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RESEARCH ARTICLE

A Potential-Theoretic Construction of the Schwarz-Christoffel Map for Finitely Connected Domains

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We explicitly construct the Schwarz-Christoffel map from a (bounded or unbounded) finitely connected Jordan domain to a (bounded or unbounded) finitely connected polygonal domain. The map is derived in terms of Green's function and the harmonic measure functions of the Jordan domain which need not be a canonical multiply connected domain.

Keywords: conformal mapping; Schwarz-Christoffel maps; potential theory

AMS Subject Classification: 30C20, 30C85, 31A05

1. Introduction

The Schwarz-Christoffel (S-C) map from a finitely connected circular domain (*i.e.* a domain whose complement consists of a finite number of closed disks) to a finitely connected polygonal domain has been the object of intense study since the publication of the milestone paper [1] (see also [2], [3], [4], [5]). Since the Riemann Mapping Theorem ensures that all simply connected domains are conformally equivalent to the unit disk, the classical S-C formula is derived from the unit disk or the upper halfplane. Analogously, the annulus is the domain of the S-C map in the doubly connected case ([6], [7], [8]).

If the connectivity is three or higher, there are several canonical domains: circular, a slit disk (a disk with concentric circular slits), a slit annulus (an annulus with concentric circular slits), a circular slit domain, and others [9]. The first example of an S-C map for domains of connectivity three or higher seems to be in [10], where an S-C map is derived from a circular domain onto a polygonal domain consisting of convex polygons. In [1], [4], and [5], the S-C map is from a circular domain, and the derivation characterizes the global pre-Schwarzian on the Riemann sphere via the Reflection Principle. The derivative of the map is expressed as an infinite product. In [2] and [3], the S-C map is also from a circular domain, and the derivation uses Schottky-Klein prime functions. In both approaches, the convergence of the resulting infinite products is proved for sufficiently separated circular domains.

Our construction of the S-C function incorporates, similar to Crowdy's, an intermediate conformal mapping ψ from the Jordan domain D_0 to a slit disk or a slit

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annulus in the bounded case, or to a circular slit domain in the unbounded case. The S-C mapping is then rendered in the form $f_0 = f_1 \circ \psi$ where f_1 has the form

$$f_1(z) = \int_0^z \exp(u_{1,2}(\zeta) + iv_{1,2}(\zeta))d\zeta. \tag{1}$$

The function $v_{1,2} \circ \psi$ is harmonic on D_0 and its harmonic conjugate, $u_{1,2} \circ \psi$, is single-valued. The boundary values of $v_{1,2} \circ \psi$ are a combination of the boundary values of $\arg \psi$, the harmonic measures of the boundary components of D_0, and of the boundary data of the polygonal domain. The details are presented in Section 2.2 and Section 4.

Our paper is organized as follows. Our main result is stated in Section 6. In Section 2.1, we present two observations, Lemma 2.2 and Lemma 2.3, which motivate our construction of the S-C map from bounded Jordan domains to bounded polygonal domains. We verify the construction in Section 3. While the construction can be applied to any Jordan domain, in Section 4 we show that when it is applied to a bounded analytic domain it yields a compact formula for the S-C map onto a bounded polygonal domain. In Section 5, we discuss the slight modifications of the construction for the unbounded case. The results up to this point are summarized in Section 6. The next two sections contain a derivation from our results of the classical formulas for the unit disk and the annulus. Section 9 discusses the context of our results and prospects for applications.

2. An explicit construction for the bounded case

Let $\mathrm{Ext}(\Gamma)$ denote the exterior of the Jordan curve Γ. We identify a Jordan curve with any of its parametrizations.

We assume we are given the following data.

Given Data 2.1

(1) A bounded domain D_0 bounded by Jordan curves $\Gamma_0, \ldots, \Gamma_M$, the outermost of which is Γ_0.
(2) A selection of *prevertices* $\{a_k^{(j)}\}_{j=0,\ldots,M,k=1,\ldots,n_j}$ such that $a_k^{(j)} \in \Gamma_j$ and $\Gamma_j^{-1}(a_1^{(j)}) < \ldots < \Gamma_j^{-1}(a_{n_j}^{(j)})$.
(3) A selection of reals in $(0,1)$, $\{\beta_k^{(j)}\}_{j=0,\ldots,M,k=1,\ldots,n_j}$, such that

$$\sum_{k=1}^{n_j}(1 - \beta_k^{(j)}) = 2.$$

(4) A $\zeta_0 \in D_0$.

We say that these given data are *satisfactory* if there is a conformal map ϕ_0 onto a bounded polygonal domain P such that

- ϕ_0 maps D_0 onto P,
- ϕ_0 maps $a_k^{(j)}$ to a vertex of a boundary component of P, $p_k^{(j)}$,
- the exterior angle at $p_k^{(j)}$ is $\pi(1 - \beta_k^{(j)})$,
- $\phi_0'(\zeta_0) = 1$, and
- $\phi_0(\zeta_0) = 0$.

Not all choices for Given Data 2.1 are satisfactory. See, *e.g.*, page 143 of [11]. So, henceforth we assume that we are dealing with a satisfactory choice. It then follows that ϕ_0 and P are unique. Our main result is that we can derive ϕ_0 and hence P from these data. In applications, it may be the case that P is part of the given data and one wishes to compute the prevertices and thereby ϕ_0. This leads to the *prevertex problem* which is discussed in [12] and [5].

2.1. *Some preliminary definitions and observations*

Let D_1 be a slit disk or slit annulus domain for which there exists a conformal map ψ of D_0 onto D_1. We also assume that the outermost boundary curve of D_1 is $\partial\mathbb{D}$, and that the center of D_1 is 0. Let $\zeta_1 = \psi(\zeta_0)$. Let $\gamma_j = \psi[\Gamma_j]$. Let r_j be the radius of γ_j. When $j \neq 0$, let $b_0^{(j)}$ and $b_1^{(j)}$ be the two points on Γ_j that map under ψ to endpoints of γ_j. Let $\phi_1 = \phi_0 \circ \psi^{-1}$.

Let $c_k^{(0)} = a_k^{(0)}$. Suppose $1 \leq j \leq M$. Let

$$S_k^{(j)} = \{a_k^{(j)} \mid 1 \leq k \leq n_j\} \ \cup \ \{b_0^{(j)}, b_1^{(j)}\}.$$

Let $\{c_k^{(j)}\}_{k=1,\ldots,\eta_j}$ be an enumeration of $S_k^{(j)}$. We can assume this enumeration is chosen so that

$$\Gamma_j^{-1}(c_k^{(j)}) < \Gamma_j^{-1}(c_{k+1}^{(j)})$$

whenever $1 \leq k < \eta_j$.

For $k > n_j$, let $c_k^{(j)} = c_{k'}^{(j)}$ where $k' \in \{1, \ldots, \eta_j\}$ and $k \equiv k' \mod \eta_j$.

For all $k \geq 1$, let $t_k^{(j)} = \Gamma_j^{-1}(c_k^{(j)})$. We also let, for all $k \geq 1$, $A_k^{(j)}$ be the arc on Γ_j from $c_k^{(j)}$ to $c_{k+1}^{(j)}$ that does not contain any $c_{k'}^{(j)}$.

Let $E = A_k^{(j)}$. We let $\phi_{0,E}$ be an analytic extension of ϕ_0 past E whose domain does not contain any $c_k^{(j)}$. Let ψ_E be an analytic extension of ψ past E whose domain does not contain any $c_k^{(j)}$. We can assume $\mathrm{dom}(\phi_{0,E}) = \mathrm{dom}(\psi_E)$. This extension of ψ may not be one-to-one. However, there is at least a neighborhood of E, W_E, in which ψ is one-to-one. Let $\phi_{1,E} = \phi_{0,E} \circ (\psi_E|_{W_E})^{-1}$.

Let $\arg_1$ be a branch of Arg on a simply connected domain which omits $\psi(a_1^{(0)})$ and contains $\gamma_1, \ldots, \gamma_n$ as well as $\partial\mathbb{D} - \{\psi(a_1^{(0)})\}$.

