |A_r^m| < 2^r \]  
(11)
\[ \bigcup_{r=0}^{m-1} A_r^m = \{1, 2, \ldots, 2^m - 1\}. \]  

(12)

From the condition of the theorem and by (11), (12) we get

\[
W_m^{(1)} (f; x, y) = |\Delta_0^m f (x, y)_{1} | + \sum_{j=1}^{2^m-1} \frac{1}{r_m(j)} |\Delta_j^m f (x, y)_{1} |
\]

\[
\leq \omega_1 \left( f; \frac{1}{2^m} \right)_C + \sum_{r=0}^{m-1} \sum_{j \in A_r^m} \frac{1}{r_m(j)} |\Delta_j^m f (x, y)_{1} |
\]

\[
\leq \omega_1 \left( f; \frac{1}{2^m} \right)_C + \sum_{r=0}^{m-1} \sum_{j \in A_r^m} |\Delta_j^m f (x, y)_{1} |
\]

\[
\leq \omega_1 \left( f; \frac{1}{2^m} \right)_C + \sum_{r=0}^{m-1} \sum_{j \in A_r^m} |\Delta_j^m f (x, y)_{1} |
\]

\[
\leq c \left\{ \omega_1 \left( f; \frac{1}{2^m} \right)_C \eta(m) + \frac{1}{2 \eta(m)} \right\} \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty,
\]

where

\[ \min_{1 \leq n \leq 2^m - 1} \left\{ \omega_1 \left( f; \frac{1}{2^m} \right)_C \eta + \frac{1}{2n} \right\} = \omega_1 \left( f; \frac{1}{2^m} \right)_C \eta(m) + \frac{1}{2 \eta(m)}, \]

consequently,

\[ \|W_m^{(1)} (f)\|_C \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty, \]  

(13)

analogously

\[ \|W_n^{(2)} (f)\|_C \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]  

(14)
We write

\[ W_{mn} (f; x, y) = |\Delta_{00}^{mn} f (x, y)| + \sum_{i=1}^{2^n-1} \frac{1}{\tau_n (i)} |\Delta_{0i}^{mn} f (x, y)| \]

\[ + \sum_{j=1}^{2^n-1} \frac{1}{\tau_m (j)} |\Delta_{j0}^{mn} f (x, y)| + \sum_{j=1}^{2^n-1} \sum_{i=1}^{2^m-1} \frac{1}{\tau_n (i)} \frac{1}{\tau_m (j)} |\Delta_{ji}^{mn} f (x, y)| \]

\[ = I + II + III + IV. \tag{15} \]

It is evident that

\[ |\Delta_{00}^{mn} f (x, y)| \leq \omega_{1,2} \left( f; \frac{1}{2^m}, \frac{1}{2^n} \right)_C \rightarrow 0 \quad \text{as} \quad n, m \rightarrow \infty. \tag{16} \]

Since

\[ |\Delta_{j0}^{mn} f (x, y)| \leq |\Delta_j^{m} f (x, y)| + |\Delta_j^{n} f (x, y) \odot 2^{-n-1}|, \]

from (13) we get

\[ \|II\|_C \leq 2 \|W_m^{(1)} (f)\|_C \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty. \tag{17} \]

Analogously,

\[ \|III\|_C \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty. \tag{18} \]

From (11) and (12) we get

\[ IV = \sum_{r=0}^{m-1} \sum_{s=0}^{n-1} \sum_{j \in A^n} \sum_{i \in A^n} \frac{1}{\tau_n (i)} \frac{1}{\tau_m (j)} |\Delta_{ji}^{mn} f (x, y)| \]

\[ \leq \sum_{r=0}^{m-1} \frac{1}{2^r} \sum_{s=0}^{n-1} \frac{1}{2^s} \sum_{j \in A^n} \sum_{i \in A^n} |\Delta_{ji}^{mn} f (x, y)|, \tag{19} \]

