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Tian-Xiao He
Louis W. Shapiro

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Tian-Xiao He[1] and Louis Shapiro[2]

[1]Department of Mathematics
Illinois Wesleyan University
Bloomington, IL 61702-2900, USA

[2] Mathematical Department
Howard University
Washington, DC, 20059, USA

Abstract

Here we use row sum generating functions and alternating sum generating functions to characterize Riordan arrays and subgroups of the Riordan group. Numerous applications and examples are presented which include the construction of Girard-Waring type identities. We also show the extensions to weighted sum (generating) functions, called the expected value (generating) functions of Riordan arrays.

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1 Introduction

Riordan arrays are infinite, lower triangular matrices defined by two generating function. They form a group, called the Riordan group (see Shapiro et al. [34]). Some other main results on the Riordan group and its application to combinatorial sums and identities can be found in [2]–[4], [6]–[7], [11]–[15], [17]–[18], [21]–[33], and [36]–[41].

More formally, consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[\![t]\!]$; the order of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ ($f_k \in \mathbb{R}$), is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$. The set of formal power series of order r is denoted by $\mathcal{F}_r$. It is known that $\mathcal{F}_0$ is the set of invertible f.p.s. and $\mathcal{F}_1$ is the set of compositionally invertible f.p.s., that is, the f.p.s. $f(t)$ for which the compositional inverse $\bar{f}(t)$ exists such that $f(\bar{f}(t)) = \bar{f}(f(t)) = t$. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t), h(t))$ defines the (proper) Riordan array $D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$, where

$$d_{n,k} = [t^n]d(t)h(t)^k \tag{1}$$

or, in other words, $d(t)h(t)^k$ is the generating function for the entries of column k.

Let $[f_0, f_1, f_2, \ldots]^T$ be a column vector with $f(t) = \sum_{n\geq 0} f_n t^n$. We then have the fundamental theorem of Riordan arrays

$$(d(t),\, h(t))f(t) = d(t)f(h(t)). \tag{2}$$

It follows quickly that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d_1(t),\, h_1(t)) * (d_2(t),\, h_2(t)) = (d_1(t)d_2(h_1(t)),\, h_2(h_1(t))). \tag{3}$$

The Riordan array $I = (1,\, t)$ is everywhere 0 except for all 1's on the main diagonal; it can be easily proved that I acts as an identity for this product, that is, $(1,\, t) * (d(t),\, h(t)) = (d(t),\, h(t)) * (1, t) = (d(t),\, h(t))$. Let $(d(t), h(t))$ be a Riordan array. Then its inverse is

$$(d(t),\, h(t))^{-1} = \left(\frac{1}{d(\bar{h}(t))},\, \bar{h}(t),\right) \tag{4}$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$, i.e., $(h \circ \bar{h})(t) = (\bar{h} \circ h)(t) = t$. In this way, the set $\mathcal{R}$ of proper Riordan arrays forms a group (see [34]).

Multiplying a matrix by the column vector $[1, 1, 1, \ldots]^T$ yields the column of row sums which we denote as R^+. Since $f(t) = 1/(1-t)$ is the corresponding generating function, then (2) presents the generating function R^+ of the row sum sequence of the Riordan array $(d(t),\, h(t))$, i.e.,

$$R^+(t) \quad := \quad (d(t), h(t))\frac{1}{1-t} = \frac{d(t)}{1-h(t)}. \tag{5}$$

More briefly this can be written as

$$R^+ \quad = \quad \frac{d}{1-h} \tag{6}$$

Similarly, the alternating sum sequence, R^-, called the alternating sum (generating) function, is

$$R^-(t) := (d(t), h(t))\frac{1}{1+t} = \frac{d(t)}{1+h(t)}. \text{ i.e. } R^- = \frac{d}{1+h}. \tag{7}$$

Thus R^- can be considered as the row sum of the matrix $(d, h)(1, -t) = (d, -h)$. Two well known properties of Pascal's triangle are that the row sums are 2^n while the alternating row sums are 0 for $n > 0$.

It is easy to see that we have, reminiscent of Pythagorean triples,

$$d = \frac{2R^+R^-}{R^+ + R^-}, \quad h = \frac{R^+ - R^-}{R^+ + R^-}. \tag{8}$$

Thus we can use the sum function R^+ and the alternating sum function R^- to characterize Riordan arrays.

Here is a list of several subgroups of $\mathcal{R}$ together with their R^+ and R^- functions

- The *Appell subgroup* , $A = \{(d(t),\, t)\}$. The subgroup A is a normal subgroup in the Riordan group.

$$R^+ = \frac{1+t}{1-t}R^-,\ R^- = \frac{1-t}{1+t}R^+ \text{ and } d = R^+ - tR^+ = R^- + tR^-. \tag{9}$$

- The *Associated subgroup (or Lagrange subgroup)*, $L = \{(1,\, h(t))\}$.

$$R^+ = \frac{R^-}{2R^- - 1},\ R^- = \frac{R^+}{2R^+ - 1} \text{ and } h\,(t) = \frac{R^+ - 1}{R^+} = \frac{1 - R^-}{R^-} \tag{10}$$

- The *Bell subgroup* , $B = \{(d(t),\, td(t))\}$.

