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Application of Faà di Bruno's Formula in the Construction of Combinatorial Identities

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Dedicated to Professor Goong Chen on the occasion of his 65th birthday

Abstract

Chou, Hsu, and Shiue in [3] use Faà di Bruno's formula give a class of composite series expansions. In this paper, we apply those expansions to construct a class of identities for Catalan numbers, large Schröder numbers, small Schröder numbers, parametric Catalan numbers, Stirling numbers, binomial numbers, some other recursive number sequences, and recursive polynomial sequences.

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1. Introduction

Throughout this paper the following notations will be used. Let $\mathbb{N}$ be the set of positive integers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Denote by $\sigma(n)$ the set of partition of n ($n \in \mathbb{N}$) with k ($1 \le k \le n$) parts, represented by $1^{k_1}2^{k_2}\cdots n^{k_n}$ with $k_1 + 2k_2 + \cdots + nk_n = n$ and $k_1 + k_2 + \cdots + k_n = k$, $k_i \in \mathbb{N}_0$. Any given formal power series $\phi(t) = \sum_{n\ge 0} a_n t^n$ over real number field $\mathbb{R}$ may be conveniently written as

$$\phi(t) = \sum_{n\ge 0} \begin{bmatrix}\phi\\ n\end{bmatrix} t^n.$$

Sometimes we denote $\begin{bmatrix}\phi\\ n\end{bmatrix}$ by $[t^n]\phi(t)$.

Let f and ϕ be n-th differentiable functions. Then the n-th derivative of their composition function of $f \circ \phi$ can be presented by the Faà di Bruno's formula (see, for example, [4, 15])

$$\begin{aligned} & D^n\left((f\circ\phi)(t)\right) \\ = & \sum_{\sigma(n)} \frac{n!}{k_1!k_2!\cdots k_n!}\left(D^k f\right)\left(\frac{D^1\phi(t)}{1!}\right)^{k_1}\left(\frac{D^2\phi(t)}{2!}\right)^{k_2}\cdots\left(\frac{D^n\phi(t)}{n!}\right)^{k_n}. \end{aligned} \tag{1}$$

Assume that f has the inverse, denoted by f^{-1}. Considering $(f\circ\phi)$ and $f^{-1}\circ(f\circ\phi)$, inspired by Hsu [14] and Chou, Hsu, and Shiue [3], we may use Faa di Bruno's formula to establish

$$\begin{bmatrix}f\circ\phi\\ n\end{bmatrix} = \sum_{\sigma(n)}\left(D^k f(x)\right)_{x=\phi(0)} \frac{\begin{bmatrix}\phi\\ 1\end{bmatrix}^{k_1}\begin{bmatrix}\phi\\ 2\end{bmatrix}^{k_2}\cdots\begin{bmatrix}\phi\\ n\end{bmatrix}^{k_n}}{k_1!k_2!\cdots k_n!} \tag{2}$$

and

$$\begin{bmatrix}\phi\\ n\end{bmatrix} = \sum_{\sigma(n)}\left(D^k f^{-1}(x)\right)_{x=(f\circ\phi)(0)} \frac{\begin{bmatrix}f\circ\phi\\ 1\end{bmatrix}^{k_1}\begin{bmatrix}f\circ\phi\\ 2\end{bmatrix}^{k_2}\cdots\begin{bmatrix}f\circ\phi\\ n\end{bmatrix}^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{3}$$

Let $f(x) = e^x$ and $\phi(t) = \sum_{n\ge 0} a_n t^n$, and let $b_n = \begin{bmatrix}f\circ\phi\\ n\end{bmatrix}$. If $\phi(0) = 0$, i.e., $a_0 = 0$, then (2) and (3) become

$$b_n = \sum_{\sigma(n)} \frac{a_1^{k_1}a_2^{k_2}\cdots a_n^{k_n}}{k_1!k_2!\cdots k_n!} \tag{4}$$

and

$$a_n = \sum_{\sigma(n)} (-1)^{k-1}(k-1)! \frac{b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n}}{k_1! k_2! \cdots k_n!}, \tag{5}$$

respectively.

Let $f(x) = x^r$ ($r \neq 0$) and $\phi(t) = \sum_{n\geq 0} a_n t^n$, and let $b_n = \begin{bmatrix} f \circ \phi \\ n \end{bmatrix}$. If $\phi(0) = 0$, i.e., $a_0 = 0$, then (2) and (3) become

$$b_n = \sum_{\sigma(n,r)} r! \frac{a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}}{k_1! k_2! \cdots k_n!}, \tag{6}$$

where $\sigma(n, r)$ is the set of partition of n ($n \in \mathbb{N}$) with exact r parts. If $\phi(0) = 1$, i.e., $a_0 = 1$, then (2) and (3) become

$$b_n = \sum_{\sigma(n)} \frac{r!}{k!} \frac{a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}}{k_1! k_2! \cdots k_n!} \tag{7}$$

and

$$a_n = \sum_{\sigma(n)} \left(\frac{1}{r}\right)_k \frac{b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n}}{k_1! k_2! \cdots k_n!}, \tag{8}$$

respectively, where $(t)_k = t(t-1)\ldots(t-k+1)$.

In next section, by selecting different ϕ and f we will apply (4) to (8) to construct numerous identities for Catalan numbers, large Schröder numbers, small Schröder numbers, parametric Catalan numbers, Stirling numbers, binomial numbers, some other recursive number sequences, and recursive polynomial sequences.