Lemma 2.2: *Suppose $E = A_k^{(j)}$. Then, $\zeta \mapsto \mathrm{Arg}(\psi(\zeta)\phi'_{1,E}(\psi(\zeta)))$ is constant on E.*

Proof: We begin by choosing θ_1, θ_2 so that $r_j e^{i\theta_1} = c_k^{(j)}$ and $r_j e^{i\theta_2} = c_{k+1}^{(j)}$. We assume without loss of generality that $\theta_1 < \theta_2$. When $\theta_1 < \theta < \theta_2$, let

$$\sigma(\theta) = \phi_{1,E}(r_j e^{i\theta}).$$

Therefore, σ traces a line segment. Hence, the unit tangent vector to the curve σ is the same at every point. But, the unit tangent vector to σ at $(\theta, \sigma(\theta))$ is $\exp(i\,\mathrm{Arg}(\sigma'(\theta)))$. It follows that $\mathrm{Arg}(\sigma')$ is constant. On the other hand,

$$\sigma'(\theta) = \phi'_{1,E}(r_j e^{i\theta}) r_j i e^{i\theta}.$$

Hence,

$$\mathrm{Arg}(\sigma'(\theta)) = \mathrm{Arg}(\psi(\theta)\phi'_{1,E}(\psi(\theta))) + \frac{\pi}{2}.$$

The Lemma follows. □

Let $C_k^{(j)}$ be the value of $\mathrm{Arg}(\psi \cdot \phi'_{1,E} \circ \psi)$ on $A_k^{(j)}$.

If $c_k^{(j)} = a_{k'}^{(j)}$ and if $\psi(c_k^{(j)})$ is not an arc endpoint in D_1, then let $\delta_k^{(j)} = \beta_{k'}^{(j)}$. If $\psi(c_k^{(j)})$ is an arc endpoint in D_1, but if $c_k^{(j)}$ is not a prevertex, then let $\delta_k^{(j)} = 2$. Finally, if $\psi(c_k^{(j)})$ is an arc endpoint in D_1, and if $c_k^{(j)} = a_{k'}^{(j)}$, then let $\delta_{k'}^{(j)} = \beta_{k'}^{(j)} + 1$.

Write $x \equiv y$ if x is equivalent to y modulo 2π.

Lemma 2.3: $C_{k+1}^{(j)} - C_k^{(j)} \equiv \pi(1 - \delta_{k+1}^{(j)})$.

Proof: We first choose $\theta_1, \theta_2, \theta_3 \in [0, 2\pi)$ so that

$$r_j e^{i\theta_1} = \psi(c_k^{(j)})$$
$$r_j e^{i\theta_2} = \psi(c_{k+1}^{(j)})$$
$$r_j e^{i\theta_3} = \psi(c_{k+2}^{(j)}).$$

Let

$$\gamma(t) = \begin{cases} t(\theta_2 - \theta_1) + \theta_1 & 0 \le t \le 1 \\ (t-1)(\theta_3 - \theta_2) + \theta_2 & 1 \le t \le 2 \end{cases}$$

Let $E_1 = A_k^{(j)}$, and let $E_2 = A_{k+1}^{(j)}$. Let

$$\sigma(t) = \begin{cases} \phi_{1,E_1}(r_j e^{i\gamma(t)}) & 0 \le t \le 1 \\ \phi_{1,E_2}(r_j e^{i\gamma(t)}) & 1 \le t \le 2 \end{cases}$$

We first consider the case where $c_{k+1}^{(j)}$ is a preimage of an arc endpoint (under ψ) but is not a prevertex. It follows that, $\pi(1 - \delta_{k+1}^{(j)}) = -\pi$. It also follows that σ traces a line segment. Hence, $\mathrm{Arg}(\sigma')$ is constant. However, in this case, $\theta_2 - \theta_1$ and $\theta_3 - \theta_2$ have opposite sign. So, when $0 < t < 1$, it follows that $\mathrm{Arg}(\sigma'(t)) = C_k^{(j)} \pm \pi/2$, and when $1 < t < 2$ it follows that $\mathrm{Arg}(\sigma'(t)) = C_{k+1}^{(j)} \mp \pi/2$. Hence, $C_{k+1}^{(j)} - C_k^{(j)} = \pm\pi \equiv \pi(1 - \delta_{k+1}^{(j)})$.

Now, suppose $c_{k+1}^{(j)}$ is a prevertex but not a ψ-preimage of an arc endpoint. In this case, $\theta_2 - \theta_1$ and $\theta_3 - \theta_2$ have the same sign. It follows that when $0 < t < 1$, $\mathrm{Arg}(\sigma'(t)) = C_k^{(j)} \pm \pi/2$, and when $1 < t < 2$, $\mathrm{Arg}(\sigma'(t)) = C_{k+1}^{(j)} \pm \pi/2$. It then follows that $C_{k+1}^{(j)} - C_k^{(j)}$ is the exterior angle at $p_{k+1}^{(j)}$ which is $\pi(1 - \delta_{k+1}^{(j)})$.

The remaining case follows by a combination of the above cases. □

2.2. *The construction*

We now construct a sequence of harmonic functions. Some of these harmonic functions will be constructed to have domain D_0, and some will be constructed to have domain D_1. We begin in D_0. Let $\theta_k^{(j)} = \pi(1 - \delta_k^{(j)})$. Let:

$$\sigma_1^{(j)} = 0$$
$$\sigma_2^{(j)} = \theta_2^{(j)}$$
$$\sigma_k^{(j)} = \sigma_{k-1}^{(j)} + \theta_k^{(j)} \qquad 1 < k \le \eta_j$$

Let $v_{0,1}$ be the harmonic function on D_0 whose limit at each $\zeta \in A_k^{(j)}$ is

$$\sigma_k^{(j)} - \arg_1(\psi(\zeta)). \tag{2}$$

The existence and uniqueness of $v_{0,1}$ is guaranteed by say Theorem II.1.1 of [13].

Let ω denote the harmonic measure function, and let $\omega_j = \omega(\cdot, \Gamma_j, D_0)$. There exist unique $b_1, \ldots, b_M$ such that

$$v_{0,2} =_{df} v_{0,1} + \sum_{j=1}^{M} b_j \omega_j \tag{3}$$

has a single-valued harmonic conjugate. These numbers can be expressed in terms of the Riemann matrix of D_0 and the harmonic measure functions of D_0. See, *e.g.*, Lemma B.2 of [13].

For all $z \in D_0$, let

$$u_{0,2}(z) = -\int_{\zeta_0}^{z} \frac{\partial v_{0,2}}{\partial n} ds.$$

Hence, the conjugate of $u_{0,2}$ is $v_{0,2}$.

We now move to the slit disk domain. Let:

$$\begin{aligned} u_{1,2} &= u_{0,2} \circ \psi^{-1} \\ v_{1,2} &= v_{0,2} \circ \psi^{-1} \end{aligned} \tag{4}$$

It follows that the conjugate of $u_{1,2}$ is $v_{1,2}$.

Let

$$H(z) = \exp(u_{1,2}(z) + i v_{1,2}(z)).$$

For all $z \in D_1$, let

$$f_1(z) = \int_{\zeta_1}^{z} H(\zeta) d\zeta.$$

Let $f_0 = f_1 \circ \psi$. Hence, for the moment at least, f_0 and f_1 are only multi-valued functions. However, we will show in Section 3 that they are in fact single-valued, and moreover f_0, after normalization, is ϕ_0. The situation thus far when D_1 is a slit disk and $M = 2$ is diagrammed in Figure 1.

3. Verification of the construction

For the sake of the verification, we can assume that all boundary curves are analytic. See, for example, Lemma II.2.2 of [13].

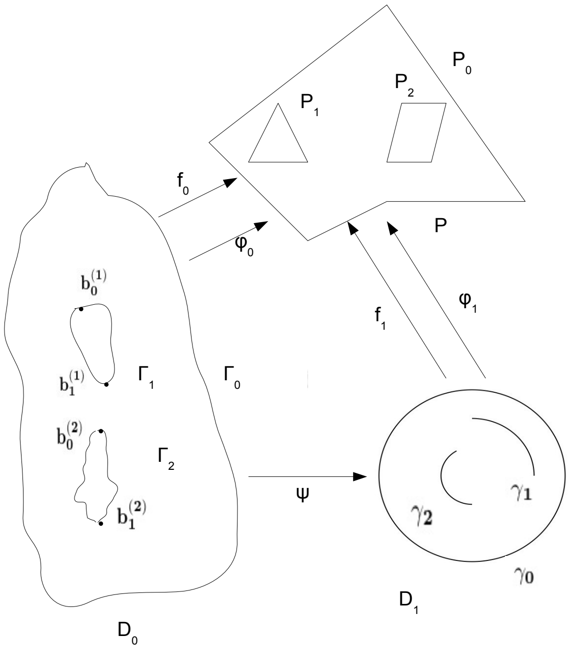

Figure 1. The construction

Let $E = A_k^{(j)}$. We can assume W_E is closed under the reflection map of Γ_j. Let:

$$\begin{aligned} U_E &= W_E \cap D_0 \\ A_E &= \psi[E] \\ C_E &= \psi[U_E] \\ S_E &= \psi[W_E] \end{aligned}$$

Let C'_E be the reflection of C_E about γ_j. Hence, $S_E = C_E \cup A_E \cup C'_E$. These sets are illustrated in Figure 2.