$$R^+ = \frac{R^-}{1 - 2tR^-} \quad \textit{and} \quad R^- = \frac{R^+}{1 + 2tR^+} \text{ and } d = \frac{R^+}{1 + tR^+} = \frac{R^-}{1 - tR^-} \tag{11}$$

- The *checkerboard subgroup*, $\mathcal{C} = \{(d(t),\, h(t) \mid d\,(t)$ an even function, $h\,(t)$ an odd function$\}$.

$$R^+(t) = R^-(-t). \tag{12}$$

- The *stochastic subgroup*, $\mathcal{S} = \{(d(t),\ h(t))\}$ where their row sums are one, i.e.,

$$R^+ = 1/\,(1-t)\,, \quad R^- = \frac{d}{2 - (1-t)d} = \frac{1-h}{(1-t)(1+h)}. \tag{13}$$

- The *hitting-time subgroup* , $\mathcal{H} = \{(th'(t)/h\,(t)\,,\, h(t))\}$.

$$R^+ = \frac{th'}{h(1-h)} \quad \textit{and} \quad R^- = \frac{th'}{h(1+h)}, \tag{14}$$

  which implies

$$t\frac{D_t R^+}{R^+} - R^+ = t\frac{D_t R^-}{R^-} - R^-.$$

- The *derivative subgroup*, $\mathcal{D} = \{(h'(t),\, h(t))\}$

$$R^+ = \frac{h'}{1-h} = -D_t \ln|1 - h(t)| \quad \textit{and} \quad R^- = \frac{h'}{1+h} = D_t \ln|1 + h(t)|. \tag{15}$$

  This implies that

$$2 = e^{\int R^+(t)dt} + e^{\int R^-(t)dt}.$$

We summarize those characterizations of subgroups of Riordan group as follows:

Proposition 1.1 *Let $R^+(t)$ and $R^-(t)$ be the sum and alternating sum function of a Riordan array $(d(t), h(t))$. If $(d(t), h(t))$ is an element of the subgroups of Appell, associated, Bell, checkboard, stochastic, hitting-time, and derivative, then the characterizations of $R^+(t)$ and $R^-(t)$ of $(d(t), h(t))$ are shown in* (9), (10), (11), (12), (13), (14), *and* (15), *respectively.*

As a heuristic principle two pieces of information will determine an element in the Riordan group. As examples we have: $d(t)$ and $h(t)$, R^+ and R^-, $A(t)$ and $Z(t)$ (see equations (17) and (19)). Next are some examples where the first piece is R^+ and the second is a subgroup. For all these examples $R^+ = C(t) = C = 1 + tC^2 = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} t^n = 1 + t + 2t^2 + 5t^3 + 14t^4 + \cdots$, the Catalan numbers. Hence, we obtain various decompositions of Catalan numbers in terms of k by using different type Riordan arrays. Those Riordan array decompositions can be applied to other number sequences similarly, which provide a way to establish identities of Catalan numbers and other numbers.

$$\text{Appell, } \left(1 + t^2 C^3, t\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & \\ 3 & 1 & 0 & 1 & 0 & 0 & \cdots \\ 9 & 3 & 1 & 0 & 1 & 0 & \\ 28 & 9 & 3 & 1 & 0 & 1 & \\ & & & \cdots & & & \end{bmatrix}$$

$$\text{Lagrange, } (1, tC) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 2 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 5 & 5 & 3 & 1 & 0 & \\ 0 & 14 & 14 & 9 & 4 & 1 & \\ & & & \cdots & & & \end{bmatrix} \qquad (16)$$

$$\text{Bell, } (F, tF) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & \\ 2 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 6 & 4 & 3 & 0 & 1 & 0 & \\ 18 & 13 & 6 & 4 & 0 & 1 & \\ & & & \cdots & & & \end{bmatrix} \text{ where } F = \frac{C}{1+tC}, \text{ the GF for the Fine numbers.}$$

$$\text{Checkerboard, } \left(\frac{2C(t)C(-t)}{C(t)+C(-t)}, \frac{C(t)-C(-t)}{C(t)+C(-t)}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 4 & 0 & 1 & 0 & 0 & \cdots \\ 6 & 0 & 7 & 0 & 1 & 0 & \\ 0 & 31 & 0 & 10 & 0 & 1 & \\ & & & \cdots & & & \end{bmatrix}$$

$$\text{Derivative, } \left(\frac{C}{1+(C-1)e^{1-C}}, \frac{(C-1)e^{1-C}}{1+(C-1)e^{1-C}}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \\ \frac{5}{2} & \frac{3}{2} & 0 & 1 & 0 & 0 & 0 & \cdots \\ \frac{23}{3} & \frac{10}{3} & 2 & 0 & 1 & 0 & 0 & \\ \frac{193}{8} & \frac{245}{24} & \frac{25}{6} & \frac{5}{2} & 0 & 1 & 0 & \\ \frac{1571}{20} & \frac{629}{20} & 13 & 5 & 3 & 0 & 1 & \\ & & & \cdots & & & & \end{bmatrix}$$