2. Construction of Identities

2.1. Identities for Catalan numbers and Schröder numbers

Catalan numbers, $C_n = \binom{2n}{n}/(n+1)$, form a sequence of integers that occur in the solutions of many counting problems. The book Enumerative Combinatorics: Volume 2 [21] by Stanley contains a set of exercises of Chapter 6 which describe 66 different interpretations of the Catalan numbers. Some complementary materials of the exercises of chapter 6 are collected in [22]. The small and large Schröder numbers are defined by $\{1, 1, 3, 11, 45, 197, \ldots\}$ and $\{1, 2, 6, 22, 90, 394, \ldots\}$, respectively. A survey regarding those numbers can be found in [23] by Stanley. Like Catalan numbers, Schröder numbers occur in various counting problems, often involving recursively defined objects, such as dissections of a convex polygon, certain polyominoes, various lattice paths, Lukasiewicz words, permutations avoiding given patterns, and, in particular, plane trees (see, for example, [7, 17]). Deutsch and Shapiro [5] present many interesting identities of Catalan numbers. Our intention in this paper is to construct identities for the Catalan numbers and Schröder numbers by using the approach shown in (4) and (5) based on Faà di Bruno's formula.

Theorem 1. *Denote by C_n the Catalan numbers. Then there hold*

$$C_n := \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\sum_{\sigma(n)} \frac{2^{2n-k}}{1^{k_1}k_1!2^{k_2}k_2!\cdots n^{k_n}k_n!} \tag{1}$$

and

$$4^n = 2n\sum_{\sigma(n)}(-1)^{k-1}(k-1)!\Pi_{j=1}^{n}\frac{\binom{2j}{j}^{k_j}}{k_j!}. \tag{2}$$

Furthermore, we have

$$C_n = \frac{1}{n+1}\sum_{[n/2]\le k\le n}\left(-\frac{1}{2}\right)_k \frac{(-4)^{n-k}}{(2k-n)!(n-k)!}. \tag{3}$$

Proof. Let $f(x) = e^x$ and

$$\begin{aligned}\phi(t) &= \ln\frac{1}{\sqrt{1-4t}} = -\frac{1}{2}\ln(1-4t)\\ &= \frac{1}{2}\left(4t+\frac{1}{2}(4t)^2+\frac{1}{3}(4t)^3+\cdots\right).\end{aligned}$$

Then

$$\begin{bmatrix}\phi\\ n\end{bmatrix} = \frac{1}{2n}4^n.$$

Since

$$(f\circ\phi)(t) = \frac{1}{\sqrt{1-4t}} = \sum_{n\ge 0}\binom{2n}{n}t^n, \tag{4}$$

from (4) the coefficient of the nth term can be written as

$$\begin{aligned} b_n &= \binom{2n}{n} = \sum_{\sigma(n)}\frac{\left(\frac{4}{2\cdot 1}\right)^{k_1}\left(\frac{4^2}{2\cdot 2}\right)^{k_2}\cdots\left(\frac{4^n}{2\cdot n}\right)^{k_n}}{k_1!k_2!\cdots k_n!}\\ &= \sum_{\sigma(n)}\frac{4^{k_1+2k_2+\cdots+nk_n}}{2^{k_1+k_2+\cdots+k_n}1^{k_1}k_1!2^{k_2}k_2!\cdots n^{k_n}k_n!}\\ &= \sum_{\sigma(n)}\frac{4^n}{2^k 1^{k_1}k_1!2^{k_2}k_2!\cdots n^{k_n}k_n!},\end{aligned}$$

which implies (1). Similarly, noting $f^{-1}(x) = \ln(x)$ and

$$D^k\ln(x) = (-1)^{k-1}(k-1)!,$$

from (5) we may obtain (2).

Let $f(x) = x^{-1/2}$ and $\phi(t) = 1 - 4t$. Then $(f \circ \phi)(t)$ can be expanded as (4). Thus, substituting $b_n = \binom{2n}{n}$ and $a_1 = 1$ and $a_2 = -4$ into (7) and noticing $k_1 = 2k - n$ and $k_2 = n - k$, we obtain (3). $\square$

Remark 2.1 The combinatorial identities about 4^n can be found from many references, for instance, Chen and Xu [2], Duarte and Guedes de Oliverira [6], Petkovšep, Wilf, and Zeilberger [16], and Sved [25].

Remark 2.2 The right hand-sides of (1) and (3) give two methods to evaluate the non-crossing partitions of an n-element set.

Recently, one of the authors define a generalization of Catalan numbers and Catalan triangles associated with two parameters based on the sequence characterization of Bell-type Riordan arrays. Among the generalized Catalan numbers, a class of large generalized Catalan numbers and a class of small generalized Catalan numbers are defined, which can be considered as an extension of large Schröder numbers and small Schröder numbers, respectively. The generating function of the parametric Catalan numbers with parameters c and r is defined in [10] by

$$d_{c,r}(t) = \frac{1 - (c-r)t - \sqrt{1 - 2(c+r)t + (c-r)^2t^2}}{2rt}. \tag{5}$$

In particular, when $(c, r) = (1, 1), (2, 1)$, and $(1, 2)$, we obtain $d_{1,1}(t) =: C(t) = (1 - \sqrt{1-4t})/(2t)$, the classical Catalan function, $d_{2,1}(t) =: S(t) = (1 - t - \sqrt{1 - 6t + t^2})/(2t)$, the large Schröder function, and

$$d_{1,2}(t) =: s(t) = \frac{1 + t - \sqrt{1 - 6t + t^2}}{4t},$$

the small Schröder function, respectively.

