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Abstract

In this paper, we discuss a linear operator T defined in Riordan group $\mathcal{R}$ by using the upper shift matrix U and lower shift matrix U^T, namely for each $R \in \mathcal{R}$, $T : R \mapsto URU^T$. Some isomorphic properties of the operator T and the structures of its range sets for different domains are studied. By using the operator T and the properties of Bell subgroup of $\mathcal{R}$, the Riordan type Chu-Vandermonde identities and the Riordan equivalent identities of Format Last Theorem and Beal Conjecture are shown. The applications of the shift operators to the complementary Riordan arrays and to the Riordan involutions and Riordan pseudo-involutions are also presented.

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Key Words and Phrases: Riordan arrays, Riordan group, generating function, shift matrices, production matrix, Bell subgroup, Chu-Vandermonde identity, Format Last Theorem, Beal Conjecture, Riordan involutions, and Riordan pseudo-involutions.

1 Introduction

Riordan arrays are infinite, lower triangular matrices defined by the generating function of their columns. They form a group, called *the Riordan group* (see Shapiro, Getu, W. J. Woan and L. Woodson [16]).

More formally, let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[\![t]\!]$; the *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ ($f_k \in \mathbb{R}$), is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t), h(t))$ defines the *(proper) Riordan array* $D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ having

$$d_{n,k} = [t^n]d(t)h(t)^k \tag{1}$$

or, in other words, having $d(t)h(t)^k$ as the generating function whose coefficients make-up the entries of column k.

From the *fundamental theorem of Riordan arrays* (see [15]), it is immediate to show that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d_1(t),\, h_1(t))(d_2(t),\, h_2(t)) = (d_1(t)d_2(h_1(t)),\, h_2(h_1(t))). \tag{2}$$

The Riordan array $I = (1,\, t)$ acts as an identity for this product.

Several subgroups of $\mathcal{R}$ are important and have been considered in the literature:

- the set $\mathcal{A}$ of *Appell arrays* is the collection of all Riordan arrays $R = (d(t),\, t)$ in $\mathcal{R}$;
- the set $\mathcal{L}$ of *Lagrange arrays* is the collection of all Riordan arrays $R = (1,\, h(t))$ in $\mathcal{R}$;
- the set $\mathcal{B}$ of *Bell* or *renewal arrays* is the collection of all Riordan arrays $R = (d(t),\, td(t))$ in $\mathcal{R}$;
- the set $\mathcal{C}$ of the *checkboard arrays* is the collection of all Riordan arrays $R = (d(t),\, h(t))$ for which $d(t)$ is an even function and $h(t)$ is an odd function;
- the set $\mathcal{E}$ of the Riordan arrays $R = ((h(t)/t)^r h'(t)^s, h(t))$ for real or complex r and s in $\mathcal{R}$ is called *Luźon-Merlini-Morón-Sprugnoli (LMMS) subgroup*, denoted by $\mathcal{E}[r, s]$, which includes $\mathcal{D} = \mathcal{E}[0, 1]$ as its special case (see [10]).

From [14], an infinite lower triangular array $[d_{n,k}]_{n,k\in\mathbb{N}} = (d(t), h(t))$ is a Riordan array if and only if an *A-sequence* $A = (a_0 \neq 0, a_1, a_2, \ldots)$ exists such that for every $n, k \in \mathbb{N}$ there holds

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + \cdots + a_n d_{n,n}, \tag{3}$$

which is equivalent to (see, for example, [8])

$$h(t) = tA(h(t)). \tag{4}$$

Here, $A(t)$ is the generating function of the A-sequence. In [12] it is also shown that a unique *Z-sequence* $Z = (z_0, z_1, z_2, \ldots)$ exists such that every element in column 0 can be expressed as the linear combination

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + \cdots + z_n d_{n,n}, \tag{5}$$

or equivalently (see, for example, [8]),

$$d(t) = \frac{d_{0,0}}{1 - tZ(h(t))}. \tag{6}$$

We may write (6) and (4) as

$$\frac{d(t) - d_{0,0}}{td(t)} = Z(h(t)), \quad \frac{d(t)h^n(t)}{t} = t^{n-1}A(h(t)). \tag{7}$$

Denote the *upper shift matrix* by U, i.e.,

$$U = (\delta_{i+1,j})_{i,j\geq 0} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$P = \begin{bmatrix} z_0 & a_0 & 0 & 0 & 0 & \cdots \\ z_1 & a_1 & a_0 & 0 & 0 & \cdots \\ z_2 & a_2 & a_1 & a_0 & 0 & \cdots \\ z_3 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \left(Z(t), A(t), tA(t), t^2A(t), \ldots\right), \tag{8}$$

where the rightmost expression is the representation of P by using its column generating functions. Here, P is called the *production matrix* or P*- matrix characterization* or simply P *matrix*.

The transpose of U is

$$\begin{aligned} U^T &= (\delta_{i+1,j})^T_{i,j\in N_0} = (\delta_{i,j+1})_{i,j\in N_0} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \end{aligned}$$

which is called the *lower shift matrix*. Thus, $UU^T = I$ and $U^TU = I - diag(1,0,0,\ldots)$, because the shift arrays U and U^T are infinite. However, for an $n \times n$ shift matrix U_n, we have $U_nU_n^T = I - diag(0,\ldots,0,1)_n$ and $U_n^TU_n = I - diag(1,0,\ldots,0)_n$. Hence, U^T is the right inverse of U, and by abuse of notation - if not otherwise specified - we will represent the right inverse U^T simply by U^{-1} sometimes. Thus, (7) can be written in a matrix form by using upper shift matrix U:

$$U(d(t), h(t)) = (d(t), h(t))P \tag{9}$$

because its left-hand side and right-hand side are the same (see also [5]).