Lemma 3.1: *Suppose $E = A_k^{(j)}$. Then, there is an analytic extension of $f'_0|_{U_E}$ to W_E, $f'_{0,E}$.*

Proof: To construct $f'_{0,E}$, we first take a primitive of H on C_E, F_E. That is, $F'_E = H$. We will first show that for every $\zeta \in A_E$,

$$\lim_{z \to \zeta, z \in C_E} H(z) \tag{5}$$

exists and that the resulting extension of H to $C_E \cup A_E$ is continuous. We then demonstrate the resulting extension of F_E maps A_E into a line. This allows us to extend F_E to S_E and define $f'_{0,E}$ to be $(F'_E \circ \psi_E) \cdot \psi'_E$.

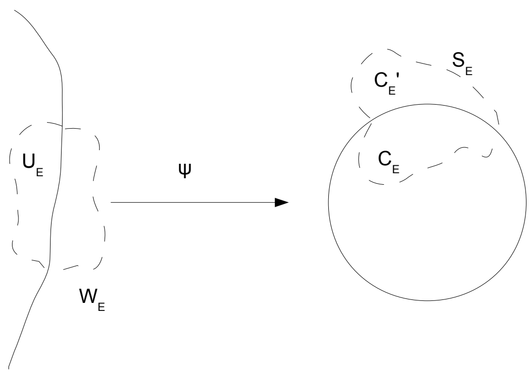

Figure 2. The sets W_E, U_E, S_E

Let $\zeta \in A_E$. To prove (5), it suffices to show that

$$\lim_{z \to \zeta, z \in C_E} u_{1,2}(z) \tag{6}$$

exists. To this end, let ψ_1 be a conformal map of C_E onto $\mathbb{D}$. Let:

$$\begin{aligned} u &= u_{1,2} \circ \psi_1^{-1} \\ v &= v_{1,2} \circ \psi_1^{-1} \\ z &= re^{i\phi} \in \mathbb{D} \end{aligned}$$

Then,

$$\begin{aligned} -u(z) &= \int_{-\pi}^{\pi} \frac{2r\sin(\phi-\theta)v(e^{i\theta})}{1-2r\cos(\phi-\theta)+r^2}\frac{d\theta}{2\pi} \\ &= \int_{\psi_1[A_E]} \frac{2r\sin(\phi-\theta)v(e^{i\theta})}{1-2r\cos(\phi-\theta)+r^2}\frac{d\theta}{2\pi} \\ &\quad + \int_{\partial\mathbb{D}-\psi_1[A_E]} \frac{2r\sin(\phi-\theta)v(e^{i\theta})}{1-2r\cos(\phi-\theta)+r^2}\frac{d\theta}{2\pi}. \end{aligned}$$

(See, *e.g.*, Exercise II.11 of [13].) Let $e^{i\theta_1} = \psi_1(\zeta)$. The denominator of the latter integral is therefore bounded away from 0 as (r, ϕ) approaches $(1, \theta_1)$. When $e^{i\theta} \in \psi_1[A_E]$, $v(e^{i\theta}) = C + \arg_1(\psi_1^{-1}(e^{i\theta}))$ by (2), (3), and (4). Since the conjugate of a constant is a constant, it follows that

$$\lim_{(r,\phi)\to(1,\theta_1)} \int_{\psi_1[A_E]} \frac{2r\sin(\phi-\theta)C}{1-2r\cos(\phi-\theta)+r^2}\frac{d\theta}{2\pi}$$

exists. Since A_E does not contain a prevertex, we can assume C_E small enough so as to be contained in dom($\arg_1$). Hence,

$$\int_{-\pi}^{\pi} \frac{2r\sin(\phi-\theta)\arg_1(\psi_1^{-1}(e^{i\theta}))}{1-2r\cos(\phi-\theta)+r^2}\frac{d\theta}{2\pi} = \ln|\psi_1^{-1}(re^{i\phi})|.$$

And, $\ln|\psi_1^{-1}(e^{i\theta_1})|$ is defined. It follows that $u_{1,2}$ has a limit at each point of A_E.

It now follows that the resulting extension of H to $C_E \cup A_E$ is continuous. Hence, the extension of F_E to $C_E \cup A_E$ is continuous. By construction, $\zeta \mapsto \text{Arg}(\zeta F_E'(\zeta))$ is constant on A_E. Hence, F_E maps A_E into a line segment. $\square$

With $E = A_k^{(j)}$, we let

$$\left(\frac{f_0'}{\phi_0'}\right)_E = \frac{f_{0,E}'}{\phi_{0,E}'}.$$

Lemma 3.2: *Let $E = A_k^{(j)}$. Then,* $\text{Arg}\left(\frac{f_0'}{\phi_0'}\right)_E$ *is constant on E.*

Proof: For each $\zeta \in E$,

$$\arg\left(\frac{f_{0,E}'(\zeta)}{\phi_{0,E}'(\zeta)}\right) = \arg\left(\frac{\psi(\zeta)H(\psi(\zeta))}{\psi(\zeta)\phi_{1,E}(\psi(\zeta))}\right).$$

From the construction along with (2), (3), and (4), we infer that the argument of $\psi \cdot (H \circ \psi)$ is constant on E. It follows from Lemma 2.2 that the argument of $\psi \cdot (\phi_{1,E}' \circ \psi)$ is constant on E. $\square$

Lemma 3.3: *Let $E = A_k^{(j)} \cup A_{k+1}^{(j)}$. Then, there is an analytic extension of (f_0'/ϕ_0') to a neighborhood of E, $(f_0'/\phi_0')_E$. Furthermore, the argument of this extension is constant on E.*

Proof: Let:

$$E_1 = A_k^{(j)}$$
$$E_2 = A_{k+1}^{(j)}$$

Let λ_E be a curve contained in D_0 such that $\lambda_E \cup \overline{E}$ is a Jordan curve, and let U_E be the interior of $\lambda_E \cup \overline{E}$. It follows from what has been shown that f_0'/ϕ_0' has a continuous extension to $\overline{U_E} - \{b\}$. Let h denote this extension. We note that if $\zeta_1 \in E_1$ and $\zeta_2 \in E_2$, then

$$\arg(\psi_{E_1}(\zeta_1)H(\psi_{E_1}(\zeta_1)) \equiv \arg(\psi_{E_2}(\zeta_2)H(\psi_{E_2}(\zeta_2)) + \pi(1-\delta_k^{(j)}), \text{ and}$$
$$\arg(\psi_{E_1}(\zeta_1)\phi_{1,E_1}'(\psi_{E_1}(\zeta_1)) \equiv \arg(\psi_{E_2}(\zeta_2)\phi_{1,E_2}'(\psi_{E_2}(\zeta_2)) + +\pi(1-\delta_k^{(j)}).$$

It follows that h maps E_1, E_2 into a line l through 0. Let α be such that $e^{i\alpha}h$ maps E_1 and E_2 into $\mathbb{R}$. Hence, $Im(e^{i\alpha}h) = 0$ on $E - \{b\}$. It is undefined at b. But, if we set it to 0 at b, we obtain the same harmonic function on U_E as $Im(e^{i\alpha}h)$. So, we can assume $Im(e^{i\alpha}h) = 0$ on E. Hence, h has a limit at b. Let W_E be the image of U_E under the reflection map of E. Then, h extends analytically to W_E. Denote this extension by $(f_0'/\phi_0')_E$. Note that the argument of $(f_0'/\phi_0')_E$ is constant on E. $\square$

It now follows that f_0'/ϕ_0' has an analytic extension to a neighborhood of $\overline{D_0}$. It also follows that the argument of f_0'/ϕ_0' is constant on each Γ_j.