Theorem 2. *Denote by C_n the Catalan numbers. Then there hold*

$$C_n = \sum_{\sigma(n)} \Pi_{j=1}^{n} \frac{1}{j^{k_j} k_j!} \left(\frac{\binom{2k_j - 1}{k_j}}{k_j} \right)^{k_j} \tag{6}$$

and

$$\binom{n+1}{2} = n \sum_{\sigma(n)} (-1)^{k-1}(k-1)! \Pi_{j=1}^{n} \frac{C_j^{k_j}}{k_j!}. \tag{7}$$

Proof. Let $f(x) = e^x$ and

$$\phi(t) = \ln\left(\frac{1 - \sqrt{1-4t}}{2t} \right).$$

Then from the Taylor expansion of the above $\phi(t)$

$$\phi(t) = \sum_{n \geq 0} \binom{2n-1}{n} \frac{t^n}{n}$$

we have

$$\begin{bmatrix}\phi\\ n\end{bmatrix}=\frac{1}{n}\binom{2n-1}{n}$$

for $n \geq 1$, which is Sequence A001700 shown in [18]. Since

$$(f\circ\phi)(t)=\frac{1-\sqrt{1-4t}}{2t}=\sum_{n\geq 0}C_n t^n,$$

by substituting

$$a_n=\frac{1}{n}\binom{2n-1}{n}\quad\text{and}\quad b_n=C_n$$

into (4) and (5), we obtain (6) and (7). □

Theorem 3. *Denote by S_n the large Schröder numbers. Then there hold*

$$S_n=\sum_{\sigma(n)}\Pi_{j=1}^{n}\frac{2^k}{j^{k_j}k_j!}\left(\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}\binom{j+\ell}{\ell}\right)^{k_j}\tag{8}$$

and

$$\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\binom{n+\ell}{\ell}=\frac{n}{2}\sum_{\sigma(n)}(-1)^{k-1}(k-1)!\Pi_{j=1}^{n}\frac{S_j^{k_j}}{k_j!},\tag{9}$$

where

$$2\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\binom{n+\ell}{\ell}$$

are elements of sequence $A002003$, $\{d_n\}_{n\geq 0}=\{0,2,8,38,192,1002,\ldots\}$.

Proof. Let $f(x)=e^x$ and $\phi(t)=\ln S(t)$. Then from the Taylor expansion of $\ln S(t)$

$$\phi(t)=2\sum_{n\geq 1}\left(\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\binom{n+\ell}{\ell}\right)\frac{t^n}{n}$$

we have

$$a_n:=\begin{bmatrix}\phi\\ n\end{bmatrix}=\frac{2}{n}\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\binom{n+\ell}{\ell}$$

for $n \geq 1$, which is Sequence A002003 shown in [19]. Since

$$(f\circ\phi)(t)=\frac{1-t-\sqrt{1-6t+t^2}}{2t}=\sum_{n\geq 0}S_n t^n,$$

by substituting a_n and $b_n=S_n$ into (4) and (5), one may have (8) and (9). □

Theorem 4. *Denote by s_n the small Schröder numbers. Then there hold*

$$s_n = \sum_{\sigma(n)} \Pi_{j=1}^{n} \frac{1}{j^{k_j} k_j!} \left(\sum_{\ell=0}^{j-1} \binom{j}{\ell+1} \binom{j+\ell}{\ell} \right)^{k_j} \tag{10}$$

and

$$\sum_{\ell=0}^{n-1} \binom{n}{\ell+1} \binom{n+\ell}{\ell} = n \sum_{\sigma(n)} (-1)^{k-1}(k-1)! \Pi_{j=1}^{n} \frac{s_j^{k_j}}{k_j!}, \tag{11}$$

where

$$\sum_{\ell=0}^{n-1} \binom{n}{\ell+1} \binom{n+\ell}{\ell}$$

are elements of sequence $A002002$*,* $\{e_n\}_{n\geq 0} = \{0, 1, 5, 25, 129, 681, \ldots\}$.

Proof. Let $f(x) = e^x$ and $\phi(t) = \ln s(t)$. Then from the Taylor expansion of $\ln s(t)$

$$\phi(t) = \sum_{n\geq 1} \left(\sum_{\ell=0}^{n-1} \binom{n}{\ell+1} \binom{n+\ell}{\ell} \right) \frac{t^n}{n}$$

we have

$$a_n := \begin{bmatrix} \phi \\ n \end{bmatrix} = \frac{1}{n} \sum_{\ell=0}^{n-1} \binom{n}{\ell+1} \binom{n+\ell}{\ell}$$

for $n \geq 1$, which is Sequence A002002 shown in [20]. Since

$$(f \circ \phi)(t) = \frac{1 + t - \sqrt{1 - 6t + t^2}}{4t} = \sum_{n\geq 0} s_n t^n,$$

(10) and (11) follow. □

Remark 2.3 a_n shown in Theorem 3 is the number of order-preserving partial self maps of $\{1, ..., n\}$. And b_n shown in Theorem 4 is the number of ordered trees with $2n$ edges, having root of even degree, non-root nodes of outdegree at most 2 and branches of odd length.

