August, 2013

An Operator-Theoretic Existence Proof of Solutions to Planar Dirichlét Problems

Timothy H. McNicholl, *Lamar University*

AN OPERATOR-THEORETIC EXISTENCE PROOF OF SOLUTIONS TO PLANAR DIRICHLÉT PROBLEMS

TIMOTHY H. MCNICHOLL

ABSTRACT. By using some elementary techniques from operator theory, we prove constructively prove the existence of solutions to Dirichlét problems for planar Jordan domains with at least two boundary curves. An iterative method is thus obtained, and explicit bounds on the error in the resulting approximations are given. Finally, a closed form for the solution is given. No amount of differentiability of the boundary is assumed.

1. INTRODUCTION

Suppose we are given a bounded Jordan domain $D \subseteq \mathbb{C}$ and a piecewise continuous function $f : \partial D \to \mathbb{R}$. The resulting *Dirichlét problem* is to find a harmonic function on D, u, such that

$$(1.1) \qquad \lim_{z \to \zeta} u(z) \;\; = \;\; f(\zeta)$$

for all $\zeta \in \partial D$ at which f is continuous. It is well-known that this Dirichlét problem has a solution. Once the existence of u is demonstrated, uniqueness follows immediately from the Maximum Principle for harmonic functions. That is, there is exactly one harmonic function on D, u, for which Equation (1.1) holds. See, *e.g.*, Chapters I and II of [4]. Accordingly, we denote this function u_f. Dirichlét problems for other kinds of domains exist but will not be considered here.

When D is simply connected, that is when D is the interior of a Jordan curve, it is fairly straightforward to prove the existence of u_f. In particular, when D is the unit disk $\mathbb{D}$,

$$u_f(z) = \frac{1}{2\pi} \int_{\partial \mathbb{D}} f(\zeta) P(z, \zeta) ds_\zeta,$$

where

$$P(z, \zeta) = \frac{1 - |z|^2}{|z - \zeta|^2}$$

and ds_ζ is the differential of arc length with respect to the variable ζ. The function P is of course called the *Poisson kernel*. Since the composition of a harmonic function with an analytic function yields a harmonic function, the existence of u_f for simply connected Jordan domains now follows from the Riemann Mapping Theorem and the Carathéodory Theorem.

When D is bounded by more than one Jordan curve, there are at least two methods available to prove the existence of u_f. One is an extreme-value argument as in Section 6.4.2 of [1]. If one desires a constructive proof, a natural choice is the

Schwarz alternating method which is described in Section IV.2 of [2] and in Chapter II of [4]. A sequence of harmonic functions $u_1 \geq u_2 \geq u_3 \ldots$ that converges to u is produced thereby. However, the proof does not give any information about the rate of convergence which is essential for error control when designing a numerical method. Nevertheless, it is hinted in the exercises of [4] that the Schwarz alternating method can be turned into an integral operator which in turn leads to a numerical method with error control.

Here, we will turn this chain of ideas on its head and give what appears to be a new and constructive proof of the existence of solutions to Dirichlét problems for planar Jordan domains based on some fairly elementary ideas from operator theory. Namely, we will first define an integral operator and then show it has a fixed point u. We will then show that this fixed point extends to a harmonic function on D for which Equation (1.1) holds. We will also give explicit bounds on the error in the resulting sequence of iterations. Finally, we use these results to derive a closed form for the solution to the Dirichlét problem. All results hold for an arbitrary Jordan domain with at least two boundary curves, even those whose boundary is nowhere differentiable.

2. A few preliminaries

When r is a positive real and $z_0 \in \mathbb{C}$, let $D_r(z_0)$ denote the open disk whose center is z_0 and whose radius is r. Let $\mathbb{D} = D_1(0)$.

Let BHarm(D) denote the space of bounded harmonic functions on D with the sup norm.