Definition 1.1 *[7, 10] Let $\mathcal{R}$ be the Riordan group. $T : \mathcal{R} \mapsto \mathcal{R}$ is a linear operator defined by*

$$TR = URU^T \tag{10}$$

for every $R \in \mathcal{R}$.

It can be seen that the operator T is well defined because, for any $R = (d(t), h(t)) \in \mathcal{R}$, there holds

$$\begin{aligned}
&TR = URU^T \\
= \; &(\delta_{i+1,j})_{i,j\in\mathbb{N}_0}(d(t), d(t)h(t), d(t)h^2(t), \ldots)(\delta_{i,j+1})_{i,j\in\mathbb{N}_0} \\
= \; &\left(\frac{d(t)h(t)}{t}, \frac{d(t)h^2(t)}{t}, \frac{d(t)h^3(t)}{t}, \ldots\right) = \left(\frac{d(t)h(t)}{t}, h(t)\right) \in \mathcal{R}. \qquad (11)
\end{aligned}$$

We now give an alternative definition of T based on the observation of (11).

Definition 1.2 *Let $\mathcal{R}$ be the Riordan group. For every $R = (d(t), h(t)) \in \mathcal{R}$, the right multiply linear operator $T_r : \mathcal{R} \mapsto \mathcal{R}$ is defined by*

$$T_r R := R(A(t), t) = (d(t), h(t))\left(\frac{t}{\bar{h}(t)}, t\right), \qquad (12)$$

where $A(t)$ is the generating function of the A-sequence characterization, $\bar{h}(t)$ is the compositional inverse of $h(t)$, and the last equation comes from $A(t) = t/\bar{h}(t)$ shown in Corollary 3.2 of [6]. The left multiply linear operator $T_l : \mathcal{R} \mapsto \mathcal{R}$ is defined by

$$T_l R := (A(h(t), t)R = \left(\frac{h(t)}{t}, t\right)(d(t), h(t)), \qquad (13)$$

where $A(h(t)) = h(t)/t$ is from the sequence characterization of Riordan arrays (see, for example, [8]).

From the definitions of the shift operators shown above, we may obtain

Theorem 1.3 *Definitions 1.1 and 1.2 are equivalent in the sense that*

$$TR = T_r R = T_l R = \left(d(t)\frac{h(t)}{t}, h(t)\right) \qquad (14)$$

for every $R \in \mathcal{R}$. Both T_r and T_l are invertible. Their inverses, denoted by T_r^{-1} and T_l^{-1}, respectively, are linear operators given by

$$T_r^{-1} R = R\left(\frac{1}{A(t)}, t\right) = (d(t), h(t))\left(\frac{\bar{h}(t)}{t}, t\right) \quad \text{and} \qquad (15)$$

$$T_l^{-1} R = \left(\frac{1}{A(h(t))}, t\right) R = \left(\frac{t}{h(t)}, t\right)(d(t), h(t)). \qquad (16)$$

Remark 1.1 Let $R \equiv (d_{n,k})_{n,k\in bN_0}$ and $TR \equiv (e_{n,k})_{n,k\in bN_0}$. We can prove $TR = \left(d(t)\frac{h(t)}{t}, h(t)\right)$ directly by showing $e_{n,k} = d_{n+1,k+1}$ as follows:

$$e_{n,k} = [t^n]d(t)\frac{h(t)}{t}h(t)^k = [t^{n+1}]d(t)h(t)^{k+1} = d_{n+1,k+1}.$$

Corollary 1.4 *Each of T, T_r, and T_l maps $\mathcal{A}$ and $\mathcal{C}$ to themselves, maps $\mathcal{L}$ to $\mathcal{B}$, and maps $\mathcal{E}[r,s]$ to $\mathcal{E}[r+1,s]$. $\mathcal{L}$ and $\mathcal{B}$ have one-to-one correspondence and are isomorphic.*

Proof. Corollary 1.4 includes Propositions 2.4 and 2.5 of [7] as special cases, but it proof is similar to the proofs of Propositions 2.4 and 2.5. Hence, we omit it.

■

Theorem 1.5 *Let $R = (d(t), h(t)) \in \mathcal{R}$, and let P be the matrix characterizing R with $Z(t)$ and $A(t)$ as the generating functions of Z- and A-sequences characterizing R, respectively. Then TR defined by (10) is in $\mathcal{R}$ with characterization matrix*

$$\tilde{P} = \left[\tilde{Z}(t), \tilde{A}(t), t\tilde{A}(t), \ldots\right], \tag{17}$$

where

$$\tilde{Z}(t) = \frac{td(\bar{h}(t)) - a_0 d(0)\bar{h}(t)}{t\bar{h}(t)d(\bar{h}(t))} \quad \text{and} \quad \tilde{A}(t) = \frac{t}{\bar{h}(t)}, \tag{18}$$

where $a_0 = A(0)$, $\tilde{Z}(t)$ and $\tilde{A}(t)$ are the generating functions of the Z- and A-sequences characterizing TR, respectively, and $\bar{h}(t)$ is the compositional inverse of $h(t)$. In addition, the relationships between $Z(t)$ and $\tilde{Z}(t)$ and $A(t)$ and $\tilde{A}(t)$ are

$$\tilde{Z}(t) = \frac{A(t) - a_0}{t} + a_0 \frac{Z(t)}{A(t)} \quad \text{and} \quad \tilde{A}(t) = A(t). \tag{19}$$

Furthermore, the characterization matrix $\tilde{P}$ of TR can be presented as

$$\tilde{P} = TP + Q, \tag{20}$$

where $TP = UPU^T$ and $Q = (a_0 Z(t)/A(t), A(t), tA(t), \ldots)$.