Another extension of Catalan numbers by parametrization,

$$G_{a,b}(t) = \frac{1 - at - \sqrt{1 - 2at + (a^2 - 4b)t^2}}{2bt^2}, \tag{12}$$

is given by Catlan in[1], which includes Catalan numbers, Motzkin numbers, and small Schröder numbers as its special cases, but not large Schröder numbers. In fact,

$$G_{2,1} = \frac{1-2t-\sqrt{1-4t}}{2t^2},$$
$$G_{1,1} = \frac{1-t-\sqrt{1-2t-3t^2}}{2t^2},$$
$$G_{3,2} = \frac{1-3t-\sqrt{1-6t+t^2}}{4t^2}$$

are generating functions of Catalan numbers ($\{\binom{2n}{n-1}/n\}_{n\geq 1}$), Motzkin numbers, and small Schröder numbers, respectively. $G_{a,b}(t)$ has the following expansion formula shown in [1]:

$$G_{a,b}(t) = \sum_{n\geq 0}\left(\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k a^{n-2k}b^k\right)x^n. \tag{13}$$

We now give a uniform identity of the coefficient sequence of $G_{a,b}(t)$ in terms of all real numbers a and b by using Faà di Brune formula.

Theorem 5. *Let $G_{a,b}(t)$ be the function defined as* (12) *with coefficient sequence*

$$G_{a,b;n} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k a^{n-2k}b^k. \tag{14}$$

Then we have

$$G_{a,b;n} = \sum_{\sigma(n)}\Pi_{j=1}^{n}\frac{1}{j^{k_j}k_j!}\left(\sum_{\ell=0}^{\lfloor j/2\rfloor}\binom{j}{2\ell}\binom{2\ell}{\ell}a^{j-2\ell}b^{\ell}\right)^{k_j} \tag{15}$$

and

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}a^{n-2k}b^k = n\sum_{\sigma(n)}(-1)^{k-1}(k-1)!\Pi_{j=1}^{n}\frac{G_{a,b;j}^{k_j}}{k_j!}. \tag{16}$$

In particular, for $(a,b)=(2,1),(1,1)$, and $(3,2)$, we have

$$C_n = \sum_{\sigma(n)}\Pi_{j=1}^{n}\frac{1}{j^{k_j}k_j!}\left(\sum_{\ell=0}^{\lfloor j/2\rfloor}\binom{j}{2\ell}\binom{2\ell}{\ell}2^{j-2\ell}\right)^{k_j},$$
$$M_n = \sum_{\sigma(n)}\Pi_{j=1}^{n}\frac{1}{j^{k_j}k_j!}\left(\sum_{\ell=0}^{\lfloor j/2\rfloor}\binom{j}{2\ell}\binom{2\ell}{\ell}\right)^{k_j},$$
$$s_n = \sum_{\sigma(n)}\Pi_{j=1}^{n}\frac{1}{j^{k_j}k_j!}\left(\sum_{\ell=0}^{\lfloor j/2\rfloor}\binom{j}{2\ell}\binom{2\ell}{\ell}3^{j-2\ell}2^{\ell}\right)^{k_j},$$

respectively, where C_n, M_n, and s_n are Catalan numbers, Motzkin numbers, and small Schröder numbers.

Remark 2.4 The number $G_{a,b;n}$ is called generalized Motzkin number by Z.-W. Sun [24]. In [24] and [26], Sun and Wang et al. studied the congruence properties and combinatorial properties of $G_{a,b,n}$, respectively.

Proof. Consider functions $f(x) = e^x$ and

$$\phi(t) = \ln G_{a,b}(t),$$

where $G_{a,b}(t)$ is presented as (12). Then we have

$$b_n = \begin{bmatrix} f \circ \phi \\ n \end{bmatrix} = G_{a,b;n} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k a^{n-2k} b^k.$$

To find

$$a_n = \begin{bmatrix} \phi \\ n \end{bmatrix},$$

we compute

$$D_x \ln G_{a,b}(t) = \frac{1}{t}\left(\frac{1}{\sqrt{1-2at+(a^2-4b)t^2}} - 1\right).$$

By denoting

$$e_n = [t^n]\frac{1}{\sqrt{1-2at+(a^2-4b)t^2}},$$

we have

$$\ln G_{a,b}(t) = \int \frac{1}{t}\left(\frac{1}{\sqrt{1-2at+(a^2-4b)t^2}} - 1\right) dt$$

$$= \int \sum_{n\geq 1} e_n t^{n-1} dt = \sum_{n\geq 1} \frac{e_n}{n} t^n,$$

which implies

$$a_n = \frac{e_n}{n}.$$

From [1], one may know that

$$e_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} a^{n-2k} b^k.$$

Hence, we finally obtain

$$a_n = \frac{1}{n}\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} a^{n-2k} b^k.$$

(15) and (16) and other special cases follow from (4) and (5) by substituting the above a_n and b_n properly. □

Remark 2.5 Theorem 5 gives a relationship between Theorems 1 and 2 of [1], i.e., the relationship between the coefficients of the series expansions of functions

$$\frac{1}{\sqrt{1-2at+(a^2-4b)t^2}} \quad \text{and} \quad \frac{1-at-\sqrt{1-2at+(a^2-4b)t^2}}{2bt^2}.$$

The above two functions are studied in [13] by Sprugnoli and one of the authors as two generating functions of a hitting-time Riordan array. The corresponding combinatorial explanation is also given in the paper.