Proposition 2.1. *If D is open, then BHarm(D) is complete.*

Proof. Let $u_1, u_2, \ldots$ be a Cauchy sequence in BHarm(D). It follows that this sequence converges uniformly to a function $u : D \to \mathbb{R}$. It follows that u is continuous and bounded. It remains to show that u is harmonic. To do so, we use the mean value property. Accordingly, suppose $\overline{D_r(z_0)} \subseteq D$. Then,

$$
\begin{aligned}
u(z_0) &= \lim_{n \to \infty} u_n(z_0) \\
&= \lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} u_n(z_0 + re^{i\theta}) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \lim_{n \to \infty} u_n(z_0 + re^{i\theta}) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.
\end{aligned}
$$

It follows that u is harmonic. $\square$

3. An operator-theoretic existence proof

Suppose D is a bounded Jordan domain. Suppose $f : \partial D \to \mathbb{R}$ is piecewise continuous.

Let $\sigma_1, \ldots, \sigma_{n-1}, \tau_1, \ldots, \tau_{n-1}$ be pairwise disjoint arcs such that $D_1 =_{df} D - \bigcup \tau_j$ and $D_2 =_{df} D - \bigcup \sigma_j$ are simply connected. The case when D is bounded by four curves is illustrated in Figure 1. Let $\sigma = \bigcup_j \sigma_j$, and let $\tau = \bigcup_j \tau_j$.

Let ϕ_j be a continuous map of $\overline{\mathbb{D}}$ onto $\overline{D_j}$ that is conformal on $\mathbb{D}$. Thus, each point of $\overline{D} - \sigma$ has exactly one preimage under ϕ_1, and each point of $\overline{D} - \tau$ has
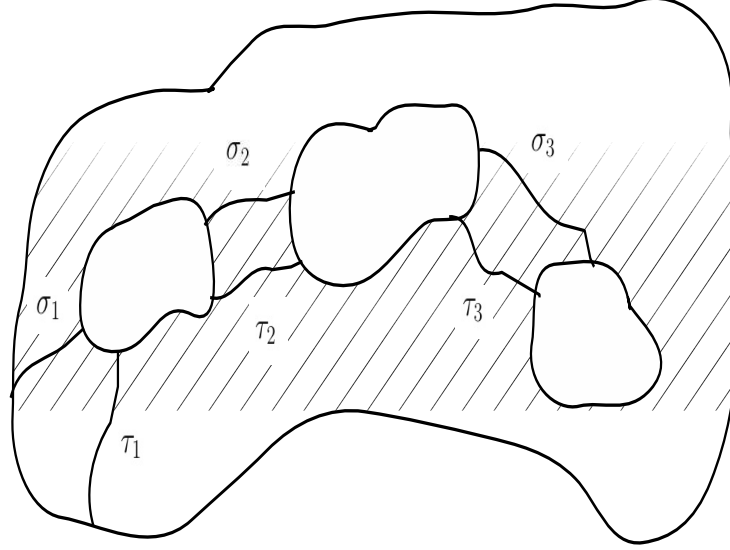
FIGURE 1.

exactly one preimage under ϕ_2. The existence of these maps follows from Theorem 2.1 of [5]. Let:

$$
\begin{aligned}
A_j &= \phi_j^{-1}[\partial D] \\
B_1 &= \phi_1^{-1}[\sigma] \\
B_2 &= \phi_2^{-1}[\tau]
\end{aligned}
$$

Let H_1 be the harmonic function on D_1 determined by the boundary conditions

$$
H_1(\zeta) = \begin{cases} f(\zeta) & \text{if } \zeta \in \partial D \\ 0 & \text{if } \zeta \in \sigma - \partial D \end{cases},
$$

and let H_2 be the harmonic function on D_2 determined by the boundary conditions

$$
H_2(\zeta) = \begin{cases} f(\zeta) & \text{if } \zeta \in \partial D \\ 0 & \text{if } \zeta \in \tau - \partial D \end{cases}
$$

It follows that H_1 and H_2 have the integral forms:

$$H_1(z) \;=\; \frac{1}{2\pi} \int_{A_1} P(\phi_1^{-1}(z),\zeta)f(\phi_1(\zeta))ds_\zeta$$

$$H_2(z) \;=\; \frac{1}{2\pi} \int_{A_2} P(\phi_2^{-1}(z),\zeta)f(\phi_2(\zeta))ds_\zeta$$