Proof. Since $TR \in \mathcal{R}$, there exists a P-matrix characterization of TR, which is denoted by $\tilde{P}$ and satisfies $U(TR) = (TR)\tilde{P}$. Furthermore, matrix $\tilde{P}$ can be written as (17), where $\tilde{Z}(t)$ and $\tilde{A}(t)$ are the generating functions of the Z- and A- sequences characterizing TR, respectively. Since $TR = (d(t)h(t)/t, h(t))$, from (11) we may have $\tilde{A}(t) = t/\bar{h}(t)$, and from (14) we may find

$$\tilde{Z}(h(t)) = \frac{d(t)\frac{h(t)}{t} - d(0)h_1}{td(t)\frac{h(t)}{t}},$$

where h_1 is the coefficient of the linear term of $h(t)$, which can be evaluated as

$$h_1 = \lim_{t\to 0} \frac{h(t)}{t} = \lim_{t\to 0} A(h(t)) = a_0.$$

Substituting $t = \bar{h}(t)$ into the last expression of $\tilde{Z}(t)$ yields

$$\tilde{Z}(t) = \frac{d(\bar{h}(t))\frac{t}{\bar{h}(t)} - d(0)a_0}{td(\bar{h}(t))},$$

which implies the first formula of (18). (14) can be written as

$$d(\bar{h}(t)) = \frac{d(0)}{1-\bar{h}(t)Z(t)}.$$

By substituting the above expression of $d(\bar{h}(t))$ into the first equation of (18), we have

$$\begin{aligned}\tilde{Z}(t) &= \frac{t - a_0\bar{h}(t) + a_0\bar{h}(t)^2 Z(t)}{t\bar{h}(t)}\\ &= \frac{\frac{t}{\bar{h}(t)} - a_0}{t} + a_0\frac{\bar{h}(t)}{t}Z(t),\end{aligned}$$

and (19) follows immediately by noticing $A(t) = t/\bar{h}(t)$. In addition, it is easy to observe that

$$TP = UPU^T = \left(\frac{A(t)-a_0}{t}, A(t), tA(t), \ldots\right),$$

which implies (20).

■

Remark 2.2 From (9), the necessary and sufficient condition for $R \in \mathcal{R}$, there holds

$$TR = URU^T = RP^T.$$

Hence,

$$U(TR) = URP^T = UR(U^TU + D(1,\bar{0}))PU^T = (TR)(TP) + URD(1,\bar{0})PU^T,$$

where $D(1,\bar{0})$ is the diagonal matrix $(1,0,0,\ldots)$, which implies

$$URD(1,\bar{0})PU^T = (TR)Q,$$

where Q is given in (20) because $TR \in \mathcal{R}$ and its matrix characterization is $TP+Q$.

As an example of Theorem 1.5, we consider $(d(t),h(t)) = (1, t/(1-t))$, which has the generating functions of Z- and A-sequences as $Z(t) = 0$ and $A(t) = 1+t$, respectively. Then $T(d(t),h(t)) = (1/(1-t), t/(1-t))$, which has the generating functions of Z- and A-sequences as $\tilde{Z}(t) = (A(t)-1)/t = 1$ and $\tilde{A}(t) = A(t) = 1+t$, respectively.

The right inverse, denoted by S, of the operator T can be defined by

$$SR = U^TRU \tag{21}$$

in the sense that for every $R \in \mathcal{R}$

$$(TS)R = U(U^TRU)U^T = R,$$

where we use $UU^T = I$. However, because of $U^TU = I - diag(1,0,0,\ldots)$ there holds

$$(ST)R = U^T(URU^T)U = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}. \tag{22}$$

Hence, we have the following result.

Proposition 1.6 *Let $\mathcal{R}$ be the Riordan group, and let S be the linear operator defined by* (21)*. Then there holds reproduce formula*

$$R = (SR + D(d_{0,0}, \bar{0}))(U^T P + D(1, \bar{0})) = U^T RP + D(d_{0,0}, \bar{0}) \tag{23}$$

for every $R \in \mathcal{R}$, where P is shown in (8)*, and $D(a, \bar{b})$ is the diagonal matrix $diag(a, b, b, \ldots)$.*

Proof. By using $RP = UR$ shown in (9), the right-hand side of the first equation of (23) can be expanded as

$$\begin{aligned} & (U^T RU)(U^T P) + D(d_{0,0}, \bar{0}) = U^T RP + D(d_{0,0}, \bar{0}) \\ = \quad & U^T UR + D(d_{0,0}, \bar{0}) = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ d_{1,0} & d_{1,1} & 0 & \cdots \\ d_{2,0} & d_{2,1} & d_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix} + D(d_{0,0}, \bar{0}), \end{aligned}$$

which implies that the rightmost side and the leftmost side of (23) are equal.

■

Corollary 1.7 *Let $\mathcal{R}$ be the Riordan group, and let $D(a, \bar{b})$ be the diagonal matrix $diag(a, b, b, \ldots)$. Then for every $R \in \mathcal{R}$ and any $n \in \mathbb{N}$ there holds the expansion formula of R*

$$R = \sum_{k=0}^{n} (U^T)^k D(d_{0,0}, \bar{0}) P^k + (U^T)^{n+1} RP^{n+1}, \tag{24}$$

where $(U^T)^{n+1} RP^{n+1}$ is the remainder, which implies that the kth row ($k \geq 0$) of R is the $d_{0,0}$ multiple of the kth row of P^k.