2.2. Identities for Stirling numbers

Theorem 6. *Denote by $S(n,r)$ the Stirling numbers of the second type, where $n, r \in \mathbb{N}$. Then we have*

$$1^n + 2^n + \cdots + r^n = n \sum_{\sigma(n)} (-1)^{k-1}(k-1)! \Pi_{j=1}^{n} \frac{S(r+j,r)^{k_j}}{k_j!} \tag{17}$$

and

$$S(n+r,r) = \sum_{\sigma(n)} \Pi_{j=1}^{n} \frac{\left(1^j + 2^j + \cdots + r^j\right)^{k_j}}{j^{k_j} k_j!}, \tag{18}$$

or equivalently

$$S(n,r) = \sum_{\sigma(n-r)} \Pi_{j=1}^{n-r} \frac{\left(1^j + 2^j + \cdots + r^j\right)^{k_j}}{j^{k_j} k_j!}. \tag{19}$$

Proof. Let $f(x) = e^x$ and

$$\phi(t) = \ln \frac{1}{(1-t)(1-2t)\cdots(1-rt)}.$$

Then $\phi(0) = 0$ and

$$(f \circ \phi)(t) = \frac{1}{(1-t)(1-2t)\cdots(1-rt)} = \sum_{n \geq r} S(n,r) t^{n-r},$$

i.e.,

$$b_n := \begin{bmatrix} f \circ \phi \\ n \end{bmatrix} = S(n+r,r).$$

Since

$$\begin{aligned}\phi(t) &= -\sum_{j=1}^{r} \ln(1-jt) \\ &= \sum_{j=1}^{r}\sum_{n\geq 1}\frac{j^n}{n}t^n = \sum_{n\geq 1}\left(\sum_{j=1}^{r}\frac{j^n}{n}\right)t^n,\end{aligned}$$

we have

$$a_n := \begin{bmatrix}\phi \\ n\end{bmatrix} = \sum_{j=1}^{r}\frac{j^n}{n} = \frac{1^n+2^n+\cdots+r^n}{n}.$$

Therefore from (4) and (5) we obtain (18) and (17), respectively. □

Remark 2.6 (17) is an extension of the following well-known identity (see P221 of [4])

$$1^n+2^n+\cdots+r^n = \sum_{k=1}^{n} k!S(n,k)\binom{r+1}{k+1}.$$

Theorem 7. *Let $n, r \in \mathbb{N}$. Then there hold*

$$\binom{n}{r} = \sum_{\sigma(r)} \frac{(-1)^{r-k}n^k}{1^{k_1}k_1!2^{k_2}k_2!\cdots r^{k_r}k_r!} \tag{20}$$

and

$$(-1)^{r-1}n = r\sum_{\sigma(r)}(-1)^{k-1}(k-1)!\frac{\binom{n}{1}^{k_1}\binom{n}{2}^{k_2}\cdots\binom{n}{r}^{k_r}}{k_1!k_2!\cdots k_r!}. \tag{21}$$

Proof. Let $f(x) = e^x$ and $\phi(t) = \ln(1+t)^n = n\ln(1+t)$. Note that $\phi(0) = 0$ and

$$(f\circ\phi)(t) = (1+t)^n = \sum_{r=0}^{n}\binom{n}{r}t^r.$$

Thus,

$$b_r := \begin{bmatrix}f\circ\phi \\ r\end{bmatrix} = \binom{n}{r}.$$

Since

$$\phi(t) = n\ln(1+t) = \sum_{r\geq 1}(-1)^{r-1}\frac{n}{r}t^r,$$

we have

$$a_r := \begin{bmatrix}\phi \\ r\end{bmatrix} = (-1)^{r-1}\frac{n}{r}.$$

By using (4) and (5), we obtain (20) and (21), respectively. □

Theorem 8. *Let* $\phi(t)=\sum_{n\geq 1} a_n t^n$, *and let* $b_n=[t^n]\,\phi(t)^r$. *Then there holds*

$$b_n = r! \sum_{\sigma(n,r)} \frac{a_1^{k_1}a_2^{k_2}\cdots a_n^{k_n}}{k_1!k_2!\cdots k_n!}, \tag{22}$$

where $k_1+k_2+\cdots+k_n=r$. *Particularly, if* $\phi(t)=\ln(1+t)$ *and* e^t-1, *then* b_n *are* $s(n,r)$ *and* $S(n,r)$, *the Stirling numbers of the first kind and the second kind, respectively. Thus* (22) *implies*

$$s(n,r) = n! \sum_{\sigma(n,r)} \frac{(-1)^{n-r}}{1^{k_1}k_1!2^{k_2}k_2!\cdots n^{k_n}k_n!} \tag{23}$$

$$S(n,r) = n! \sum_{\sigma(n,r)} \frac{1}{(1!)^{k_1}k_1!(2!)^{k_2}k_2!\cdots (n!)^{k_n}k_n!}, \tag{24}$$

where $k_1+k_2+\cdots+k_n=r$.