Let $Q = f_0'/\phi_0'$. Hence, $Q[\Gamma_j]$ is contained in a ray extending from the origin. Let $w_0 \in \mathbb{C} - Q[\partial D_0]$. Hence, the Winding Number of $Q[\Gamma_j]$ around w_0 is 0. It follows by a simple computation that

$$\int_{\Gamma_j} \frac{Q'(\zeta)}{Q(\zeta) - w_0} d\zeta = 0.$$

We assume each Γ_j is positively oriented with respect to D_0. Since Q has an analytic extension to a neighborhood of $\overline{D_0}$, it now follows from the Argument Principle that $w_0 \notin \operatorname{ran}(Q)$. Hence, the range of Q is contained in a set with empty interior. It then follows from the Open Mapping Theorem that Q is constant.

It now follows that f_0 is single-valued and

$$\phi_0(z) = \frac{f_0(z) - f_0(\zeta_0)}{f_0'(\zeta_0)}.$$

Hence, we have constructed ϕ_0 from the given data.

4. An explicit formula for analytic domains

We suppose our boundary curves are analytic so that we can use the Poisson Integral Formula (see, *e.g.*, Theorem II.2.5 of [13]). To this end, let G denote the Green's function of D_0. At the same time, we abbreviate $G(z, \zeta_0)$ by $G(z)$.

When u is harmonic in D_0, let $\hat{u}$ denote the analytic function whose domain is D_0 and whose imaginary part has the form

$$u + \sum_{j=1}^{M} b_j(u)\omega_j.$$

The operators $b_1, \ldots, b_M$ are linear. It follows that $u \mapsto \hat{u}$ is linear. We can then write

$$\begin{aligned} f_0(z) &= \int_{\zeta_1}^{\psi(z)} \exp(\hat{\Lambda}(\psi^{-1}(\zeta)) - \hat{\Omega}(\psi^{-1}(\zeta))d\zeta \\ &= \int_{\zeta_0}^{z} \exp(\hat{\Lambda}(\zeta) - \hat{\Omega}(\zeta))\psi'(\zeta)d\zeta \end{aligned}$$

where:

$$\hat{\Lambda}(z) = \sum_{k,j} \sigma_k^{(j)} \hat{\omega}(z, A_k^{(j)}, D_0)$$

$$\Omega(z) = -\frac{1}{2\pi} \int_{\partial D_0} \operatorname{Arg}(\psi(z)) \frac{\partial G(z,\zeta)}{\partial n_\zeta}$$

$$\psi(z) = \exp(-\hat{G}(z))$$

5. The unbounded case

We assume we are given the following.

Given Data 5.1

(1) Jordan curves $\Gamma_1, \ldots, \Gamma_M$ such that

$$\Gamma_j \subseteq \bigcup_{k \neq j} \overline{Ext(\Gamma_k)}.$$

(2) Prevertices $\{a_k^{(j)}\}_{j=1,\ldots,M, k=1,\ldots,n_j}$ such that
- $a_k^{(j)} \in \Gamma_j$, and
- $a_k^{(j)} \neq a_{k'}^{(j)}$ if $k \neq k'$.

(3) $\{\beta_k^{(j)}\}_{j=1,\ldots,M, k=1,\ldots,n_j}$ such that

$$\sum_{k=1}^{n_j} (1 - \beta_k^{(j)}) = 2.$$

(4) A point

$$\zeta_0 \in D_0 =_{df} \bigcap_{j=1}^{M} Ext(\Gamma_k).$$

Assume there is a conformal map ϕ_0 of D_0 onto a polygonal exterior domain P such that

- ϕ_0 maps $a_k^{(j)}$ to a vertex of a boundary component of P, $p_k^{(j)}$,
- the exterior angle at $p_k^{(j)}$ is $\pi(1 - \beta_k^{(j)})$,
- $\phi_0'(\zeta_0) = 1$, and
- $\phi_0(\zeta_0) = 0$.

It follows that ϕ_0 and P are unique. Let D_1 be a circular slit domain for which there exists a conformal map ψ of D_0 onto D_1. We can assume these slits are centered at 0. Let $\zeta_1 = \psi(\zeta_0)$. We can now proceed exactly as in Section 2.

6. Main theorem

The above results can be summarized as follows.

Theorem 6.1: *Let P be an $M+1$ connected polygonal region (bounded or unbounded) and D_0 be a conformally equivalent domain (bounded or unbounded),* *whose boundary curves are analytic. Then D_0 is mapped conformally onto P by a function of the form $af_0(z) + b$, where*

$$f_0(z) = \int_{\zeta_0}^{z} \exp(\hat{\Lambda}(\zeta) - \hat{\Omega}(\zeta))\psi'(\zeta)d\zeta,$$

$$\hat{\Lambda}(z) = \sum_{k,j} \sigma_k^{(j)} \hat{\omega}(z, A_k^{(j)}, D_0),$$

$$\Omega(z) = -\frac{1}{2\pi}\int_{\partial D_0} \operatorname{Arg}(\psi(z))\frac{\partial G(z,\zeta)}{\partial n_\zeta},$$

and ψ is as defined in the last two sections.

7. Derivation of the classical formula for the disk

Suppose $M = 0$ and $D_0 = \mathbb{D}$. Let us take $\zeta_0 = 0$. Hence, $D_1 = D_0$ and $\psi = Id_{\mathbb{D}}$. Let us assume without loss of generality that $a_1^{(0)} = -1$. Take $\arg_1$ to be a branch of Arg on $\mathbb{C} - (-\infty, 0]$. It now follows that

$$\hat{\Lambda}(z) = \sum_{k=1}^{n_0} \sigma_k^{(0)} \hat{\omega}(z, A_k^{(0)}, \mathbb{D}), \text{ and}$$

$$\Omega(z) = \int_{\partial\mathbb{D}} \arg_1(\zeta) P_z(e^{i\theta})\frac{d\theta}{2\pi}.$$

Since $D_0 = \mathbb{D}$, $v_{0,2} = v_{0,1}$.

Let Log_1 be the analytic branch of Log corresponding to $\arg_1$. It follows that for all $\zeta \in \mathbb{D}$, $\hat{\Omega}(\zeta)$ and $2\,\mathrm{Log}_1(1+\zeta)$ have the same imaginary part. We then infer that, modulo a constant, $\hat{\Omega}(z) = 2\,\mathrm{Log}_1(1+z)$.

We now analyze $\hat{\Lambda}$. Set $a_{n_j+1} = a_1^{(0)}$. It then follows that

$$\omega(z, A_k^{(0)}, \mathbb{D}) = \frac{1}{\pi}\operatorname{Arg}\left(\frac{z - a_{k+1}^{(0)}}{z - a_k^{(0)}}\right) + Const..$$

(See, *e.g.* Exercise I.1(a) of [13].) So,

$$\hat{\omega}(z, A_k^{(0)}, \mathbb{D}) = \frac{1}{\pi}\operatorname{Log}\left(\frac{z - a_{k+1}^{(0)}}{z - a_k^{(0)}}\right) + Const..$$

A fairly straightforward computation now reveals that,

$$\hat{\Lambda}(z) = \frac{1}{\pi}\left(\sum_{k=1}^{n_0}(-\theta_k^{(0)})\operatorname{Log}(z - a_k^{(0)}) + 2\pi\operatorname{Log}(z - a_1^{(0)})\right) + Const..$$

But, $z - a_1^{(0)} = 1 + z$, and so it follows that,

$$\hat{\Lambda}(z) - \hat{\Omega}(z) = \sum_{k=1}^{n_0} -(1 - \beta_k^{(0)})\operatorname{Log}(z - a_k^{(0)}) + Const.. \tag{7}$$

Since $\psi = Id_{\mathbb{D}}$, $u_{1,2} = u_{0,2}$ and $v_{1,2} = v_{0,2}$. Hence, there is a constant C such that

$$H(z) = C \prod_{k=1}^{n_0} \frac{1}{(z - a_k^{(0)})^{1-\beta_k^{(0)}}}.$$

The classical result now follows.

8. Derivation of the classical formula for the annulus

In the first part of this section we assume $D_1 = D_0$ and $\psi = Id$. In the second part, the same derivation will be carried out for the situation where D_1 is the slit disk. The derivation in the first part is a warm-up for the next derivation.