The first few rows of the Riordan array $(C^2e^{-2tC}, 1-Ce^{-2tC})$ are

$$\text{Hitting time, } (C^2e^{-2tC}, 1-Ce^{-2tC}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \\ \frac{8}{3} & \frac{4}{3} & 0 & 1 & 0 & 0 & 0 & \cdots \\ 8 & \frac{10}{3} & \frac{5}{3} & 0 & 1 & 0 & 0 & \\ \frac{376}{15} & \frac{149}{15} & 4 & 2 & 0 & 1 & 0 & \\ \frac{731}{9} & \frac{154}{5} & \frac{539}{45} & \frac{14}{3} & \frac{7}{3} & 0 & 1 & \\ & & & \cdots & & & & \end{bmatrix}.$$

Here, when we explain the Catalan numbers in terms of trees, the kth column in the Appell array is the numbers of trees with stem height k, while the k in the Lagrange array decomposition is the degree of the root. The k in the Bell array decomposition is the number of branches of the root with just one edge.

Despite the fractions appearing in these last two examples often we do get nonnegative integer entries. For instance if $h(t) = tC$ then the derivative and hitting time matrices are

$$(B, tC) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ 2 & 1 & 0 & 0 & 0 & \\ 6 & 3 & 1 & 0 & 0 & \cdots \\ 20 & 10 & 4 & 1 & 0 & \\ 70 & 35 & 15 & 5 & 1 & \\ & & \cdots & & & \end{bmatrix} \text{ and } \left(\frac{B}{C}, tC\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & \\ 3 & 2 & 1 & 0 & 0 & \cdots \\ 10 & 6 & 3 & 1 & 0 & \\ 35 & 20 & 10 & 4 & 1 & \\ & & \cdots & & & \end{bmatrix}$$

with row sum generating functions BC and B respectively. These are of combinatorial significance since B is the generating function for ordered trees with a marked vertex while $\frac{B}{C}$ is the generating function for ordered trees with a marked leave. Similar results hold as long as other kinds of ordered trees which have the same possibilities for updegrees at every vertex.

An infinite lower triangular array $[d_{n,k}]_{n,k\in\mathbb{N}} = (d(t), h(t))$ is a Riordan array if and only if a sequence $A = (a_0 \neq 0, a_1, a_2, \ldots)$ exists such that for every $n, k \in \mathbb{N}$

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + \cdots + a_n d_{n,n}. \tag{17}$$

This is equivalent to

$$h(t) = tA(h(t)) \quad \text{or} \quad t = \bar{h}(t)A(t). \tag{18}$$

This important result could be called the second fundamental theorem of the Riordan arrays, see [21, 28, 30] for more information. Here, $A(t)$ is the generating function of the A-sequence. In [21, 25] it is also shown that a unique sequence $Z = (z_0, z_1, z_2, \ldots)$ exists such that every element in column 0 can be expressed as the linear combination

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + \cdots + z_n d_{n,n}, \tag{19}$$

or equivalently,

$$d(t) = \frac{d_{0,0}}{1 - tZ(h(t))}. \tag{20}$$

In the next section, we construct the row sum functions and row alternating sum functions of inverse Riordan arrays and multiplication of Riordan arrays as well as the relationship between the sum functions and alternating sum functions and the sequence characterizations of Riordan arrays. In Section 3, an application of row sums in the construction of Girard-Waring identities will be given. In addition, several extensions of sum functions will be developed, which include weighted sum (generating) functions of Riordan arrays and sum functions of some improper Riordan arrays.

2 Row sum functions and row alternating sum functions of the inverse Riordan array and multiplication of Riordan arrays

Let $(d(t), h(t))$ be a Riordan array. Then its inverse is $(1/d(\bar{h}(t)), \bar{h}(t))$, where $\bar{h}(t)$ is the compositional inverse of $h(t)$, i.e., $(h \circ \bar{h})(t) = (\bar{h} \circ h)(t) = t$. Denote the sum function and the alternating sum function of $(1/d(\bar{h}(t)), \bar{h}(t))$ by $S^+ \equiv S^+(t)$ and $S^- \equiv S^-(t)$, respectively. Then

$$S^+ = \frac{1}{(d \circ \bar{h})(1 - \bar{h})} \quad \textit{and} \quad S^- = \frac{1}{(d \circ \bar{h})(1 + \bar{h})}. \tag{21}$$

Hence,

$$\frac{1}{d \circ \bar{h}} = \frac{2S^+S^-}{S^+ + S^-} \quad \textit{and} \quad \bar{h} = \frac{S^+ - S^-}{S^+ + S^-}. \tag{22}$$

Substituting $t = (\bar{h} \circ h)$ into the second formula of (8) yields

$$\bar{h} = \frac{R^+(\bar{h} \circ h) - R^-(\bar{h} \circ h)}{R^+(\bar{h} \circ h) + R^-(\bar{h} \circ h)}.$$

Comparing the last formula and the second formula of (22), we obtain

$$\frac{R^+(\bar{h}) - R^-(\bar{h})}{R^+(\bar{h}) + R^-(\bar{h})} = \frac{S^+(h) - S^-(h)}{S^+(h) + S^-(h)},$$

which establishes the relationship between the sum functions and alternating sum functions of $(d(t), h(t))$ and its inverse as follows:

$$\frac{R^{+}(\bar{h})}{S^{+}(h)} = \frac{R^{-}(\bar{h})}{S^{-}(h)}. \tag{23}$$

On the other hand, from the first formula of (8) and the first formula of (22), we have

$$\frac{1}{g(\bar{f})} = \frac{R^{+}(\bar{h}) + R^{-}(\bar{h})}{2R^{+}(\bar{h})R^{-}(\bar{h})} = \frac{2S^{+}S^{-}}{S^{+} + S^{-}}.$$