Let H_3 be the harmonic function on D_1 determined by the boundary conditions

$$H_3(\zeta) = \left\{ \begin{array}{ll} H_2(\zeta) & \text{if } \zeta \in \sigma \\ 0 & \text{otherwise} \end{array} \right.$$

Thus, H_3 has the integral form

$$(3.1) \qquad H_3(z) \;=\; \frac{1}{2\pi} \int_{B_1} P(\phi_1^{-1}(z),\zeta)H_2(\phi_1(\zeta))ds_\zeta.$$

Finally, let

$$(3.2) \qquad\qquad\qquad h \;=\; H_1 + H_3.$$

For each $\zeta_1 \in B_2$, let $K(\cdot,\zeta_1)$ be the harmonic function on D_1 defined by the boundary conditions

$$K(\zeta,\zeta_1) = \left\{ \begin{array}{ll} 0 & \zeta \notin \sigma \\ 2\pi P(\phi_2^{-1}(\zeta),\zeta_1) & \zeta \in \sigma \end{array} \right.$$

Thus, K has the integral form

$$K(z,\zeta_1) = \int_{B_1} P(\phi_2^{-1}\phi_1(\zeta),\zeta_1)P(\phi_1^{-1}(z),\zeta)ds_\zeta.$$

Note that $\phi_2^{-1}(\zeta)$ is bounded away from B_2 as ζ ranges over σ. Accordingly, let

$$m \;=\; \max\left\{ \int_{B_2} P(\phi_2^{-1}(\zeta),\zeta_1)ds_{\zeta_1} \;:\; \zeta \in \sigma \right\}.$$

It follows that $m < 2\pi$.

We now define an operator on $\mathrm{BHarm}(D_1)$. When v is harmonic on D_1, let $F(v)$ denote the function on D_1 defined by the equation

$$F(v)(z) = h(z) + \frac{1}{(2\pi)^2} \int_{B_2} K(z,\zeta_1)v(\phi_2(\zeta_1))ds_{\zeta_1}.$$

The key lemma is the following.

Lemma 3.1. *F is a contraction map on BHarm(D_1). In particular, for all $v_1, v_2 \in$ BHarm(D_1),*

$$\| F(v_1) - F(v_2) \|_\infty \le \frac{m}{2\pi} \| v_1 - v_2 \|_\infty .$$

Proof. We first show that $F(v)$ is a harmonic function on D_1 whenever v is. That is, F maps BHarm(D_1) into BHarm(D_1). This can be seen by expanding $F(v)(z)$

and applying Fubini's Theorem so as to obtain

$$
\begin{aligned}
F(v)(z) \quad = \quad & H_1(z) \\
+ \quad & \frac{1}{2\pi} \int_{B_1} \left[\frac{1}{2\pi} \int_{A_2} f(\phi_2(\zeta_1)) P(\phi_2^{-1}\phi_1(\zeta),\zeta_1) ds_{\zeta_1} \right] P(\phi_1^{-1}(z),\zeta) ds_\zeta \\
+ \quad & \frac{1}{2\pi} \int_{B_1} \left[\frac{1}{2\pi} \int_{B_2} v(\phi_2(\zeta_1)) P(\phi_2^{-1}\phi_1(\zeta),\zeta_1) ds_{\zeta_1} \right] P(\phi_1^{-1}(z),\zeta) ds_\zeta
\end{aligned}
$$

(3.3)

Each summand in Equation (3.3) defines a harmonic function on D_1. Thus, $F(v)$ is harmonic on D_1 whenever v is.

We now show that F is a contraction map. It follows from Fubini's Theorem that

$$
\int_{B_2} K(z,\zeta_1) ds_{\zeta_1} = \int_{B_1} \left[\int_{B_2} P(\phi_2^{-1}\phi_1(\zeta),\zeta_1) ds_{\zeta_1} \right] P(\phi_1^{-1}(z),\zeta) ds_\zeta.
$$

However,

$$
\begin{aligned}
\int_{B_1} \left[\int_{B_2} P(\phi_2^{-1}\phi_1(\zeta),\zeta_1) ds_{\zeta_1} \right] P(\phi_1^{-1}(z),\zeta) ds_\zeta \quad &\leq \quad m \int_{B_1} P(\phi_1^{-1}(z),\zeta) ds_\zeta \\
&\leq \quad 2\pi m.
\end{aligned}
$$

The conclusion follows. $\qquad\square$

So, let u be the fixed point of F.