Proof. Noting

$$\left(\phi(t)^r\right)^{(k)}(0) = (r)_k\delta_{r,k},$$

where $\delta_{r,k}$ is the Kronecker symbol, we obtain (22). Since

$$(\ln(1+t))^r = \frac{r!}{n!}\sum_{n\geq r} s(n,r)t^n$$

$$\left(e^t-1\right)^r = \frac{r!}{n!}\sum_{n\geq r} S(n,r)t^n,$$

we immediately have (23) and (24) from (22). □

2.3. Identities in q forms

The Gaussian binomial coefficients or q-binomial coefficients are q- analogs of binomial coefficients defined by (see, for example, [8])

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)} & k\leq n, \\ 0 & k>n. \end{cases} \tag{25}$$

Theorem 9. *Let* $n,r\in\mathbb{N}$. *Then*

$$q^{\binom{r}{2}}\begin{bmatrix} n \\ r \end{bmatrix}_q = \sum_{\sigma(r)} (-1)^{r-k}\Pi_{j=1}^{r}\frac{\left(\frac{1-q^{nj}}{1-q^j}\right)^{k_j}}{j^{k_j}k_j!} \tag{26}$$

and

$$\frac{q^{nr}-1}{q^r-1} = r\sum_{\sigma(r)}(-1)^{r-k}(k-1)!q^{\sum_{i=0}^{r}k_i\binom{i}{2}}\Pi_{j=0}^{r}\left(\frac{\begin{bmatrix}n\\j\end{bmatrix}_q^{k_j}}{k_j!}\right). \tag{27}$$

Proof. It is well known that (see [4], p. 118)

$$\Pi_{j=0}^{n-1}(1+q^{jt}) = \sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q q^{\binom{j}{2}}t^j.$$

Let $f(x)=e^x$ and

$$\begin{aligned}\phi(t) = \ln\left(\Pi_{j=0}^{n-1}(1+q^jt)\right) &= \sum_{j=0}^{n-1}\ln\left(1+q^jt\right)\\ = \quad \sum_{m\geq 1}&\left(\sum_{j=0}^{n-1}\frac{(-1)^{m-1}q^{jm}}{m}t^m\right).\end{aligned}$$

Thus $\phi(0)=0$ and

$$a_i = \begin{bmatrix}\phi\\i\end{bmatrix} = (-1)^{i-1}\sum_{j=0}^{n-1}\frac{q^{ji}}{i} = \frac{(-1)^{i-1}}{i}\frac{1-q^{ni}}{1-q^i}.$$

Since

$$(f\circ\phi)(t) = \Pi_{j=0}^{n-1}(1+q^{jt}) = \sum_{j=0}^{n}q^{\binom{j}{2}}\begin{bmatrix}n\\j\end{bmatrix}_q t^j,$$

we have

$$b_i = \begin{bmatrix}f\circ\phi\\i\end{bmatrix} = q^{\binom{i}{2}}\begin{bmatrix}n\\i\end{bmatrix}_q$$

for $i\leq n$. From formulas (4) and (5),

$$\begin{aligned}q^{\binom{r}{2}}\begin{bmatrix}n\\r\end{bmatrix}_q &= \begin{bmatrix}f\circ\phi\\r\end{bmatrix}\\ &= \sum_{\sigma(r)}(-1)^{\sum_{j=1}^{r}(j-1)k_j}\Pi_{j=1}^{r}\frac{\left(\frac{1-q^{nj}}{1-q^j}\right)^{k_j}}{j^{k_j}k_j!}\\ &= \sum_{\sigma(r)}(-1)^{r-k}\Pi_{j=1}^{r}\frac{\left(\frac{1-q^{nj}}{1-q^j}\right)^{k_j}}{j^{k_j}k_j!}\end{aligned}$$

and

$$\frac{q^{nr}-1}{q^r-1} = \begin{bmatrix}\phi \\ r\end{bmatrix} = r\sum_{\sigma(r)}(-1)^{r-k}(k-1)!q^{\sum_{i=0}^{r} k_i^{\binom{i}{2}}}\Pi_{j=0}^{r}\left(\frac{\begin{bmatrix}n \\ j\end{bmatrix}_q^{k_j}}{k_j!}\right).$$

□

2.4. Identities for the recursive sequences

Let $\{a_n\}_{n\geq 0}$ be the recursive number sequence of order 2 satisfying

$$a_n = ba_{n-1} + ca_{n-2} \tag{28}$$

for $n \geq 2$ and with initials a_0 and a_1. It is known that

$$\sum_{n\geq 0} a_n x^n = \frac{(a_1 - a_0 b)x + a_0}{1 - bx - cx^2}. \tag{29}$$

We now give identities of a_n inspired by (12) of [14].

Theorem 10. *Let a_n be the recursive number sequence defined by* (28)*. Assume its characteristic polynomial $1 - bx - cx^2$ has two distinct roots r_1 and r_2 and $a_0 \neq 0$. Then there holds*

$$\frac{1}{n}\left(r_1^n + r_2^n - \left(b - \frac{a_1}{a_0}\right)^n\right) = \sum_{\sigma(n)}(-1)^{k-1}(k-1)!\Pi_{j=1}^{n}\frac{b_j^{k_j}}{k_j!}, \tag{30}$$

where

$$b_j = \frac{a_0(r_1 - b) + a_1}{r_1 - r_2} r_1^j + \frac{a_0(r_2 - b) + a_1}{r_2 - r_1} r_2^j. \tag{31}$$

Particularly, if $b = c = 1$ and $a_0 = a_1 = 1$, then (30) *implies*

$$\sum_{\sigma(n)}(-1)^{k-1}(k-1)!\Pi_{j=1}^{n}\frac{F_j^{k_j}}{k_j!} = \frac{1}{n}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right), \tag{32}$$

where F_j is the j-th Fibonacci number, which is given by [14].