Noting (23), one may change the right-hand side of the above equation to prove:

Proposition 2.1 *Let $R^{+} \equiv R^{+}(t)$, $S^{+} \equiv S^{+}(t)$, $R^{-} \equiv R^{-}(t)$, and $S^{-} \equiv S^{-}(t)$. We then have*

$$\begin{aligned} \frac{2S^{+}S^{-}}{S^{+} + S^{-}} &= \frac{S^{+}(h) + S^{-}(h)}{2S^{+}(h)R^{-}(\bar{h})} \\ &= \frac{S^{+}(h) + S^{-}(h)}{2R^{+}(\bar{h})S^{-}(h)}. \end{aligned} \tag{24}$$

Note that this implies equation (23)*.*

As an example, if $(d, h) = (1, tC)$, where $C \equiv C(t)$ is the generating function of Catalan numbers, then $\bar{h}(t) = t(1-t)$, and $C(\bar{h}(t)) = 1/(1-t)$. Hence, $(1, tC)^{-1}$, the inverse of $(1, tC)$, is $(1, t(1-t))$. The sum functions and alternating sum functions of $(1, tC)$ and its inverse satisfy

$$\begin{aligned} R^{+}(\bar{h}) &= \frac{1}{1 - \bar{h}C(\bar{h})} = \frac{1}{1-t}, \quad S^{+}(h) = \frac{1}{1 - h(1-h)} = \frac{1}{1-t}, \\ R^{-}(\bar{h}) &= \frac{1}{1 + \bar{h}C(\bar{h})} = \frac{1}{1+t}, \quad S^{-}(h) = \frac{1}{1 + h(1-h)} = \frac{1}{1+t}. \end{aligned}$$

Therefore, (23) and (24) hold.

Denote by $R_1^{+} \equiv R_1^{+}(t)$, $R_2^{+} \equiv R_2^{+}(t)$, and $R_3^{+} \equiv R_3^{+}(t)$ the sum functions of the Riordan arrays $(d, h) \equiv (d(t), h(t))$, $(g, f) \equiv (g(t), f(t))$, and their product $(d(g \circ h), f \circ h)$, respectively, and by $R_1^{-} \equiv R_1^{-}(t)$, $R_2^{-} \equiv R_2^{-}(t)$, and $R_3^{-} \equiv R_3^{-}(t)$ the corresponding alternating sum functions. Then, from the second formula of (8) we have

$$f = \frac{R_2^{+} - R_2^{-}}{R_2^{+} + R_2^{-}} \quad \textit{and} \quad f \circ h = \frac{R_3^{+} - R_3^{-}}{R_3^{+} + R_3^{-}}. \tag{25}$$

Substituting $t = h(t)$ into the first formula above and comparing it with the above second formula yields

$$\frac{R_2^{+}(h) - R_2^{-}(h)}{R_2^{+}(h) + R_2^{-}(h)} = \frac{R_3^{+} - R_3^{-}}{R_3^{+} + R_3^{-}},$$

which generates

$$\frac{R_2^+(h)}{R_3^+} = \frac{R_2^-(h)}{R_3^-}. \tag{26}$$

On the other hand, from the first formula of (8), we have

$$d = \frac{2R_1^+R_1^-}{R_1^+ + R_1^-}, \quad g = \frac{2R_2^+R_2^-}{R_2^+ + R_2^-}, \tag{27}$$

and

$$d(g \circ h) = \frac{2R_3^+R_3^-}{R_3^+ + R_3^-}. \tag{28}$$

Hence, using formulas (27) and (28), we obtain

$$\frac{2R_1^+R_1^-}{R_1^+ + R_1^-}\frac{2R_2^+(h)R_2^-(h)}{R_2^+(h) + R_2^-(h)} = \frac{2R_3^+R_3^-}{R_3^+ + R_3^-}.$$

To simplify the last equation, we may write it as

$$\frac{2R_1^+R_1^-}{R_1^+ + R_1^-}\frac{2R_2^+(h)R_2^-(h)}{2R_3^+R_3^-} = \frac{R_2^+(h) + R_2^-(h)}{R_3^+ + R_3^-}.$$

Since the right-hand of the last equation can be simplified as $R_2^+(h)/R_3^+$, we obtain the following results.

Proposition 2.2 *With the notations shown above we have*

$$\frac{2R_1^+R_1^-R_2^+(h)}{R_3^+(R_1^+ + R_1^-)} = \frac{2R_1^+R_1^-R_2^-(h)}{R_3^-(R_1^+ + R_1^-)} = 1. \tag{29}$$

Note that this implies equation (26).