Lemma 3.2. *u extends to a harmonic function on D such that $f(\zeta) = \lim_{z\to\zeta} u(z)$ for all $\zeta \in \partial D$ at which f is continuous.*

Proof. By Lemma 3.1, u is harmonic on D_1. It follows from Equation (3.3) and Fubini's Theorem that

$$
u(z) = F(u)(z) = H_1(z) + \frac{1}{2\pi} \int_{B_1} g(\phi_1(\zeta)) P(\phi_1^{-1}(z),\zeta) ds_\zeta
$$

where

$$
g(z) = \frac{1}{2\pi} \int_{A_2} P(\phi_2^{-1}(z),\zeta_1) f(\phi_2(\zeta_1)) ds_{\zeta_1} + \frac{1}{2\pi} \int_{B_2} P(\phi_2^{-1}(z),\zeta_1) u(\phi_2(\zeta_1)) ds_{\zeta_1}.
$$

Thus, g is the harmonic function on D_2 defined by the boundary conditions

$$
g(\zeta) = \left\{ \begin{array}{ll} f(\zeta) & \zeta \in \partial D \\ u(\zeta) & \zeta \in \tau - \partial D \end{array} \right.
$$

It also follows that u is the harmonic function on D_1 defined by the boundary conditions

$$
u(\zeta) = \left\{ \begin{array}{ll} f(\zeta) & \zeta \in \partial D \\ g(\zeta) & \zeta \in \sigma - \partial D \end{array} \right.
$$

Hence, $g(\zeta) = u(\zeta)$ for all $\zeta \in \sigma$. Decompose D into the simply connected domains $S_1, \ldots, S_n$ as in Figure 2. It follows that $u(\zeta) = g(\zeta)$ for all $\zeta \in \partial S_j$. It then follows that $u(z) = g(z)$ for all $z \in S_j$. It then follows that u extends to a harmonic function on D that solves the given Dirichlét problem. $\qquad\square$
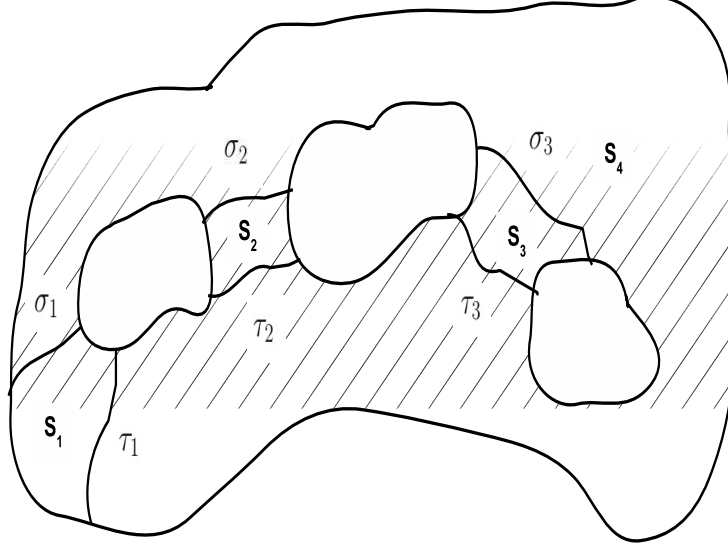
FIGURE 2.

A few moments of reflection will reveal that what the operator F does is the basic step in the Schwarz alternating method. However, by couching things in the framework of operator theory, it becomes possible to prove convergence using the contraction mapping theorem. The proof thus gives more information than would be obtained by using Harnack's Principle since it reveals the rate of convergence of the construction which is important for computation.