Preliminaries

Suppose $M = 1$. We assume $\Gamma_0 = \partial\mathbb{D}$ and $\Gamma_1 = \partial D_\mu(0)$. We assume $\zeta_0 = \tau \in \mathbb{R}$. We orient Γ_0 counterclockwise and Γ_1 clockwise. We assume without loss of generality that the points of each sequence $\{c_k^{(j)}\}_{k=1,\ldots,\eta_j}$ appear in counterclockwise order. Let $\pi\alpha_{j,k}$ be the interior angle of P at $p_k^{(0)}$.

Let

$$\Theta(z) = \prod_{k=1}^{\infty} (1 - \mu^{2k-1} z)(1 - \mu^{2k-1} z^{-1}).$$

Note that

$$\Theta(\mu^{-1} z) = \prod_{k=1}^{\infty} (1 - \mu^{2k-2} z)(1 - \mu^{2k} z^{-1})$$
$$\Theta(\mu z) = \prod_{k=1}^{\infty} (1 - \mu^{2k-2} z^{-1})(1 - \mu^{2k} z)$$

The classical formula can now be rendered

$$\phi_0(z) = Const. \times \int_{\zeta_0}^{z} \prod_{m=1}^{n_0} \left(\Theta\left(\frac{\zeta}{\mu a_m^{(0)}}\right)\right)^{\alpha_{0,m}-1} \prod_{m=1}^{n_1} \left(\Theta\left(\frac{\mu\zeta}{a_m^{(1)}}\right)\right)^{\alpha_{1,m}-1} d\zeta$$

See, *e.g.* Section 17.5 of [8]. Note that

$$\Theta\left(\frac{w}{\mu w_0}\right) = \left(1 - \frac{w}{w_0}\right) \prod_{k=1}^{\infty} \left(1 - \mu^{2k} \frac{w}{w_0}\right) \left(1 - \mu^{2k} \frac{w_0}{w}\right)$$
$$\Theta\left(\frac{\mu w}{w_0}\right) = \left(1 - \frac{w_0}{w}\right) \prod_{k=1}^{\infty} \left(1 - \mu^{2k} \frac{w}{w_0}\right) \left(1 - \mu^{2k} \frac{w_0}{w}\right)$$

The numbers $\alpha_{j,k}$ satisfy the relations

$$\sum_{k=1}^{n_1} \alpha_{1,k} = n_1 + 2$$
$$\sum_{k=1}^{n_0} \alpha_{0,k} = n_0 - 2$$

Hence, we can set $\alpha_{1,k} = 2 - \beta_k^{(1)}$ and $\alpha_{0,k} = \beta_k^{(0)}$.

Calculation of $\hat{\Lambda}$

First of all, we can write G in the form

$$G(z,\zeta) = -[\omega_0(z)\,\mathrm{Log}\,|\zeta| + \mathrm{Log}\,|p(z,\zeta)|]$$

where

$$p(z,\zeta) = \prod_{k=1}^{\infty} \frac{1-\mu^{2k-2}\zeta^{-1}z}{1-\mu^{2k-2}\overline{\zeta}z} \frac{1-\mu^{2k}\zeta z^{-1}}{1-\mu^{2k}\overline{\zeta}^{-1}z^{-1}}$$

and ω_0 is the harmonic measure function of Γ_0 with respect to D_0. (See *e.g.* page 260 of [8]; note that we use a different normalization.)

8.1. *Part 1: D_1 is an annulus*

The normal derivative of G in the direction of the outer normal with respect to a circle coincides with the partial derivative with respect to the radius. It follows from the Chain Rule that on $\partial\mathbb{D}$,

$$\frac{\partial}{\partial r_\zeta}\mathrm{Log}(1-\mu^{2k-2}\zeta^{-1}z) = -\frac{\partial}{\partial r_\zeta}\mathrm{Log}(1-\mu^{2k-2}\overline{\zeta}z), \text{ and}$$
$$\frac{\partial}{\partial r_\zeta}\mathrm{Log}(1-\mu^{2k}\zeta z^{-1}) = -\frac{\partial}{\partial r_\zeta}\mathrm{Log}(1-\mu^{2k}\overline{\zeta}^{-1}z^{-1})$$

Hence, on $\partial\mathbb{D}$,

$$\begin{aligned} &-\frac{\partial G(z,\zeta)}{\partial n_\zeta} \\ &= \omega_0(z)\frac{\partial}{\partial n_\zeta}(\mathrm{Log}\,|\zeta|) + 2\sum_{k=1}^{\infty}\frac{\partial}{\partial n_\zeta}(\mathrm{Log}\,|1-\mu^{2k-2}\zeta^{-1}z| + \mathrm{Log}\,|1-\mu^{2k}\zeta z^{-1}|). \end{aligned} \tag{8}$$

Using the Cauchy-Riemann equations for normal and tangential derivatives we obtain

$$\begin{aligned}\omega(z, A_m^{(0)}, D_0) &= -\frac{1}{2\pi}\int_{A_m^{(0)}} \frac{\partial G(z,\zeta)}{\partial n_\zeta} ds_\zeta \\ &= Const. \times \omega_0(z) \\ &+\frac{1}{\pi}\sum_{k=1}^{\infty}\left[\operatorname{Arg}\left(\frac{1-\mu^{2k-2}z/a_{m+1}^{(0)}}{1-\mu^{2k-2}z/a_m^{(0)}}\right) + \operatorname{Arg}\left(\frac{1-\mu^{2k}a_{m+1}^{(0)}/z}{1-\mu^{2k}a_m^{(0)}/z}\right)\right].\end{aligned}$$

Fix $\zeta \in \partial\mathbb{D}$. When $k \geq 1$, $|\mu^{2k}\zeta/z| < \mu$ for all $z \in D_0$. It follows that

$$z \in D_0 \mapsto \operatorname{Log}\left|1 - \mu^{2k}\frac{\zeta}{z}\right|$$

has a single-valued conjugate for all $k \geq 1$. When $k \geq 1$, $|\mu^{2k-2}z/\zeta| < 1$ for all $z \in D_0$. Thus,

$$z \in D_0 \mapsto \operatorname{Log}\left|1 - \mu^{2k-2}\frac{z}{\zeta}\right|$$

has a single-valued conjugate when $k \geq 1$. We infer

$$\begin{aligned}\hat{\omega}(z, A_m^{(0)}, D_0) &= Const. \\ &+\frac{1}{\pi}\sum_{k=1}^{\infty}\left[\operatorname{Log}\left(\frac{1-\mu^{2k-2}z/a_{m+1}^{(0)}}{1-\mu^{2k-2}z/a_m^{(0)}}\right) + \operatorname{Log}\left(\frac{1-\mu^{2k}a_{m+1}^{(0)}/z}{1-\mu^{2k}a_m^{(0)}/z}\right)\right].\end{aligned}$$

It then follows that

$$\begin{aligned}\sum_{m=1}^{n_0}\sigma_m^{(0)}\hat{\omega}(z, A_m^{(0)}, D_0) &= Const. + 2\sum_{k=1}^{\infty}\operatorname{Log}\left(1-\mu^{2k-2}\frac{z}{a_1^{(0)}}\right) \\ &+2\sum_{k=1}^{\infty}\operatorname{Log}\left(1-\mu^{2k}\frac{a_1^{(0)}}{z}\right) \\ &+\sum_{m=1}^{n_0}\sum_{k=1}^{\infty}(\beta_m^{(0)}-1)\operatorname{Log}\left(1-\mu^{2k-2}\frac{z}{a_m^{(0)}}\right) \\ &+\sum_{m=1}^{n_0}\sum_{k=1}^{\infty}(\beta_m^{(0)}-1)\operatorname{Log}\left(1-\mu^{2k}\frac{a_m^{(0)}}{z}\right).\end{aligned} \tag{9}$$

Now, on $\partial D_\mu(0)$,

$$-\frac{\partial}{\partial r_\zeta}\operatorname{Log}(1-\mu^{2k-2}\overline{\zeta}z) = \frac{\partial}{\partial r_\zeta}\operatorname{Log}(1-\mu^{2k}\zeta^{-1}z),$$

and

$$-\frac{\partial}{\partial r_\zeta}\operatorname{Log}(1-\mu^{2k}\overline{\zeta}^{-1}z^{-1}) = \frac{\partial}{\partial r_\zeta}\operatorname{Log}(1-\mu^{2k-2}\zeta z^{-1}).$$