Proof. Let $f(x) = e^x$ and

$$\phi(t) = \ln\frac{(a_1 - a_0 b)t + a_0}{1 - bt - ct^2}.$$

Using Binet formula or Proposition 2.1 of [12] yields

$$
\begin{aligned}
(f\circ\phi)(t) = \frac{(a_1-a_0b)t+a_0}{1-bt-ct^2} &= \frac{(a_1-a_0b)t+a_0}{(1-r_1t)(1-r_2t)} \\
= \quad ((a_1-a_0b)t+a_0)\sum_{n\geq 0}\frac{r_1^{n+1}-r_2^{n+1}}{r_1-r_2}t^n,
\end{aligned}
$$

which implies

$$
b_n = \begin{bmatrix} f\circ\phi \\ n \end{bmatrix} = \frac{a_0(r_1-b)+a_1}{r_1-r_2}r_1^n + \frac{a_0(r_2-b)+a_1}{r_2-r_1}r_2^n.
$$

On the other hand, we may write $\phi(t)$ as

$$
\begin{aligned}
\phi(t) = \ln a_0 + \ln\left(1-\frac{a_0b-a_1}{a_0}t\right) - \ln(1-r_1t) - \ln(1-r_2t) \\
= \quad \ln(a_0) - \sum_{n\geq 1}\left(b-\frac{a_1}{a_0}\right)^n\frac{t^n}{n} + \sum_{n\geq 1}r_1^n\frac{t^n}{n} + \sum_{n\geq 1}r_2^n\frac{t^n}{n},
\end{aligned}
$$

which gives

$$
a_n = \begin{bmatrix} \phi \\ n \end{bmatrix} = \frac{1}{n}\left(r_1^n + r_2^n - \left(b-\frac{a_1}{a_0}\right)^n\right), \quad n\geq 1.
$$

Thus, from (5) one may obtain (30). Particularly, if $b=c=1$ and $a_0=a_1=1$, then

$$
r_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad r_2 = \frac{1-\sqrt{5}}{2}
$$

and formula (30) implies (32). □

We now extend Theorem 10 to some recursive polynomial sequences, called generalized Gegenbauer-Humbert polynomials. A sequence of the generalized Gegenbauer-Humbert polynomials $\{P_n^{\lambda,y,C}(x)\}_{n\geq 0}$ is defined by the expansion (see, for example, [4], Gould [9], and Hsu and two of the authors [11])

$$
\Phi(t) \equiv (C-2xt+yt^2)^{-\lambda} = \sum_{n\geq 0}P_n^{\lambda,y,C}(x)t^n, \tag{33}
$$

where $\lambda>0$, y and $C\neq 0$ are real numbers. As special cases of (33), we consider $P_n^{1,y,1}(x)$

as follows (see [11])

$$\begin{aligned}
&P_n^{1,1,1}(x) = U_n(x),\ Chebyshev\ polynomial\ of\ the\ second\ kind,\\
&P_n^{1/2,1,1}(x) = \psi_n(x),\ Legendre\ polynomial,\\
&P_n^{1,-1,1}(x) = P_{n+1}(x),\ Pell\ polynomial,\\
&P_n^{1,-1,1}\left(\frac{x}{2}\right) = F_{n+1}(x),\ Fibonacci\ polynomial,\\
&P_n^{1,1,1}\left(\frac{x}{2}+1\right) = B_n(x),\ Morgan-Voyc\ polynomial,\\
&P_n^{1,2,1}\left(\frac{x}{2}\right) = \Phi_{n+1}(x),\ Fermat\ polynomial\ of\ the\ first\ kind,
\end{aligned}$$

where a is a real parameter, and $F_n = F_n(1)$ is the Fibonacci number. In particular, if $y = C = 1$, the corresponding polynomials are called Gegenbauer polynomials (see [4]). The generalized Gegenbauer-Humbert polynomial sequences satisfy

$$P_n^{\lambda,y,C}(x) = 2x\frac{\lambda+n-1}{Cn}P_{n-1}^{\lambda,y,C}(x) - y\frac{2\lambda+n-2}{Cn}P_{n-2}^{\lambda,y,C}(x) \tag{34}$$

for all $n \geq 2$ with initial conditions

$$\begin{aligned}
P_0^{\lambda,y,C}(x) &= \Phi(0) = C^{-\lambda},\\
P_1^{\lambda,y,C}(x) &= \Phi'(0) = 2\lambda x C^{-\lambda-1}.
\end{aligned}$$

[12] establishes the following theorem.