Proof. Repeating use of (23) n times, we have

$$\begin{aligned} & R = D(d_{0,0}, \bar{0}) + U^T RP = D(d_{0,0}, \bar{0}) + U^T(D(d_{0,0}, \bar{0}) + U^T RP)P \\ = \quad & D(d_{0,0}, \bar{0}) + U^T D(d_{0,0}, \bar{0})P + (U^T)^2 RP^2 \\ = \quad & D(d_{0,0}, \bar{0}) + U^T D(d_{0,0}, \bar{0})P + (U^T)^2(D(d_{0,0}, \bar{0}) + U^T RP)P^2 \\ = \quad & \sum_{k=0}^{2} (U^T)^k D(d_{0,0}, \bar{0}) P^k + (U^T)^3 RP^3 = \cdots \\ = \quad & \sum_{k=0}^{n} (U^T)^k D(d_{0,0}, \bar{0}) P^k + (U^T)^{n+1} RP^{n+1} \end{aligned}$$

and completing the proof of corollary.

■

In next section, by using the defined operator in terms of shift operators and the properties of Bell subgroup of $\mathcal{R}$, we give the *Riordan type Chu-Vandermonde identities* and the Riordan equivalent identities of *Format Last Theorem* and *Beal Conjecture*. We also apply the defined operator to the *complementary Riordan arrays* to present some of their properties. Finally, the application of the shift operators in the study of *Riordan involutions* and *Riordan pseudo-involutions* is discussed, which include the characterizations of those involutions in terms of the shift operators.

2 Applications of shift operators

2.1 Riordan equivalent identities to FLT and Beal Conjuncture

Let $(d(t), h(t))$ be an element of the Bell subgroup $\mathcal{B}$ of the Riordan group, i.e., it satisfies $d(t) = h(t)/t$. Then we have the following extension of *Chu-Vandermonde identity*, called the *Riordan type Chu-Vandermonde identities.*

Proposition 2.1 *Denote by $(d_{n,k})_{0\le k\le n}$ the Bell-type Riordan array $(h(t)/t, h(t))$. Then there holds the identity*

$$\sum_{n=0}^{s} d_{n,k} d_{s-n,m} = d_{s+1,k+m+1} \tag{25}$$

for $s \ge 0$ and $n \ge k \ge 0$. Particularly, if $h(t) = 1/(1-t)$, then formula (25) reduces to the classical Chu-Vandermonde identity

$$\sum_{n=0}^{s} \binom{n}{k}\binom{s-n}{m} = \binom{s+1}{k+m+1}. \tag{26}$$

Proof. From the definition of Riordan arrays and noting $d(t) = h(t)/t$, we have

$$\begin{aligned} \sum_{n=0}^{s} d_{n,k} d_{s-n,m} &= \sum_{n=0}^{s} [t^n]d(t)h(t)^k [t^{s-n}]d(t)h(t)^m \\ = \quad & [t^s]d(t)^2 h(t)^{k+m} = [t^{s+1}]d(t)h(t)^{k+m+1} = d_{s+1,k+m+1}. \end{aligned}$$

Noting

$$d_{n,k} = [t^n]d(t)h(t)^k = \binom{n}{k}$$

when $d(t) = 1/(1-t)$ and $h(t) = t/(1-t)$ (i.e., $(d(t), h(t))$ is the Pascal triangle), we immediately obtain (26) from (25).

■

From Remark 1.1 and Corollary 1.4, we may transfer Proposition 2.1 to the case of the Lagrange-type Riordan array (or Lagrange arrays).

Corollary 2.2 *Denote by* $(e_{n,k})_{0\le k\le n}$ *the Lagrange-type Riordan array* $(1, h(t))$*. Then there holds the identity*

$$\sum_{n=0}^{s} e_{n+1,k+1}e_{s-n+1,m+1} = e_{s+2,k+m+2} \tag{27}$$

for $s \ge 0$ *and* $n \ge k \ge 0$*.*

For given positive integers a, b, and c, let $N(x^a + y^b = c)$ denote the number of positive integral solutions in x and y of the Diophantine equation $x^a + y^b = c$. Let $(d(t), h(t))$ be a Riordan involution, then its entries satisfy (45). Thus, we have the following combinatorial equivalence to the *Format Last Theorem* (FLT), called the *Riordan equivalence to the FLT*.

Theorem 2.3 *Suppose* $(d(t), h(t)) = (d_{n,k} = [t^n]d(t)h(t)^k)_{n\ge k\ge 0}$ *is a Bell-type Riordan involution, where* $d(t) = h(t)/t$ *and* $h(t) = \bar{h}(t) \in Z[[t]]$*, the compositional inverse of* $h(t)$*. Then the number* $N(x^a + y^b = c)$ *can be calculated in terms of the elements of* $(h(t)/t, h(t))$ *via the following combinatorial sum of rank* 4*:*

$$N(x^a + y^b = c) = \sum_{x\ge 1}\sum_{y\ge 1}\sum_{u\ge 1}\sum_{v\ge 1} d_{u,x^a}d_{v,y^b}d_{c+1,u+v+1}. \tag{28}$$

Particularly, for $c = z^n$ *and* $a = b = n \ge 3$*, where* z *is an integer greater* 2*, there holds the identity*

$$\sum_{x=1}^{z}\sum_{y=1}^{z}\sum_{u=1}^{z^n-1}\sum_{v=1}^{z^n-1} d_{u,x^a}d_{v,y^b}d_{z^n+1,u+v+1} = 0, \tag{29}$$

which is equivalent to that the Fermat equation $x^n + y^n = z^n$ *with* $n \ge 3$ *and* $z \ge 2$ *has no solution; i.e.,* (29) *implies FLT and vice versa.*

Proof. Making the substitution $s = u+v$ on the right-hand side of (28) and applying formulas (25) and (45) successively, we find that the right-hand side of (28) can be evaluated as

$$\begin{aligned}
& \sum_{x\ge 1}\sum_{y\ge 1}\sum_{s=2}^{c}\left(\sum_{u=1}^{s} d_{u,x^a}d_{s-u,y^b}\right)d_{c+1,s+1} \\
= & \sum_{x\ge 1}\sum_{y\ge 1}\sum_{s=2}^{c} d_{s+1,x^a+y^b+1}d_{c+1,s+1} \\
= & \sum_{x\ge 1}\sum_{y\ge 1}\delta_{c+1,x^a+y^b+1} = \sum_{x\ge 1}\sum_{y\ge 1}\delta_{c,x^a+y^b} \\
= & N(x^a + y^b = c).
\end{aligned}$$

Since FLT infers that $N(x^n + y^n = z^n) = 0$ for $n \ge 3$ and $z \ge 2$, by substituting $a = b = n$ and $c = z^n$ in (28) we obtain (29).