At the same time, on $\partial D_\mu(0)$,

$$\frac{\partial}{\partial r_\zeta}\operatorname{Log}\left|1-\frac{z}{\zeta}\right|-\frac{\partial}{\partial r_\zeta}\operatorname{Log}\left|1-\frac{\zeta}{z}\right|=\frac{\partial}{\partial r_\zeta}\operatorname{Log}\left|\frac{z}{\zeta}\right|=-\frac{1}{\mu}.$$

So, on $\partial D_\mu(0)$,

$$\begin{aligned}&-\frac{\partial G(z,\zeta)}{\partial n_\zeta}\\&=\omega_0(z)\frac{\partial}{\partial n_\zeta}\operatorname{Log}|\zeta|+2\sum_{k=1}^{\infty}\frac{\partial}{\partial n_\zeta}(\operatorname{Log}|1-\mu^{2k-2}\zeta^{-1}z|+\operatorname{Log}|1-\mu^{2k}\zeta z^{-1}|).\end{aligned}\tag{10}$$

Again, using the Cauchy-Riemann equations for normal and tangential derivatives we obtain

$$\begin{aligned}&\hat{\omega}(z,A_m^{(1)},D_0)\\&=-\frac{1}{2\pi}\int_{A_m^{(1)}}\frac{\partial G(z,\zeta)}{\partial n_\zeta}ds_\zeta\\&=Const.\\&+\frac{1}{\pi}\sum_{k=1}^{\infty}\left[\operatorname{Arg}\left(\frac{1-\mu^{2k-2}z/c_m^{(1)}}{1-\mu^{2k-2}z/c_{m+1}^{(1)}}\right)+\operatorname{Arg}\left(\frac{1-\mu^{2k}c_m^{(1)}/z}{1-\mu^{2k}c_{m+1}^{(1)}/z}\right)\right].\end{aligned}\tag{11}$$

It now follows that

$$\begin{aligned}\sum_{m=1}^{n_1}\sigma_m^{(1)}\hat{\omega}(z,A_m^{(1)},D_0)=Const.&-2\sum_{k=1}^{\infty}\operatorname{Log}\left(1-\mu^{2k-2}\frac{z}{a_1^{(1)}}\right)\\&-2\sum_{k=1}^{\infty}\operatorname{Log}\left(1-\mu^{2k}\frac{a_1^{(1)}}{z}\right)\\&+\sum_{m=1}^{n_1}\sum_{k=1}^{\infty}(1-\beta_m^{(1)})\operatorname{Log}\left(1-\mu^{2k-2}\frac{z}{a_m^{(1)}}\right)\\&+\sum_{m=1}^{n_1}\sum_{k=1}^{\infty}(1-\beta_m^{(1)})\operatorname{Log}\left(1-\mu^{2k}\frac{a_m^{(1)}}{z}\right).\end{aligned}\tag{12}$$

We now obtain:

$$\begin{aligned}&\exp(\hat{\Lambda})(z)\\&=\frac{\Theta(\mu^{-1}z/a_1^{(0)})^2}{\Theta(\mu^{-1}z/a_1^{(1)})^2}\prod_{m=1}^{n_0}\Theta\left(\mu^{-1}\frac{z}{a_m^{(0)}}\right)^{\beta_m^{(0)}-1}\prod_{m=1}^{n_1}\Theta\left(\mu^{-1}\frac{z}{a_m^{(1)}}\right)^{1-\beta_m^{(1)}}.\end{aligned}\tag{13}$$

Calculation of $\hat{\Omega}$

Since $\psi = Id$, we have

$$\Omega(z) = -\frac{1}{2\pi}\int_{\partial D_0} \operatorname{Arg}(\zeta)\frac{\partial G(z,\zeta)}{\partial n_\zeta} ds_\zeta.$$

Let

$$\begin{aligned}
\mathcal{E}_0^1(z) &= -\frac{1}{2\pi}\int_{\partial D_\mu(0)} \operatorname{Arg}(\zeta)\frac{\partial G(z,\zeta)}{\partial n_\zeta} ds_\zeta \\
\mathcal{E}_1^1(z) &= -\frac{1}{2\pi}\int_{\partial \mathbb{D}} \operatorname{Arg}(\zeta)\frac{\partial G(z,\zeta)}{\partial n_\zeta} ds_\zeta
\end{aligned}$$

It follows from the Chain Rule that on $\partial\mathbb{D}$

$$\begin{aligned}
\frac{\partial}{\partial n_\zeta} \operatorname{Log}(1-\mu^{2k-2}\zeta^{-1}z) &= \frac{\mu^{2k-2}\zeta^{-1}z}{1-\mu^{2k-2}\zeta^{-1}z} \\
\frac{\partial}{\partial n_\zeta} \operatorname{Log}(1-\mu^{2k}\zeta z^{-1}) &= -\frac{\mu^{2k}\zeta z^{-1}}{1-\mu^{2k}\zeta z^{-1}}
\end{aligned}$$

We can then conclude from (8) that on $\partial\mathbb{D}$,

$$-\frac{\partial G(z,\zeta)}{\partial n_\zeta} = \omega_0(z) + 2\operatorname{Re}\sum_{k,n=1}^{\infty}[(\mu^{2k-2}z\zeta^{-1})^n - (\mu^{2k}z^{-1}\zeta)^n] \tag{14}$$

Let $a_1^{(0)} = e^{i\theta_1}$. We conclude that

$$\begin{aligned}
&\mathcal{E}_1^1(z) \\
&= \omega_0(z) + \frac{\operatorname{Re}}{\pi}\int_{\theta_1}^{\theta_1+2\pi} t\sum_{k,n=1}^{\infty}[(\mu^{2k-2}z)^n e^{-int} - (\mu^{2k}z^{-1})^n e^{int}]dt \\
&= \omega_0(z) + \frac{\operatorname{Re}}{\pi}\sum_{n,k=1}^{\infty} 2\pi i\left[\frac{(\mu^{2k-2}z/a_1^{(0)})^n}{n} + \frac{(\mu^{2k}a_1^{(0)}/z)^n}{n}\right] \\
&= \omega_0(z) - 2\operatorname{Re} i\sum_{k=1}^{\infty}\left[\operatorname{Log}\left(1-\mu^{2k-2}\frac{z}{a_1^{(0)}}\right) + \operatorname{Log}\left(1-\mu^{2k}\frac{a_1^{(0)}}{z}\right)\right] \\
&= \omega_0(z) + 2\sum_{k=1}^{\infty}\left[\operatorname{Arg}\left(1-\mu^{2k-2}\frac{z}{a_1^{(0)}}\right) + \operatorname{Arg}\left(1-\mu^{2k}\frac{a_1^{(0)}}{z}\right)\right].
\end{aligned} \tag{15}$$

It now follows that modulo a constant

$$\exp(\hat{\mathcal{E}}_1^1)(z) = \Theta\left(\mu^{-1}z/a_1^{(0)}\right)^2 \tag{16}$$

To compute $\mathcal{E}_0^1$ we note that on $\partial\mathbb{D}_\mu$

$$\frac{\partial}{\partial n_\zeta}\operatorname{Log}(1-\mu^{2k-2}\zeta^{-1}z) = \frac{-1}{\mu}\frac{\mu^{2k-2}\zeta^{-1}z}{1-\mu^{2k-2}\zeta^{-1}z}$$
$$\frac{\partial}{\partial n_\zeta}\operatorname{Log}(1-\mu^{2k}\zeta z^{-1}) = \frac{1}{\mu}\frac{\mu^{2k-2}\zeta z^{-1}}{1-\mu^{2k}\zeta z^{-1}}$$

and from (10) that

$$-\frac{\partial G(z,\zeta)}{\partial n_\zeta} = \mathit{Const.} - \frac{\omega_0(z)}{\mu} - \frac{2}{\mu}\operatorname{Re}\sum_{k=1}^{\infty}[(\mu^{2k-2}\zeta^{-1}z|)^n - (\mu^{2k-2}\zeta z^{-1})^n].$$

By a calculation identical to (15) we obtain that

$$\mathcal{E}_0^1(z) = \mathit{Const.} - \omega_0(z) - 2\sum_{k=1}^{\infty}\left[\operatorname{Arg}\left(1-\mu^{2k-2}\frac{z}{a_1^{(1)}}\right) + \operatorname{Arg}\left(1-\mu^{2k}\frac{a_1^{(1)}}{z}\right)\right].$$

Therefore

$$\exp(\hat{\mathcal{E}}_0^1)(z) = \Theta\left(\mu^{-1}z/a_1^{(1)}\right)^{-2}$$

and

$$\exp(-\hat{\Omega})(z) = \frac{\Theta\left(\mu^{-1}z/a_1^{(1)}\right)^2}{\Theta\left(\mu^{-1}z/a_1^{(0)}\right)^2} \tag{17}$$

Pulling it all together

It now follows from (13) and (17) that

$$f_0'(z) = \prod_{m=1}^{n_0}\Theta\left(\mu^{-1}\frac{z}{a_m^{(0)}}\right)^{\beta_m^{(0)}-1}\Theta\left(\mu^{-1}\frac{z}{a_m^{(1)}}\right)^{1-\beta_m^{(1)}},$$

which is equivalent to the classical formula.