Theorem 11. *([12]) Let $x \neq \pm\sqrt{Cy}$. The generalized Gegenbauer-Humbert polynomials $\{P_n^{1,y,C}(x)\}_{n\geq 0}$ defined by expansion (33) can be expressed as*

$$P_n^{1,y,C}(x) = C^{-n-2}\frac{\left(x+\sqrt{x^2-Cy}\right)^{n+1} - \left(x-\sqrt{x^2-Cy}\right)^{n+1}}{2\sqrt{x^2-Cy}}. \tag{35}$$

Thus we have

$$P_n^{1,y,1}(x) = 2xP_{n-1}^{1,y,1}(x) - yP_{n-2}^{1,y,1}(x) \tag{36}$$

for all $n \geq 2$ that satisfies initial conditions

$$\begin{aligned}
P_0^{1,y,1}(x) &= \Phi(0) = 1,\\
P_1^{1,y,1}(x) &= \Phi'(0) = 2x.
\end{aligned}$$

In addition,

$$P_n^{1,y,1}(x) = \frac{\left(x+\sqrt{x^2-y}\right)^{n+1} - \left(x-\sqrt{x^2-y}\right)^{n+1}}{2\sqrt{x^2-y}}. \tag{37}$$

Theorem 12. *Let* $P_n^{1,y,1}(t)$ *be defined by* (36), $n \in \mathbb{N}$, *and let its characteristic polynomial* $1 - 2xt + yt^2$ *have two district roots*

$$r_1(x) = x + \sqrt{x^2 - y} \quad \textit{and} \quad r_2(x) = x - \sqrt{x^2 - y}.$$

Then there holds

$$\frac{r_1^n(x) + r_2^n(x)}{n} = \sum_{\sigma(n)} (-1)^{k-1}(k-1)!\Pi_{j=1}^n \frac{b_j(x)^{k_j}}{k_j!}, \tag{38}$$

where

$$b_j(x) = \frac{1}{2\sqrt{x^2 - y}} r_1(x)^{j+1} - \frac{1}{2\sqrt{x^2 - y}} r_2(x)^{j+1}. \tag{39}$$

Proof. Let $f(x) = e^x$ and

$$\phi(t) = \ln \frac{1}{1 - 2xt + yt^2} = \ln \frac{1}{(1 - r_1(x)t)(1 - r_2(x)t)}.$$

Then $(f \circ \phi)(t) = 1/(1 - 2xt + yt^2)$, which implies

$$b_j(x) = \begin{bmatrix} f \circ \phi \\ j \end{bmatrix} = \frac{r_1^{j+1}(x)}{r_1(x) - r_2(x)} + \frac{r_2^{j+1}(x)}{r_2(x) - r_1(x)}.$$

Since

$$a_n(x) := \begin{bmatrix} \phi \\ n \end{bmatrix} = [t^n] \left(-\ln(1 - r_1(x)t) - \ln(1 - r_2(x)(t))\right)$$
$$= [t^n] \left(\sum_{j \geq 1} \frac{r_1(x)^j t^j}{j} + \sum_{j \geq 1} \frac{r_2(x)^j t^j}{j} \right)$$
$$= \frac{r_1^n(x) + r_2^n(x)}{n},$$

from (5) we obtain (38). □

2.5. Miscellaneous Application

The techniques presented in the previous subsection can also be applied to some well known number sequences, such as the number sequences shown in [1],

$$H_{a,b}(t) = \frac{1}{\sqrt{1 - 2at + (a^2 - 4b)t^2}}, \tag{40}$$

which was used in the proof of Theorem 5, where we cited Theorem 1 of [1] to give

$$e_n := [t^n]H_{a,b}(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} a^{n-2k} b^k, \tag{41}$$

where the numbers e_n are called the generalized central trinomial coefficients by Sun in [24]. Hence, we may establish

Theorem 13. *Let $H_{a,b}(t)$ be the function defined by* (40) *with coefficients of its ordinary series expansion shown as* (41). *Then we have*

$$\frac{r_1^n + r_2^n}{2n} = \sum_{\sigma(n)} \frac{e_1^{k_1} e_2^{k_2} \cdots e_n^{k_n}}{k_1! k_2! \cdots k_n!} \tag{42}$$

and

$$e_n = \sum_{\sigma(n)} (-1)^{k-1}(k-1)! \frac{(r_1+r_2)^{k_1}(r_1^2+r_2^2)^{k_2} \cdots (r_1^n+r_2^n)^{k_n}}{2^k 1^{k_1} k_1! 2^{k_2} k_2! \cdots n^{k_n} k_n!}, \tag{43}$$

where e_n are given in (41)*,*

$$r_1 = \frac{a+2\sqrt{b}}{a^2-4b} \quad \textit{and} \quad r_2 = \frac{a-2\sqrt{b}}{a^2-4b} \tag{44}$$

with $a^2 - 4b \neq 0$.

Proof. Let $f(x) = e^x$ and

$$\phi(t) = \ln \frac{1}{\sqrt{1-2at+(a^2-4b)t^2}}.$$

Then

$$b_n = \begin{bmatrix} f \circ \phi \\ n \end{bmatrix} = e_n,$$

which are shown in (41), and

$$\begin{aligned} a_n = \begin{bmatrix} \phi \\ n \end{bmatrix} &= [t^n] \left(-\frac{1}{2}\right) (\ln(1-r_1 t) + \ln(1-r_2 t)) \\ &= \frac{1}{2} \left(\frac{r_1^n + r_2^n}{n}\right). \end{aligned}$$

Thus, from (4) and (5), we obtain (43) and (44), respectively. □

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