As an example, let $(d, h) = (C, tC)$ and $(g, f) = (1, tC)$. Then $(d, h)(g, f) = (C, (1 - \sqrt{1 - 4tC})/2)$. After some computation we have

$$\begin{aligned}
R_2^+(h) &= \frac{1}{1 - tC(t)C(tC)} = \frac{2}{1 + \sqrt{1 - 4tC(t)}} \\
&= 1 + t + 3t^2 + 11t^3 + 44t^4 + 185t^5 + 804t^6 + \cdots, \\
R_2^-(h) &= \frac{1}{1 + tC(t)C(tC)} = \frac{2}{3 - \sqrt{1 - 4tC(t)}} \\
&= 1 - t - t^2 - 3t^3 - 10t^4 - 37t^5 - 146t^6 + \cdots, \\
R_3^+ &= \frac{C}{1 - tC(t)C(tC)} = \frac{2C}{1 + \sqrt{1 - 4tC(t)}} \\
&= 1 + 2t + 6t^2 + 21t^3 + 80t^4 + 322t^5 + 1348t^6 + \cdots, \\
R_3^- &= \frac{C}{1 + tC(t)C(tC)} = \frac{2C}{3 - \sqrt{1 - 4tC(t)}} \\
&= 1 - t^3 - 6t^4 - 30t^5 - 142t^6 - \cdots,
\end{aligned}$$

where $R_2^+(h)$ and R_3^+ are the generating functions of the sequences $A127632$ and $A121988$ in [35], respectively. In this case (26) holds because both sides are equal to $1/C$. Since

$$R_1^+ = \frac{C}{1-tC} = C^2, \quad R_1^- = \frac{C}{1+tC} = F,$$

the generating function of Fine numbers, we may easily find that

$$\frac{2R_1^+R_1^-R_2^+(h)}{R_3^+(R_1^+ + R_1^-)} = \frac{2R_1^+R_1^-}{R_1^+ + R_1^-}\frac{R_2^+(h)}{R_3^+} = \frac{C}{C} = 1.$$

Similarly, we have

$$\frac{2R_1^+R_1^-R_2^-(h)}{R_3^-(R_1^+ + R_1^-)} = 1$$

due to (26).

Finally, we discuss the relationship between the sum functions and the sequence characterizations of Riordan arrays. Suppose $A \equiv A(t)$ and $Z \equiv Z(t)$ are the generating functions of the A-sequence and Z-sequence of a Riordan array (d,h). From (18), we have the $A = t/\bar{h}$, where $\bar{h}$ is the compositional inverse of h. Hence, there is a relation between A and the sum functions of $(d,h)^{-1}$. Thus

$$A = \frac{t(S^+ + S^-)}{S^+ - S^-}. \tag{30}$$

From (20), we may express Z as

$$Z = \frac{(d \circ \bar{h}) - d_{0,0}}{\bar{h}(d \circ \bar{h})}.$$

However, via (21) we have

$$d \circ \bar{h} = \frac{S^+ - S^-}{2S^+S^-} = \bar{h}(d \circ \bar{h}),$$

respectively. Hence,

$$Z = \frac{S^+ + S^- - 2d_{0,0}S^+S^-}{S^+ - S^-}. \tag{31}$$

As an example recall that the inverse of Pascal array is

$$\left(\frac{1}{1-t}, \frac{t}{1-t}\right)^{-1} = \left(\frac{1}{1+t}, \frac{t}{1+t}\right),$$

which has sum function and alternating sum function

$$S^+ = 1 \quad \text{and} \quad S^- = \frac{1}{1+2t},$$

respectively. Then (30) and (31) give us

$$A(t) = 1 + t \quad \text{and} \quad Z(t) = 1$$

for the Pascal array.

Conversely, the generating functions of A-sequence and Z-sequence of a Riordan array can be used to derive the sum function and alternating sum function of the inverse of the Riordan array directly, i.e., without computing the inverse array:

$$S^{+}=\frac{A-tZ}{A-t}\quad\text{and}\quad S^{-}=\frac{A-tZ}{A+t}. \tag{32}$$

Equation (32) can be easily proved by observing

$$A=\frac{t}{\bar{\bar{h}}},\quad Z=\frac{1}{\bar{\bar{h}}}\left(1-\frac{1}{d\circ\bar{\bar{h}}}\right)$$

and the definitions of functions S^{+} and S^{-}.

Since the A-sequence and Z-sequence of the Pascal array are $\{1,1,0,\ldots\}$ and $\{1,0,\ldots\}$, respectively, we obtain quickly that the sum function and alternating sum function of the inverse of Pascal array are

$$S^{+}(t)=\frac{(1+t)-t}{(1+t)-t}=1\quad\text{and}\quad S^{-}(t)=\frac{(1+t)-t}{(1+t)+t}=\frac{1}{1+2t}.$$

3 Applications

3.1 Girard-Waring identities

What can be done by generalizing the ring of integers $\mathbb{Z}$ to the ring of polynomials $\mathbb{Z}[x]$ or $\mathbb{Z}[x,y]$ or $\mathbb{Z}[x_1,x_2,\cdots,x_n]$? There is a survey paper [10] of Henry Gould that includes a history of the Girard-Waring identities in the rings of polynomials $\mathbb{Z}[x,y]$ and $\mathbb{Z}[x,y,z]$. As an example consider the venerable identity of Girard-Waring.