8.2. *Part 2: D_1 is a Slit Disk*

The derivation follows the pattern in Part 1. Formulas (9) and (11) are the same as in Part 1. Since the inner circle of the annulus is mapped under ψ onto a circular slit, let $b_0^{(1)}$ and $b_1^{(1)}$ be two points that are mapped under ψ to the endpoints of

γ_1. Hence (12) takes the form

$$\begin{aligned}\sum_{m=1}^{n_1}\sigma_m^{(1)}\hat{\omega}(z,A_m^{(1)},D_0) = \mathit{Const}&\\ &+\sum_{m=1}^{n_1}\sum_{k=1}^{\infty}(1-\beta_m^{(1)})\operatorname{Log}\left(1-\mu^{2k}\frac{a_m^{(1)}}{z}\right)\\ &+\sum_{m=1}^{n_1}\sum_{k=1}^{\infty}(1-\beta_m^{(1)})\operatorname{Log}\left(1-\mu^{2k-2}\frac{z}{a_m^{(1)}}\right)\\ &-\sum_{k=1}^{\infty}\operatorname{Log}\left(1-\mu^{2k-2}\frac{z}{b_0^{(1)}}\right)\\ &-\sum_{k=1}^{\infty}\operatorname{Log}\left(1-\mu^{2k-2}\frac{z}{b_1^{(1)}}\right)\\ &-\sum_{k=1}^{\infty}\operatorname{Log}\left(1-\mu^{2k}\frac{b_0^{(1)}}{z}\right)\\ &-\sum_{k=1}^{\infty}\operatorname{Log}\left(1-\mu^{2k}\frac{b_1^{(1)}}{z}\right).\end{aligned}$$

Combining the last result with (9), we now obtain:

$$\exp(\hat{\Lambda})(z) = \frac{\Theta(\mu^{-1}z/a_1^{(0)})^2}{\Theta(\mu^{-1}z/b_0^{(1)})\Theta(\mu^{-1}z/b_1^{(1)})}\prod_{m=1}^{n_0}\Theta\left(\mu^{-1}\frac{z}{a_m^{(0)}}\right)^{\beta_m^{(0)}-1}\prod_{m=1}^{n_1}\Theta\left(\mu^{-1}\frac{z}{a_m^{(1)}}\right)^{1-\beta_m^{(1)}}.$$

Calculation of $\hat{\Omega}$

We first calculate $\hat{G}$ for the sake of calculating ψ. For the harmonic measure ω_1 with value 1 on Γ_1 and with value 0 on Γ_0, we have on $\partial\mathbb{D}$ that

$$\frac{\partial\omega_1}{\partial n} = \frac{\partial\omega_1}{\partial r} = \frac{1}{\operatorname{Log}\mu}$$

and thus the period of ω_1 with respect to $\partial\mathbb{D}$ is

$$\int_{\partial\mathbb{D}}\frac{\partial\omega_1}{\partial n}ds = \int_0^{2\pi}\frac{1}{\operatorname{Log}\mu}dt = \frac{2\pi}{\operatorname{Log}\mu}.$$

Since

$$\frac{1}{2\pi}(\hat{\omega}_1(w)-1)\operatorname{Log}|\tau| - \frac{\operatorname{Log}\mu}{2\pi}\hat{\omega}_1(w)\omega_1(\tau) = -\frac{1}{2\pi}\operatorname{Log}\tau,$$

it follows that $\hat{G}(z) = -\operatorname{Log}(p(z,\tau)) - \frac{1}{2\pi}\operatorname{Log}\tau$. Hence,

$$\Omega(z) = -\frac{1}{2\pi}\int_{\partial D_0}\operatorname{Arg}(p(\zeta,\tau))\frac{\partial G(z,\zeta)}{\partial n_\zeta}ds_\zeta.$$

We have

$$
\begin{aligned}
\operatorname{Arg}(p(z,\tau)) = \operatorname{Arg}\left(1-\frac{z}{\tau}\right) \\
&+\sum_{k=1}^{\infty}\operatorname{Arg}\left(1-\mu^{2k}\frac{z}{\tau}\right) \\
&+\sum_{k=1}^{\infty}\operatorname{Arg}\left(1-\mu^{2k}\frac{\tau}{z}\right) \\
&-\sum_{k=1}^{\infty}\operatorname{Arg}\left(1-\mu^{2k-2}\tau z\right) \\
&-\sum_{k=1}^{\infty}\operatorname{Arg}\left(1-\mu^{2k}\tau^{-1}z^{-1}\right)
\end{aligned}
$$

It follows, as in the calculation of $\hat{\Lambda}$, that the last four terms are all single-valued and have single-valued conjugates.

We now let

$$\mathcal{E}(z) = -\frac{1}{2\pi}\int_{\partial D_0}\operatorname{Arg}\left(1-\frac{z}{\tau}\right)\frac{\partial G(z,\zeta)}{\partial n_\zeta}ds_\zeta.$$

We can now write

$$
\begin{aligned}
\hat{\Omega}(z) = \hat{\mathcal{E}}(z) \\
&+\sum_{k=1}^{\infty}\operatorname{Log}\left(1-\mu^{2k}\frac{z}{\tau}\right) \\
&+\sum_{k=1}^{\infty}\operatorname{Log}\left(1-\mu^{2k}\frac{\tau}{z}\right) \\
&-\sum_{k=1}^{\infty}\operatorname{Log}\left(1-\mu^{2k-2}\tau z\right) \\
&-\sum_{k=1}^{\infty}\operatorname{Log}\left(1-\mu^{2k}\tau^{-1}z^{-1}\right).
\end{aligned}
$$

Hence,

$$\exp(\hat{\Omega}(z)) = \exp(\hat{\mathcal{E}}(z))\frac{p(z,\tau)}{1-\frac{z}{\tau}}. \tag{18}$$

8.3. *Calculation of $\hat{\mathcal{E}}$*

We first note that

$$\operatorname{Arg}\left(1-\frac{z}{\tau}\right) = \operatorname{Arg}\left(1-\frac{\tau}{z}\right) + \pi + \operatorname{Arg}(z).$$

Accordingly, we split $\mathcal{E}$ as

$$\mathcal{E} = Const. + \mathcal{E}_0 + \mathcal{E}_1^0 + \mathcal{E}_1^1$$

where

$$\mathcal{E}_0(z) = -\frac{1}{2\pi}\int_{\partial D_\mu(0)} \operatorname{Arg}\left(1-\frac{\zeta}{\tau}\right)\frac{\partial G(z,\zeta)}{\partial n_\zeta} ds_\zeta$$
$$\mathcal{E}_1^0(z) = -\frac{1}{2\pi}\int_{\partial\mathbb{D}} \operatorname{Arg}\left(1-\frac{\tau}{\zeta}\right)\frac{\partial G(z,\zeta)}{\partial n_\zeta} ds_\zeta$$
$$\mathcal{E}_1^1(z) = -\frac{1}{2\pi}\int_{\partial\mathbb{D}} \operatorname{Arg}(\zeta)\frac{\partial G(z,\zeta)}{\partial n_\zeta} ds_\zeta$$