■

Corollary 2.4 *Suppose $(e_{n,k} = [t^n]h(t)^k)_{n\geq k\geq 0}$ is a Lagrange-type Riordan involution $(1, h(t))$, where $h(t) = \bar{h}(t) \in Z[[t]]$, its compositional inverse. Then the number $N(x^a + y^b = c)$ can be calculated in terms of the elements of $(1, h(t))$ via the following combinatorial sum of rank 4:*

$$N(x^a + y^b = c) = \sum_{x\geq 1}\sum_{y\geq 1}\sum_{u\geq 1}\sum_{v\geq 1} e_{u+1,x^a+1}e_{v+1,y^b+1}e_{c+2,u+v+2}. \tag{30}$$

Particularly, for $c = z^n$ and $a = b = n \geq 3$, where z is an integer greater 2, there holds the identity

$$\sum_{x=1}^{z}\sum_{y=1}^{z}\sum_{u=1}^{z^n-1}\sum_{v=1}^{z^n-1} e_{u+1,x^a+1}e_{v+1,y^b+1}e_{z^n+2,u+v+2} = 0, \tag{31}$$

which is equivalent to that the Fermat equation $x^n + y^n = z^n$ with $n \geq 3$ and $z \geq 2$ has no solution; i.e., (31) implies FLT and vice versa.

For $c \geq 2$, denote $P_c := \{(x, y) \in \mathbb{N}^2 : gcd(x, y, c) = 1\}$, and denote by $N(x^a + y^b = z^c, (x, y) \in P_z)$ the number of positive integral solutions in x and y of the Diophantine equation $x^a + y^b = z^c$ that satisfies $gcd(x, y, z) = 1$. The *Beal Conjecture* is that if the equation has positive integral solution x, y and z for positive integers a, b, and c greater than 2, then x, y and z must have a common factor. Using a same approach, we may have

Proposition 2.5 *For $a, b, c \geq 3$, let $N(x^a + y^b = z^c, (x, y) \in P_z)$ be defined as before, and let $(d(t), h(t)) = (d_{n,k} = [t^n]d(t)h(t)^k)_{n\geq k\geq 0}$ be a Bell-type Riordan involution, namely $d(t) = h(t)/t$ and $h(t) = \bar{h}(t)$. Then there holds*

$$N(x^a + y^b = z^c, (x, y) \in P_z) = \sum_{(x,y)\in P_c}\sum_{u=1}^{z^c-1}\sum_{v=1}^{z^c-1} d_{u,x^a}d_{v,y^b}d_{z^c+1,u+v+1}. \tag{32}$$

Hence, the identity

$$\sum_{(x,y)\in P_c}\sum_{u=1}^{z^c-1}\sum_{v=1}^{z^c-1} d_{u,x^a}d_{v,y^b}d_{z^c+1,u+v+1} = 0 \tag{33}$$

implies the Beal Conjecture and vice versa.

Proof. Making the substitution $s = u+v$ on the right-hand side of (28) and applying formulas (25) and (45), we find that the right-hand side of (28) can be written as

$$
\begin{aligned}
& \sum_{(x,y)\in P_c} \sum_{s=2}^{c} \left(\sum_{u\geq 1} d_{u,x^a} d_{s-u,y^b} \right) d_{c+1,s+1} \\
= & \sum_{(x,y)\in P_c} \sum_{s=2}^{c} d_{s+1,x^a+y^b+1} d_{c+1,s+1} \\
= & \sum_{(x,y)\in P_c} \delta_{c+1,x^a+y^b+1} = \sum_{(x,y)\in P_c} \delta_{c,x^a+y^b} \\
= & N(x^a+y^b=z^c,(x,y)\in P_z).
\end{aligned}
$$

Since Beal Conjecture infers that $N(x^a+y^b=z^c,(x,y)\in P_z)=0$ for $a,b,c\geq 3$, by substituting $c=z^n$ in (32) we obtain (33).

■

By using $d(t)=1/(1-t)$ we can obtain the special cases of (29) and (33) shown in Hsu and Shiue [9] and Yin [19], respectively.

Corollary 2.6 *For $a,b,c\geq 3$, let $N(x^a+y^b=z^c,(x,y)\in P_z)$ be defined as before, and let $(e_{n,k}=[t^n]h(t)^k)_{n\geq k\geq 0}$ be a Lagrange-type Riordan involution $(1,h(t))$, namely $h(t)=\bar{h}(t)$. Then there holds*

$$
N(x^a+y^b=z^c,(x,y)\in P_z)=\sum_{(x,y)\in P_c} \sum_{u=1}^{z^c-1} \sum_{v=1}^{z^c-1} e_{u+1,x^a+1} e_{v+1,y^b+1} e_{z^c+2,u+v+2}. \tag{34}
$$

Hence, the identity

$$
\sum_{(x,y)\in P_c} \sum_{u=1}^{z^c-1} \sum_{v=1}^{z^c-1} e_{u+1,x^a+1} e_{v+1,y^b+1} e_{z^c+2,u+v+2} = 0 \tag{35}
$$

implies and is implied by the Beal Conjecture.