$$\sum_{0\le k\le n/2}(-1)^k\binom{n-k}{k}(xy)^k(x+y)^{n-2k}=\frac{x^{n+1}-y^{n+1}}{x-y}, \tag{33}$$

where $xy\neq 0$ and $x\neq y$. This can be put into standard triangular form by reversing and aerating the rows by replacing $\binom{n-k}{k}$ by $\binom{\frac{n+k}{2}}{\frac{n-k}{2}}$ when $n\equiv k\,(mod\ 2)$ and 0 otherwise. We now look at the Riordan matrix

$$\left(\frac{1}{1+xyt^2},\frac{(x+y)\,t}{1+xyt^2}\right)=$$
$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1\,(x+y) & 0 & 0 & 0 & 0\\
-1\,(xy) & 0 & 1\,(x+y)^2 & 0 & 0 & 0\\
0 & -2\,(xy)\,(x+y) & 0 & 1\,(x+y)^3 & 0 & 0\\
(xy)^2 & 0 & -3\,(xy)\,(x+y)^2 & 0 & 1\,(x+y)^4 & 0\\
0 & 3\,(xy)^2\,(x+y) & 0 & -4\,(xy)\,(x+y)^3 & 0 & 1\,(x+y)^5\\
 & & & \cdots & &
\end{bmatrix}$$

Notice that for any Riordan array that we can find row sums by multiplying by the column vector $[1,1,1,\cdots]^T$ with the generating function $1/(1-z)$. Then we have

$$(g,f)\,\frac{1}{1-z}=g\cdot\frac{1}{1-f}=R^+,\ \text{the generating function for the row sums.}$$

For the Girard-Waring matrix the row sum generating function is

$$\begin{aligned}
\left(\frac{1}{1-xyz^2},\frac{(x+y)\,z}{1+xyz^2}\right)\frac{1}{1-z} &= \frac{1}{1-xyz^2}\cdot\frac{1}{1-\frac{(x+y)z}{1+xyz^2}}\\
&= \frac{1}{1+xyz^2-z\,(x+y)}\\
&= \frac{1}{(1-xz)\,(1-yx)}\\
&= \frac{x}{x-y}\cdot\frac{1}{1-xz}-\frac{y}{x-y}\cdot\frac{1}{1-yz}\\
&= \frac{1}{x-y}\sum_{n\geq 0}\left(x^{n+1}-y^{n+1}\right)z^n.
\end{aligned}$$

Equating coefficients of z^n finishes the proof.

We now look at a more general case by considering the Riordan arrays $(1/(1+at^2), bt/(1+at^2))$ with $b^2\geq 4a>0$ thus obtaining a generalized Binet formula. The entries of the Riordan array are

$$\begin{aligned}
d_{n,k} &= [t^n]\frac{b^kt^k}{(1+at^2)^{k+1}}\\
&= [t^n]\sum_{j\geq 0}\binom{j+k}{j}(-at^2)^jb^kt^k\\
&= [t^n]\sum_{j\geq 0}\binom{j+k}{j}(-a)^jb^kt^{2j+k}\\
&= \begin{cases}\binom{\frac{n+k}{2}}{\frac{n-k}{2}}(-1)^{(n-k)/2}a^{(n-k)/2}b^k, & \text{when } n\equiv k\ (mod\ 2),\\ 0, & \text{when } n\not\equiv k\ (mod\ 2).\end{cases}
\end{aligned}\tag{34}$$

For the last step we set $j=(n-k)/2$ when $n\equiv k(mod\ 2)$. Hence, the row sum of $(1/(1+at^2), bt/(1+at^2))=(d_{n,k})_{n,k\geq 0}$ is

$$\begin{aligned}
\sum_{k=0}^{n}d_{n,k} &= \sum_{0\leq k\leq n,n\equiv k\ (mod\ 2)}\binom{\frac{n+k}{2}}{\frac{n-k}{2}}(-1)^{(n-k)/2}a^{(n-k)/2}b^k\\
&= \sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}a^kb^{n-2k}.
\end{aligned}\tag{35}$$

Here the last step results from the transform $(n-k)/2\to k$ when $n\equiv k\ (mod\ 2)$.

The generating function of the row sums of $(1/(1+at^2), bt/(1+at^2))$ can be presented as

$$\left(\frac{1}{1+at^2}, \frac{bt}{1+at^2}\right)\frac{1}{1-t} = \frac{\frac{1}{1+at^2}}{1-\frac{bt}{1+at^2}} = \frac{1}{at^2-bt+1}. \tag{36}$$

Since $b^2 \geq 4a > 0$, we have

$$at^2 - bt + 1 = a(t-t_1)(t-t_2),$$

where

$$t_1 = \frac{b+\sqrt{b^2-4a}}{2a}, \quad t_2 = \frac{b-\sqrt{b^2-4a}}{2a}. \tag{37}$$