Since

\[ \sum_{j \in A^n} \sum_{i \in A^n} |\Delta_{ji}^{mn} f (x, y)| \leq 2 |A^n| \sup_{x \in [0,1]} \sum_{i \in A^n} |\Delta_i^{n} f (x, y)|, \]
\[
\sum_{j \in A^n} \sum_{i \in A^m} |\Delta_{ji}^m f(x, y)| \leq 2 |A^n| \sup_{y \in [0, 1]} \sum_{j \in A^n} |\Delta_j^m f(x, y)|,
\]

we have
\[
\sum_{j \in A^n} \sum_{i \in A^m} |\Delta_{ji}^m f(x, y)| = \left( \sum_{j \in A^n} \sum_{i \in A^m} |\Delta_{ji}^m f(x, y)| \right)^{1/2}
\times \left( \sum_{j \in A^n} \sum_{i \in A^m} |\Delta_{ji}^m f(x, y)| \right)^{1/2} \leq 2 [A^n] [A^m].
\]

After substituting (20) in (19) we obtain by (11) and from the condition of the theorem that
\[
IV \leq 2 \sum_{r=0}^{m-1} \frac{1}{2^{r/2}} \left( \sup_{y \in [0, 1]} \sum_{j \in A^n} |\Delta_j^m f(x, y)| \right)^{1/2}
\times \sum_{s=0}^{n-1} \frac{1}{2^{s/2}} \left( \sup_{x \in [0, 1]} \sum_{i \in A^m} |\Delta_i^n f(x, y)| \right)^{1/2}
= 2 \left\{ \sum_{r=0}^{\varphi(m)-1} \frac{1}{2^{r/2}} \left( \sup_{y \in [0, 1]} \sum_{j \in A^n} |\Delta_j^m f(x, y)| \right)^{1/2} \right\}
+ \sum_{r=\varphi(m)}^{m-1} \frac{1}{2^{r/2}} \left( \sup_{y \in [0, 1]} \sum_{j \in A^n} |\Delta_j^m f(x, y)| \right)^{1/2}
\times \left\{ \sum_{s=0}^{\psi(n)-1} \frac{1}{2^{s/2}} \left( \sup_{x \in [0, 1]} \sum_{i \in A^m} |\Delta_i^n f(x, y)| \right)^{1/2} \right\}
+ \sum_{s=\psi(n)}^{n-1} \frac{1}{2^{s/2}} \left( \sup_{x \in [0, 1]} \sum_{i \in A^m} |\Delta_i^n f(x, y)| \right)^{1/2}
\]
\[
\leq c \left\{ \sqrt{\omega_1 \left( f; \frac{1}{2^m} \right) \varphi (m) + \frac{1}{2^{\varphi(m)/2}}} \right\} \\
\times \left\{ \sqrt{\omega_2 \left( f; \frac{1}{2^n} \right) \psi (n) + \frac{1}{2^{\psi(n)/2}}} \right\} \to 0 \quad \text{as} \quad m, n \to \infty,
\]

(21)

where

\[
\min_{1 \leq \varphi \leq 2^{m-1}} \left\{ \sqrt{\omega_1 \left( f; \frac{1}{2^m} \right) \varphi + \frac{1}{2^{\varphi/2}}} \right\} = \sqrt{\omega_1 \left( f; \frac{1}{2^m} \right) \varphi (m) + \frac{1}{2^{\varphi(m)/2}}}
\]

and

\[
\min_{1 \leq \psi \leq 2^{n-1}} \left\{ \sqrt{\omega_2 \left( f; \frac{1}{2^n} \right) \psi + \frac{1}{2^{\psi/2}}} \right\} = \sqrt{\omega_2 \left( f; \frac{1}{2^n} \right) \psi (n) + \frac{1}{2^{\psi(n)/2}}}.
\]

Combining (15)-(18) and (21) we have

\[
\|W_{mn}(f)\|_C \to 0 \quad \text{as} \quad n, m \to \infty.
\]

(22)

From (13), (14) and (22) we complete the proof of Theorem 4.

References


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