It can be shown (see, for example, [11]) that if the abc conjecture holds, then the Beal conjecture holds for large enough exponents. Here, the abc conjecture can be formulated as follows. Let $r(a,b,c)$ be the square free part of the product abc, i.e., the product of the prime divisors of a, b, and c with each divisor counted only once. For each $\epsilon>0$, there exists a constant $\mu>1$ such that if a and b are relatively prime (or coprime) and $c=a+b$, then $c<\mu r(a,b,c)^{1+\epsilon}$. Furthermore, Darmon and Granville [4] showed in 1995 that if the positive integers a, b, and c are such that $1/a+1/b+1/c<1$, then there exist only finitely many triples of coprime integers x, y, z satisfying $x^a+y^b=z^c$; or equivalently, there exist only finitely many triples of coprime integers x, y, z such that (33) holds for positive integers a, b, and c satisfying $1/a+1/b+1/c<1$. Hence, there are only finitely many identities (33) or (35) need to be checked.

2.2 Applications to complementary Riordan arrays

A Riordan array $R=(d(t),h(t))=(d_{n,k})_{n,k\in\mathbb{N}_0}$ can be extended to the setting of $n,k\in\mathbb{Z}$. The corresponding array is an infinite array centered at $d_{0,0}$ and its entries are defined by $d_{n,k}=[t^n]d(t)h(t)^k$, where $n,k\in\mathbb{Z}$ and $d(t)h(t)^k$ is a formal Laurent series (see [10]). The extended Riordan array of R is denoted by $\tilde{R}:=(d_{n,k}=[t^n]d(t)h(t)^k)_{n,k\in\mathbb{Z}}$. As an example, we give the following Pascal array, which is also shown in [10].

$$
(d_{n,k})_{n,k\in\mathbb{Z}}=\left(\frac{1}{1-t},\frac{t}{1-t}\right)
$$

$$
\begin{array}{cc}
 & (k=0) \\
=\quad (n=0) & \begin{bmatrix}
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 3 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{bmatrix},
\end{array}
$$

where the center entry of the above array is $d_{0,0}=1$. It is clearly that the set of all Riordan arrays $(d(t),h(t))=(d_{n,k})_{n,k\in\mathbb{Z}}$ associated with matrix multiplication

$$
(d_{n,k})_{n,k\in\mathbb{Z}}(e_{n,k})_{n,k\in\mathbb{Z}}=(r_{n,k}=\sum_j d_{n,j}e_{j,k})_{n,k\in\mathbb{Z}}
$$

form a group called the *extended Riordan group*. The A- and Z-sequence characterizations of $(d_{n,k})_{n,k\in\mathbb{Z}}$ are the same as the original Riordan array $R=(d(t),h(t))=(d_{n,k})_{n,k\in\mathbb{N}_0}$. Hence, $\tilde{R}=(d(t),h(t))=(d_{n,k})_{n,k\in\mathbb{Z}}$ is also called *recursive matrix* in [10] accordingly. For instance, the above Pascal array has Z-sequence $\{1,0,0,\ldots\}$ and A-sequence $\{1,1,0,\ldots\}$. Hence, we have the P-matrix characterization of $\tilde{R}=(d(t),h(t))=(d_{n,k})_{n,k\in\mathbb{Z}}$ as follows:

$$
\begin{array}{cc}
 & (k=0) \\
P=(n=0) & \begin{bmatrix}
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{bmatrix}
\end{array}
$$

We use $n = 0$ and $k = 0$ to separate the Pascal array of integer indices into four quadrants. Luzón et al. [10] defines $[m]$-*complementary matrix* from the recursive matrix $\tilde{R} = (d(t), h(t)) = (d_{n,k})_{n,k\in\mathbb{Z}}$. We now simplify their construction as below. A $[m]$-complementary matrix, denoted by $(d^{[m]}_{n,k})_{n\geq k\geq 0}$ associated with the recursive matrix $\tilde{R} = (d(t), h(t)) = (d_{n,k})_{n,k\in\mathbb{Z}}$ is defined by

$$d^{[m]}_{n,k} = d_{-k-m,-n-m} \tag{36}$$

for $n, k \in \mathbb{N}_0$, which is derived by using the following three step process: (1) Rotating $(d(t), h(t)) = (d_{n,k})_{n,k\in\mathbb{Z}}$ $\pi/2$-counterclockwisely around $d_{0,0}$; (2) reflecting the part of the resulting matrix in the third quadrant to the fourth quadrant about the zero column, which is denoted by $(d^{[0]}_{n,k})_{n\geq k\geq 0}$; and (3) applying operator T defined in Definition 1.1 to the matrix $(d^{[0]}_{n,k})_{n\geq k\geq 0}$ in the fourth quadrant m times to obtain $(d^{[m]}_{n,k})_{n\geq k\geq 0}$. Hence (see [10]),

$$(d^{[m]}_{n,k})_{n\geq k\geq 0} = T^m (d^{[0]}_{n,k})_{n\geq k\geq 0}. \tag{37}$$

From Theorem 3.1 of [10], the $[m]$-complementary array of Riordan array $R = (d(t), h(t))$ is also a Riordan array, which is denoted by $R^{[m]}$. From (36) and (37), we have

$$R^{[m]} = \left(d(\bar{h}(t))\bar{h}'(t) \left(\frac{t}{\bar{h}(t))} \right)^{m+1}, \bar{h}(t) \right), \tag{38}$$

which is given in [10] and proved by using Lagrange Inversion Formula (See [3, 17, 18]).

When we apply linear operator T to the extension of R, some results presented before may not hold because $d_{n,k}$ may not be zero for $k < 0$. However, there is no problem to apply T to $[m]$-complementary array $R^{[m]}$. From (14), it is easy to see that

$$T^{m+1} R^{[m]} = \left(d(\bar{h}(t))\bar{h}'(t), \bar{h}(t) \right),$$

which means that T^{m+1} maps the $[m]$-complementary array of $(1, h(t))$, an element in $\mathcal{L}$, to $(\bar{h}'(t), \bar{h}(t))$, an element of $\mathcal{D}$. The mappings between the $[m]$-complementary arrays of some subgroups and other subgroups can be established similarly. The interested readers may derive them and similar results shown before.