Hence, (36) can be written as

$$\begin{aligned}
& \left(\frac{1}{1+at^2}, \frac{bt}{1+at^2}\right)\frac{1}{1-t} = \frac{1}{a(t-t_1)(t-t_2)} \\
= & \frac{1}{a(t_1-t_2)}\left(\frac{1}{t-t_1} - \frac{1}{t-t_2}\right) \\
= & \frac{1}{at_2(t_1-t_2)}\frac{1}{1-\frac{t}{t_2}} - \frac{1}{at_1(t_1-t_2)}\frac{1}{1-\frac{t}{t_1}} \\
= & \frac{1}{at_2(t_1-t_2)}\sum_{n\geq 0}\left(\frac{t}{t_2}\right)^n - \frac{1}{at_1(t_1-t_2)}\sum_{n\geq 0}\left(\frac{t}{t_1}\right)^n \\
= & \sum_{n\geq 0}\frac{1}{a(t_1-t_2)}\left(\frac{1}{t_2^{n+1}} - \frac{1}{t_1^{n+1}}\right)t^n. \qquad (38)
\end{aligned}$$

From the last expression of the row sum generating function and the row sums shown in (35), we obtain the identity

$$\sum_{k=0}^{n/2}(-1)^k\binom{n-k}{k}a^k b^{n-2k} = \frac{1}{a(t_1-t_2)}\left(\frac{1}{t_2^{n+1}} - \frac{1}{t_1^{n+1}}\right), \tag{39}$$

where t_1 and t_2 are given in (37).

We note that if we set $a = 1$ and $b = -2x$ then the row sums are the Chebyshev polynomials of the second kind. We now have $t_1 = \frac{-2x+\sqrt{4x^2-4}}{2} = -x+\sqrt{x^2-1}$ and $t_2 = -x-\sqrt{x^2-1}$ and $t_1^{-1} = t_2, t_2^{-1} = t_1.$ so

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}}.$$

Similarly, if we set $b = -2x$, $a = -1$ and 2, then the corresponding row sums are the Pell polynomials, $\{P_{n+1}\}_{n\geq 0}$, and the Fermat polynomials of the first kind, $\{F_{n+1}\}_{n\geq 0}$, respectively:

$$
\begin{aligned}
P_{n+1}(x) &= \frac{\left(x+\sqrt{x^2+1}\right)^{n+1}-\left(x-\sqrt{x^2+1}\right)^{n+1}}{2\sqrt{x^2+1}},\\
F_{n+1}(x) &= \frac{\left(x+\sqrt{x^2-2}\right)^{n+1}-\left(x-\sqrt{x^2-2}\right)^{n+1}}{2\sqrt{x^2-2}}.
\end{aligned}
$$

The above polynomials can be sorted into the class of the generalized Gegenbauer-Humbert polynomials (see, for example, [9] and [19]). Their expressions constructed by using Binet formula can be found from [20].

We now can extend identity (39) to $\mathbb{Z}[x,y,z]$ or higher cases using a similar argument.

Let the polynomial at^3+ct^2-bt+1 have three distinct roots $t_1=1/x$, $t_2=1/y$, and $t_3=1/z$. Then a three variable Girard-Waring identity is

$$
\begin{aligned}
&\sum_{0\le k\le n/3}\binom{n-2k}{k}(xyz)^k(x+y+z)^{n-3k}\\
=\;& \frac{x^{n+2}}{(x-y)(x-z)}+\frac{y^{n+2}}{(y-x)(y-z)}+\frac{z^{n+2}}{(z-x)(z-y)} \qquad (40)
\end{aligned}
$$

with x,y,z distinct and nonzero and $xy+yz+zx=0$. This can be proved using row sums as follows:

$$
\left(\frac{1}{1+at^3},\frac{bt}{1+at^3}\right)\frac{1}{1-t}=\frac{\frac{1}{1+at^3}}{1-\frac{bt}{1+at^3}}=\frac{1}{1-bt+at^3}
$$

with roots t_1, t_2, and t_3 satisfying $t_1+t_2+t_3=0$. Then let $t_1=1/x$, $t_2=1/y$, and $t_3=1/z$ so that $a=-xyz$ and $b=x+y+z$.

In general, for integer $\ell\ge 2$, by using the technique shown in [16], we have identities

$$
\begin{aligned}
&\sum_{0\le k\le n/\ell}\binom{n-(\ell-1)k}{k}(-1)^{\ell+1}\left(\Pi_{i=1}^{\ell}x_i\right)^k\left(\sum_{i=1}^{\ell}x_i\right)^{n-\ell k}\\
=\;&\sum_{i=1}^{\ell}\frac{x_i^{n+\ell-1}}{\Pi_{j=1,j\ne i}^{\ell}(x_i-x_j)}, \qquad (41)
\end{aligned}
$$

where the x_i $(1,2,\ldots,\ell)$ are distinct and $\Pi_{i=1}^{\ell}x_i\ne 0$ and

$$
\sum_{0\le i_1<i_2<\cdots<i_{\ell-j}\le\ell}x_{i_1}x_{i_2}\cdots x_{i_{\ell-j}}=0
$$

for $j=1,2,\ldots,\ell-2$. Obviously, when $\ell=2$ and 3, we have identities (33) and (40), respectively.