The formula for $\hat{\mathcal{E}}_1^1$ is identical to (16) computed earlier. Combining (14) and, on $\partial\mathbb{D}$,

$$\begin{aligned}\operatorname{Arg}\left(1-\frac{\tau}{\zeta}\right) &= \frac{1}{2i}\left[\operatorname{Log}\left(1-\frac{\tau}{z}\right) - \operatorname{Log}\left(1-\frac{\tau}{\overline{z}}\right)\right]\\ &= \frac{1}{2i}\sum_{n=1}^{\infty}\left[-\frac{1}{n}\tau^n\zeta^{-n} + \frac{1}{n}\tau^n\zeta^n\right]\end{aligned}$$

we obtain from a lengthy but straightforward computation that

$$\begin{aligned}\mathcal{E}_1^0(z) &= \operatorname{Re}\frac{1}{i}\sum_{k,n=1}^{\infty}\left[\frac{1}{n}\left(\frac{\mu^{2k}\tau}{z}\right)^n + \frac{1}{n}\left(\mu^{2k-2}\tau z\right)^n\right]\\ &= -\operatorname{Re}\frac{1}{i}\sum_{k=1}^{\infty}\left[\operatorname{Log}\left(1-\mu^{2k}\frac{\tau}{z}\right) + \operatorname{Log}\left(1-\mu^{2k-2}\tau z\right)\right]\\ &= -\sum_{k=1}^{\infty}\left[\operatorname{Arg}\left(1-\mu^{2k}\frac{\tau}{z}\right) + \operatorname{Arg}\left(1-\mu^{2k-2}\tau z\right)\right].\end{aligned}$$

To compute $\mathcal{E}_0$, we need a slightly different expression than (10) for the normal derivative of G on $\partial D_\mu(0)$. We start by observing that:

$$\frac{\partial \operatorname{Log}\left(1-\mu^{2k-2}\zeta^{-1}z\right)}{\partial n_\zeta} = \frac{\partial \operatorname{Log}\left(1-\mu^{2k}\zeta^{-1}z\right)}{\partial n_\zeta} + \frac{\partial \operatorname{Log}\left(1-\zeta^{-1}z\right)}{\partial n_\zeta}$$
$$-\frac{\partial \operatorname{Log}\left(1-\mu^{2k-2}\overline{\zeta}z\right)}{\partial n_\zeta} = \frac{\partial \operatorname{Log}\left(1-\mu^{2k}\zeta^{-1}z\right)}{\partial n_\zeta}$$
$$-\frac{\partial \operatorname{Log}\left(1-\mu^{2k}\overline{\zeta}^{-1}z^{-1}\right)}{\partial n_\zeta} = \frac{\partial \operatorname{Log}\left(1-\mu^{2k}\zeta z^{-1}\right)}{\partial n_\zeta} + \frac{\partial \operatorname{Log}\left(1-\zeta/z\right)}{\partial n_\zeta}$$
$$\frac{\partial \operatorname{Log}\left(1-z/\zeta\right)}{\partial n_\zeta} + \frac{\partial \operatorname{Log}\left(1-\zeta/z\right)}{\partial n_\zeta} = \frac{\partial\left[\operatorname{Log}\left(-z/\zeta\right) + 2\operatorname{Log}\left(1-\zeta/z\right)\right]}{\partial n_\zeta}$$

It now follows that on $\partial\mathbb{D}_\mu(0)$

$$-\frac{\partial G(z,\zeta)}{\partial n_\zeta} = \frac{\omega_0\left(z\right)}{\mu} + 2\,\mathrm{Re}\sum_{k=1}^{\infty}\frac{\partial}{\partial n_\zeta}\left[\mathrm{Log}\left(1-\mu^{2k}\zeta^{-1}z\right)+\mathrm{Log}\left(1-\mu^{2k-2}\zeta z^{-1}\right)\right]$$

$$= \frac{\omega_0\left(z\right)}{\mu} + 2\mu^{-1}\,\mathrm{Re}\sum_{k,n=1}^{\infty}\left[\left(\mu^{2k}z\right)^n\zeta^{-n}-\left(\mu^{2k-2}/z\right)^n\zeta^n\right].$$

If $\zeta = \mu e^{it}$, then

$$\mathrm{Arg}\left(1-\frac{\zeta}{\tau}\right) = \frac{1}{2i}\left[\mathrm{Log}\left(1-\frac{\zeta}{\tau}\right)-\mathrm{Log}\left(1-\frac{\overline{\zeta}}{\tau}\right)\right]$$

$$= \frac{1}{2i}\sum_{n=1}^{\infty}\left[-\frac{1}{n}\left(\frac{\mu}{\tau}\right)^n e^{int}+\frac{1}{n}\left(\frac{\mu}{\tau}\right)^n e^{-int}\right].$$

We now obtain by a lengthy but fairly straightforward computation that

$$\mathcal{E}_0(z) = \omega_0(z) + \mathrm{Re}\,\frac{1}{i}\sum_{n,k=1}^{\infty}\left[\frac{1}{n}\left(\frac{\mu^{2k}z}{\tau}\right)^n+\frac{1}{n}\left(\frac{\mu^{2k}}{z\tau}\right)^n\right]$$

$$= \omega_0(z) - \mathrm{Re}\,\frac{1}{i}\sum_{k=1}^{\infty}\left[\mathrm{Log}\left(1-\mu^{2k}\frac{z}{\tau}\right)+\mathrm{Log}\left(1-\frac{\mu^{2k}}{z\tau}\right)\right]$$

$$= \omega_0(z) - \sum_{k=1}^{\infty}\left[\mathrm{Arg}\left(1-\mu^{2k}\frac{z}{\tau}\right)+\mathrm{Arg}\left(1-\frac{\mu^{2k}}{z\tau}\right)\right].$$

Notice that

$$\exp(\hat{\mathcal{E}}_1^0(z))\exp(\hat{\mathcal{E}}_0(z)) = \left(1-\frac{z}{\tau}\right)\Theta(\mu^{-1}z/\tau)^{-1}\Theta(\mu z/\tau)^{-1}.$$

Hence,

$$\exp(\hat{\Omega}(z)) = \Theta(\mu^{-1}z/a_1^{(0)})^2\left(1-\frac{z}{\tau}\right)\Theta(\mu^{-1}z/\tau)^{-1}\Theta(\mu z/\tau)^{-1}$$

Pulling it all together

It now follows that

$$f_0'(z) = N(z)\prod_{m=1}^{n_0}\Theta\left(\mu^{-1}\frac{z}{a_m^{(0)}}\right)^{\beta_m^{(0)}-1}\Theta\left(\mu^{-1}\frac{z}{a_m^{(1)}}\right)^{1-\beta_m^{(1)}}$$

where

$$N(z) = \frac{\Theta(\mu^{-1}z/\tau)\Theta(\mu z/\tau)\hat{G}'(z)}{\Theta(\mu^{-1}z/b_0^{(1)})\Theta(\mu^{-1}z/b_1^{(1)})}$$

By direct calculation, $\Theta(\mu^2 z) = -z^{-1}\Theta(z)$. We then conclude from our expression for $\hat{G}$ that $\hat{G}(z) = \hat{G}(\mu^2 z)$. Hence, $\hat{G}'(\mu^2 z) = \hat{G}'(z)\mu^{-2}$. Since τ is real, $\overline{\psi(z)} = \psi(\overline{z})$.

It then follows from elementary calculus that $b_1^{(1)} = \overline{b_0^{(1)}}$. We then infer that

$$N(\mu^2 z) = \mu^{-4} N(z).$$

Hence, $z \mapsto z^2 N(z)$ is a loxodromic function that is analytic except possibly at 0. Thus, by Liouiville's theorem for loxodromic functions, this function is constant. (See, *e.g.* [14].) Let C denote this constant. It then follows that

$$\begin{aligned} f_0'(z) &= Cz^{-2} \prod_{m=1}^{n_0} \Theta\left(\mu^{-1}\frac{z}{a_m^{(0)}}\right)^{\beta_m^{(0)}-1} \Theta\left(\mu^{-1}\frac{z}{a_m^{(1)}}\right)^{1-\beta_m^{(1)}} \\ &= C\phi_0'(z). \end{aligned}$$

The derivation is complete.

9. Discussion

The previous two sections provide strong evidence that Theorem 6 provides a simple unified framework for deriving explicit formulae for Schwarz-Christoffel mappings from analytic domains which are bounded or unbounded to polygonal domains which are bounded or unbounded. Another consequence of our work is that it is possible to contruct Schwarz-Christoffel mappings in the spirit of M. Schiffer; that is, in terms of the functions of potential theory [15]. In particular, if the boundary curves are analytic, such a map can be explicitly expressed in terms of the Green's function of D_0. A formula for the Green's function of a circular domain in terms of elementary functions is given in [16]. Another formula based on a convergent approach can be found in [**?**].

In a forthcoming paper, we will demonstrate a computational relationship between ϕ_0 and the Given Data 2.1.

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