2.3 Riordan involutions and pseudo-involutions

If g is an element of a group G, then the smallest positive integer n such that $g^n = e$, the identity of the group, if it exists, is called the *order* of g. If there is no such integer, then g is said to have *infinite order*. It is well-known (see [15]) that if we restrict all entries of a Riordan array to be integers, then any element of finite order in the Riordan group must have order 1 or 2, and each element of order 2 generates a subgroup of order 2. We now give the P-matrix characterization of Riordan arrays of order 2. In [2], an element of order 2 in the Riordan group is called a *Riordan involution*. Some structures of a Riordan involution were presented in [1, 2].

Proposition 2.7 *Let $R \in \mathcal{R}$, and let P be its P-matrix characterization. Then R has order 2 if and only if its P matrix satisfies*

$$P = RUR, \tag{39}$$

where U is the upper shift matrix $(\delta_{i+1,j})_{i,j\geq 0}$.

Proof. From (9) we have

$$UR^2 = RPR.$$

Hence, $R^2 = I$ implies $P = R^{-1}UR^{-1} = RUR$. In addition, (9) and $P = RUR$ yields

$$TR = URU^T = RPU^T = R^2URU^T = R^2TR,$$

which implies $R^2 = I$.

■

Theorem 2.8 *Let R be a lower triangular matrix with non-zero diagonal entries. Then R is a Riordan involution if and only if there exits a matrix $P = (Z(t), A(t), tA(t), \ldots)$ such that both $UR = RP$ and $RU = PR$ hold, where*

$$A(t) = \frac{t}{h(t)} \qquad \text{and} \qquad Z(t) = \frac{\delta - d(t)}{\delta h(t)} = \frac{1-\delta d(t)}{h(t)}, \tag{40}$$

where $\delta = d_{0,0} = d(0)$.

Proof. We know $R \in \mathcal{R}$ if and only if it satisfies $UR = PR$ for some P shown in the statement. Furthermore, $R \in \mathcal{R}$ and $R^2 = I$ if and only if $P = RUR$, or equivalently, $PR = PR^{-1} = RU$. Since R is a Riordan involution, from Corollary 2.2, we have

$$(d(t), h(t))U = P(d(t), h(t)). \tag{41}$$

where $P = (Z(t), A(t), tA(t), \ldots)$. The left-hand side of (41) can be written as

$$(d(t), h(t))U = (0, d(t), d(t)h(t), d(t)h(t)^2, \ldots), \tag{42}$$

where the last matrix is represented by the generating functions of its columns. Meanwhile, the right-hand side of (41) is

$$\begin{aligned}
& P(d(t), h(t)) = (Z(t), A(t), tA(t), \ldots)(d_{n,k})_{0\leq k\leq n} \\
= \; & \left(d_{0,0}Z(t) + \frac{A(t)}{t}\sum_{k\geq 1} d_{k,0}t^k,\; \frac{A(t)}{t}\sum_{k\geq 1} d_{k,1}t^k,\; \frac{A(t)}{t}\sum_{k\geq 2} d_{k,2}t^k,\; \ldots\right) \\
= \; & \left(d_{0,0}\left(Z(t) - \frac{A(t)}{t}\right) + \frac{A(t)}{t}d(t),\; \frac{A(t)}{t}d(t)h(t),\; \frac{A(t)}{t}d(t)h(t)^2,\; \ldots\right).
\end{aligned} \tag{43}$$

Comparing the rightmost terms of (42) and (43) yields

$$d_{0,0}\left(Z(t)-\frac{A(t)}{t}\right)=0 \quad and$$
$$\frac{A(t)}{t}d(t)h(t)^k=d(t)h(t)^{k-1}, \quad k=1,2,\ldots,$$

which imply (40).

■

The characterization (40) of Riordan arrays of order 2 was given in Theorem 4.3 of [8] (a modification is input here), which was proved by using a different approach.

It is easy to see that the linear operator T defined before maps a Riordan involution to an involution. Hence, we have

Proposition 2.9 *The Riordan array $(d(t),h(t))$ is an involution if and only if $d(t)$ and $h(t)$ satisfy*

$$\bar{h}(t)=h(t), \quad d(t)d(h(t))=1, \tag{44}$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$. The entries of $(d(t),h(t))$ satisfy

$$\sum_{j=k}^{n} d_{n,j}d_{j,k}=\delta_{n,k}, \tag{45}$$

where $\delta_{n,k}$ is the Kronecker symbol. In addition, the linear operator T maps a Riordan involution to an involution.

Proof. The necessary and sufficient conditions shown in (44) are readily checked via multiplication formula (2) of Riordan arrays. If $(d(t),h(t))$ is a Riordan involution, then

$$T(d(t),h(t))=\left(\frac{d(t)h(t)}{t},h(t)\right),$$

which is also an involution because $h(t)=\bar{h}(t)$ and

$$\frac{d(t)h(t)}{t}\left(\frac{d(\bar{h}(t))h(\bar{h}(t))}{\bar{h}(t)}\right)=d(t)d(h(t))=1.$$

■

If $(d(t),h(t))$ is a Riordan involution, then from Proposition 2.9 $\{T^n(d(t),h(t))\}_{n\geq 0}$ is a set of Riordan involutions.

An element R in the Riordan group has *pseudo-order 2* or is called a *Riordan pseudo-involution* (see [15]) if RM is a Riordan involution, where $M=(1,-t)$.