3.2 Expected value (generating) function

To compute R^+ we multiply by the column vector $[1, 1, 1, \ldots]^T$ giving us the total number relative to a parameter k. To find the expected value relative to k we use the column vector $[0, 1, 2, 3, \ldots]^T$ with the corresponding generating function $t/(1-t)^2$. We define $R^E = R^E(t)$ as

$$R^E \quad := \quad (d(t), h(t))\frac{t}{(1-t)^2} = \frac{d(t)h(t)}{(1-h(t))^2} = \frac{dh}{(1-h)^2}. \tag{42}$$

Thus R^E is very interesting on its own. For the Catalan matrix $(1, tC)$ we have $R^E = tC^3$ so the total root degree for ordered trees with n edges is

$$\frac{3}{2n+1}\binom{2n+1}{n-1} = \frac{3n}{n+2}C_n$$

and the average degree is $3n/(n+2) \to 3$ as $n \to \infty$.

It is easy to see that

$$d = \frac{(R^+)^2}{R^+ + R^E} \quad \text{and} \quad h = \frac{R^E}{R^+ + R^E}. \tag{43}$$

Thus we can use the sum function R^+ and the expected value function R^E to characterize a Riordan array.

We also have the following inverse relationship between (R^+, R^-) and (R^+, R^E).

$$R^- = \frac{(R^+)^2}{R^+ + 2R^E} \quad R^E = \frac{(R^+)^2 - R^+R^-}{2R^-}. \tag{44}$$

Therefore, from the characterizations of subgroups of Riordan group characterized by (R^+, R^-), we may obtain the characterizations of the subgroups with respect to (R^+, R^E) by using the relationship (44). Here are a few examples:

- The *Appell subgroup*, a normal subgroup of the Riordan group, defined by $A = \{(d(t),\, t)\}$, has the characterization:

$$R^+ = \frac{1-t}{t}R^E,\ R^E = \frac{t}{1-t}R^+, \ \text{ and } \ d = R^+(1-t) = R^E\frac{(1-t)^2}{t}. \tag{45}$$

- The *Associated subgroup (or Lagrange subgroup)*, defined by $L = \{(1,\, h(t))\}$, has the characterization:

$$(R^+)^2 - R^+ - R^E = 0 \quad \text{and} \quad h\,(t) = \frac{R^+ - 1}{R^+} = \frac{R^E}{(R^+)^2} \tag{46}$$

- The *Bell subgroup*, defined by $B = \{(d(t),\, td(t))\}$, has the characterization:

$$R^E = t(R^+)^2 \quad \text{and} \quad d = \frac{R^+}{1 + tR^+} = \frac{R^E}{t(R^E + R^+)} \tag{47}$$

For an example, by using the Pascal matrix we have

$$R^E = \frac{t}{(1-2t)^2} = \sum_{n\geq 0} n2^{n-1}t^n.$$

The most striking of these results is that (d,h) is a Bell matrix if and only if $R^E = t(R^+)^2$. For instance, consider the Catalan matrix

$$(C, tC) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & \\ 2 & 2 & 1 & 0 & 0 & \cdots \\ 5 & 5 & 3 & 1 & 0 & \\ 14 & 14 & 9 & 4 & 1 & \\ & & \cdots & & & \end{bmatrix}.$$

Then $R^+ = C^2$ so $R^E = tC^4$. The sequence

$$C^4 \longleftrightarrow \left(1, 4, 14, 48, 165, \ldots, \frac{4}{2n+4}\binom{2n+4}{n}, \ldots\right)$$

goes back at least to Cayley [1] and we discuss two settings.

Consider triangulations of a regular $(n+3)$-gon. We mark a vertex, V, and an internal diagonal, d, that has V as an end point. [For instance color d red.] The number of such triangulations has the generating function tC^4. This tC^4 also counts the number of incomplete binary trees with one red edge on the sequence of consecutive left edges leaving the root (see Figure 1)

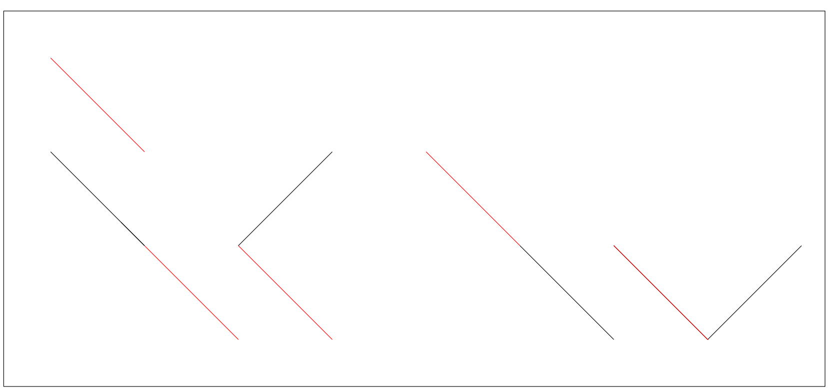

Figure 1: incomplete binary trees with one red edge on the sequence of consecutive left edges leaving the root

Since

$$[t^n]tC^4 = \frac{4}{2(n-1)+4}\binom{2(n-1)+4}{n-1} = \frac{2}{n+1}\binom{2n+2}{n-1}$$

and

$$[t^n]C^2 = \frac{2}{2n+2}\binom{2n+2}{n} = \frac{1}{n+1}\binom{2n+2}{n},$$

we have that the average length of consecutive left edges from the root is

$$\frac{[t^n]tC^4}{[t^n]C^2} = \frac{2n}{n+3} \to 2$$

as $n \to \infty$.

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