4. A CLOSED FORM

We first perform a procedure similar to kernel iteration. Let

$$
\begin{aligned}
\mathbf{K}^{(1)}(z, \zeta_1) &= K(z, \zeta_1) \\
\mathbf{K}^{(n+1)}(z, \zeta_1) &= \int_{B_2^n} K(z, \zeta_{n+1}) \prod_{j=1}^{n} K(\phi_2(\zeta_{j+1}), \zeta_j) ds_{\zeta_2} \dots ds_{\zeta_{n+1}} \quad n = 1, 2, \dots \\
\mathbf{K}(z, \zeta_1) &= \sum_{n=1}^{\infty} \frac{1}{(2\pi)^{2n}} \mathbf{K}^{(n)}(z, \zeta_1).
\end{aligned}
$$

Theorem 4.1. *Suppose D is a planar Jordan domain with at least two boundary curves and that $f : \partial D \to \mathbb{R}$ is piecewise continuous. Let h, B_j, etc. be as in*

Section 3. Then,

$$(4.1) \qquad u_f(z) \;=\; h(z) + \int_{B_2} \mathbf{K}(z, \zeta_1) h(\phi_2(\zeta_1)) ds_{\zeta_1}$$

Furthermore,

$$\left| h(z) + \sum_{n=1}^{t} \frac{1}{(2\pi)^{2n}} \int_{B_2} \mathbf{K}^{(n)}(z, \zeta_1) h(\phi_2(\zeta_1)) d_{s_{\zeta_1}} - u(z) \right| \;\leq\; \left(\frac{m}{2\pi} \right)^{t+1} \frac{2\pi}{2\pi - m} \parallel f \parallel_\infty .$$

$$(4.2)$$

Proof. Let $\mathbf{0}$ denote the zero function on D_1. Again, let u denote the fixed point of F. It follows that u is the restriction of u_f to D_1. Let $z \in D_1$. By induction on $t = 1, 2, 3, \ldots$,

$$F^{t+1}(\mathbf{0})(z) = h(z) + \sum_{n=1}^{t} \frac{1}{(2\pi)^{2n}} \int_{B_2} \mathbf{K}^{(n)}(z, \zeta_1) h(\phi_2(\zeta_1)) ds_{\zeta_1}.$$

It follows from Lemma 3.1 that

$$\parallel F^j(\mathbf{0}) - F^{j+1}(\mathbf{0}) \parallel_\infty \;\leq\; \left(\frac{m}{2\pi} \right)^j \parallel F(\mathbf{0}) \parallel_\infty$$

$$= \left(\frac{m}{2\pi} \right)^j \parallel h \parallel_\infty .$$

By the Maximum Principle, $\parallel h \parallel_\infty \leq \parallel f \parallel_\infty$. By a fairly standard calculation,

$$\parallel F^{t+1}(\mathbf{0}) - u \parallel_\infty \;\leq\; \sum_{n=t+1}^{\infty} \left(\frac{m}{2\pi} \right)^k \parallel f \parallel_\infty$$

$$= \left(\frac{m}{2\pi} \right)^{t+1} \frac{2\pi}{2\pi - m} \parallel f \parallel_\infty .$$

It follows that Equations (4.1) and (4.2) hold whenever $z \in D_1$. Hence, by continuity, they hold for all $z \in D$. $\qquad \square$

We mention here that D. Crowdy and J. Marshall have obtained closed formulas for Green's function in multiply connected domains. See [3].

References

1. Lars V. Ahlfors, *Complex analysis*, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.
2. R. Courant and D. Hilbert, *Methods of mathematical physics. Vol. II*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1989, Partial differential equations, Reprint of the 1962 original, A Wiley-Interscience Publication.
3. Darren Crowdy and Jonathan Marshall, *Green's functions for Laplace's equation in multiply connected domains*, IMA J. Appl. Math. **72** (2007), no. 3, 278–301.
4. J. B. Garnett and D. E. Marshall, *Harmonic measure*, New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2005.
5. Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 299, Springer-Verlag, Berlin, 1992.

DEPARTMENT OF MATHEMATICS, LAMAR UNIVERSITY, BEAUMONT, TEXAS 77710 USA
E-mail address: `timothy.h.mcnicholl@gmail.com`