Similar to Proposition 2.1 and Theorem 2.3, we have the following characterization of Riordan pseudo-involutions.

Theorem 2.10 *Let $R = (d(t), h(t)) = (d_{n,k})_{0\leq k\leq n}$ be a Riordan array, and let P be its P-matrix characterization. Then R is a Riordan pseudo-involution if and only if its P matrix satisfies*

$$P = (d(t), -h(t))U(d(t), -h(t)), \tag{46}$$

where U is the upper shift matrix $(\delta_{i+1,j})_{i,j\geq 0}$.

More generally, if R is a lower triangular matrix with non-zero diagonal entries. Then R is a Riordan pseudo-involution if and only if there exits a matrix $P = (Z(t), A(t), tA(t), \ldots)$ such that both $UR = RP$ and $RU = -PR$ hold, where

$$A(t) = -\frac{t}{h(t)} \qquad \text{and} \qquad Z(t) = \frac{d(t)-\delta}{\delta h(t)} = \frac{\delta d(t) - 1}{h(t)}, \tag{47}$$

where $\delta = d_{0,0} = d(0)$.

Proof. Since R is a Riordan pseudo-involution, $R(1, -t)$ is a Riordan involution. From (40), we have

$$P = R(1, -t)UR(1, -t),$$

which implies (46). From the above equation, if R is a Riordan pseudo-involution, then we also have

$$PR(1, -t) = R(1, -t)U$$

because of $(R(1, -t))^{-1} = R(1, -t)$. Hence,

$$PR = R(1, -t)U(1, -t) = -RU$$

because of $(1, -t)U(1, -t) = -U$, from which one may obtain (47) by using a similar argument in the proof of Theorem 2.3.

■

Proposition 2.11 *$(d(t), h(t))$ is a pseudo-involution in $\mathcal{R}$ if and only if*

$$(d(t), h(t))^{-1} = (d(-t), -h(-t)). \tag{48}$$

Proof. To prove (48), we only need to observe

$$((d(t), h(t))(1, -t))\,((d(t), h(t))(1, -t)) = I$$

and

$$(1, -t)(d(t), h(t))(1, -t) = (d(-t), -h(-t)),$$

which is from (15).

■

As an example of (48), we consider $(1/(1-t), t/(1-t))$ that is a pseudo-involution in $\mathcal{R}$ because

$$\left(\frac{1}{1-t}, \frac{t}{1-t}\right)^{-1} = \left(\frac{1}{1+t}, \frac{t}{1+t}\right).$$

Corollary 2.12 *$(d(t), h(t))$ is a pseudo-involution in $\mathcal{R}$ if and only if*

$$\bar{h}(t) = -h(-t) \quad \text{and} \quad d(-t)d(\bar{h}(t)) = 1, \tag{49}$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$. The last equation is equivalent to

$$d(t)d(-h(t)) = 1. \tag{50}$$

Proof. Comparing (48) and $(d(t), h(t))^{-1} = (1/d(\bar{h}(t)), \bar{h}(t))$, where $\bar{h}(t)$ is the compositional inverse of $h(t)$, we obtain the sufficient and necessary conditions (49). By substituting $t = -t$ into the last equation of (49), we have (50).

■

It is obvious that the key step to find Riordan involution is to solve the Babbage equation $h(h(t)) = t$. [1, 13] gave a nice method in solving this equation. Since $h(t) \in \mathcal{F}_1$ and $h(t) = \sum_{n\geq 0} h_n t^n$, a direct computation of $h(t)$ using Faá di Bruno's formula might be presented below.

$$h(h(t)) = \sum_{n\geq 0} c_n t^n,$$

where the coefficient c_n $(n \geq 1)$ can be expressed as a sum over composition of n or as an equivalent sum over partition of n:

$$c_n = \sum_{\mathbf{i}\in E_n} h_k h_{i_1} h_{i_2} \dots h_{i_k}, \tag{51}$$

in which the index set

$$E_n := \{(i_1, i_2, \dots, i_k) : 1 \leq k \leq n, i_1 + i_2 + \cdots + i_k = n, 1 \leq i_1, i_2, \dots, i_k \leq n\} \tag{52}$$

is the set of composition of n with k denoting the number of parts. Alternatively, we may write

$$c_n = \sum_{k=1}^{n} h_k \sum_{\pi\in P_{n,k}} \binom{k}{\pi_1, \pi_2, \dots, \pi_n} h_1^{\pi_1} h_2^{\pi_2} \cdots h_n^{\pi_n}, \tag{53}$$

where the index set

$$P_{n,k} := \{(\pi_1, \pi_2, \dots, \pi_n) : \pi_1 + \pi_2 + \cdots + \pi_n = k, \pi_1 \cdot 1 + \pi_2 \cdot 2 + \cdots + \pi_n \cdot n = n\} \tag{54}$$

is the set of partitions of n into k parts, in frequency-of-parts form. Since $h(h(t)) = 1$, there holds

$$\sum_{\mathbf{i}\in E_n} h_k h_{i_1} h_{i_2} \ldots h_{i_k} = c_n = \delta_{n,1}, \tag{55}$$

from which we may expect to solve $\{h_n\}_{n\geq 1}$ inductively. More precisely, we have the following conjecture:
Conjecture Let $h(t) = \sum_{n\geq 1} h_n t^n$ be the solution of equation $h(h(t)) = 1$, or equivalently, let its coefficients $\{h_n\}$ be the solution of (55). Then h_1 must be 1 or -1. If $h_1 = 1$, then $h(t) = t$; if $h = -1$, then all even indexed coefficients h_{2n} free and all odd indexed coefficients h_{2n+1} are determined by previous coefficients h_k $(k = 2, 3, \ldots, 2n)$, i.e., previous even indexed coefficients h_{2k} $(k = 1, 2, \ldots, n)$.

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