2016

Applications of Riordan matrix functions to Bernoulli and Euler polynomials

Tian-Xiao He

Applications of Riordan Matrix Functions to Bernoulli and Euler Polynomials

Tian-Xiao He
Department of Mathematics
Illinois Wesleyan University
Bloomington, IL 61702-2900, USA

February 29, 2016

Abstract

We define Riordan matrix functions associated with Riordan arrays and study their algebraic properties. We also give their applications in the construction of new classes of Bernoulli and Euler polynomials and Bernoulli and Euler numbers, referred to as the duals and conjugate Bernoulli and Euler polynomials and dual and conjugate Bernoulli and Euler numbers, respectively.

AMS Subject Classification: 05A05, 05A15, 15B36, 15A09, 05A30, 05A10, 05A19

Key Words and Phrases: Riordan array, Riordan group, Pascal matrix, Bernoulli polynomials and numbers, Euler polynomials and numbers, conjugate Bernoulli and Euler polynomials.

1 Introduction

Riordan arrays are infinite, lower triangular matrices defined by the generating function of their columns. They form a group, called *the Riordan group* (see Shapiro, Getu, Woan and Woodson [21]).

More formally, let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[\![t]\!]$; the *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ ($f_k \in \mathbb{R}$), is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t), h(t))$ defines the *(proper) Riordan array* $D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ having

$$d_{n,k} = [t^n]d(t)h(t)^k \tag{1}$$

or, in other words, having $d(t)h(t)^k$ as the generating function whose coefficients make-up the entries of column k.

From the *fundamental theorem of Riordan arrays* (see Shapiro [19, 20]), it is immediate to show that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d_1(t),\, h_1(t))(d_2(t),\, h_2(t)) = (d_1(t)d_2(h_1(t)),\, h_2(h_1(t))). \tag{2}$$

The Riordan array $I = (1, t)$ acts as an identity for this product. From the product, the inverse of $(d(t), h(t))$ is

$$(d(t),\, h(t))^{-1} = \left(\frac{1}{d(h^*(t))},\, \bar{h}(t)\right) \tag{3}$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$. In this way, the set $\mathcal{R}$ of proper Riordan arrays is a group.

In spited by Pascal matrix discussed in Aceto and Trigiante [1], Barry [3], Call and Velleman [4], and Edelman and Strang [9], we defined Riordan matrix functions as follows.

Definition 1.1 *Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$, and let the pair of $\{d(t), h(t)\}$ define the* (proper) Riordan array *$D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$, where $d_{n,k} = [t^n]d(t)h(t)^k$. Then the corresponding Riordan matrix function $R(x) \equiv (d(t), h(t))(x)$, $x \in \mathbb{R}$, is defined by*

$$R(x) \equiv (d(t), h(t))(x) := d_{n,k}x^{n-k} = x^{n-k}[t^n]d(t)h(t)^k. \tag{4}$$

From the definition of Riordan matrix functions shown in Definition 1.1, we may have

Theorem 1.2 *The collection of all Riordan matrix functions form a group, denoted by $\{\mathcal{R}(x)\}$ and called Riordan function group. When $x = 1$, $\{\mathcal{R}(1)\} = \{\mathcal{R}\}$, which is the Riordan group.*

As an example, we consider the function $P[x]$ defined in [4]:

$$P[x] = \begin{cases} x^{n-k}\binom{n}{k} & n \geq k \\ 0 & otherwise, \end{cases} \tag{5}$$

where $P[x]$ maps $\mathbb{R}$ into $\mathcal{R}(x)$, i.e., $P : x \mapsto (1/(1-t), t/(1-t))(x) = (\binom{n}{k})(x)$ because of $(1/(1-t), t/(1-t)) = \left(\binom{n}{k}\right)_{n\geq k\geq k\geq 0}$. Homomorphically,

$$P[x+y] = P[x]P[y] \tag{6}$$

for any $x, y \in \mathbb{R}$, which is also shown in Zhang and Liu [25]. In fact, the (i, j) entry of the right-hand side of (6) can be written as

$$\sum_{k=j}^{i} \binom{i}{k} x^{i-k} \binom{k}{j} y^{k-j}, \tag{7}$$

while the (i, j) entry of the left-hand side of (6) is

$$\binom{i}{j}(x+y)^{i-j} = \binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k} y^k x^{i-j-k} = \sum_{k=j}^{i}\binom{i}{j}\binom{i-j}{k-j} x^{i-k}y^{k-j},$$

which is exactly (7), where we use the identity

$$\binom{i}{k}\binom{k}{j} = \binom{i}{j}\binom{i-j}{k-j}.$$

Pascal matrix function $P[x]$ is studied in a numerous papers including [1, 3, 4, 6, 9, 13], etc.

We now present a Riordan array representation of $\mathcal{R}(x)$.

Theorem 1.3 *Every element $R(x) \equiv (d(t), h(t))(x)$ in $\mathcal{R}(x)$ can be written as a Riordan array $(d(xt), h(xt)/x)$. Particularly, the identity of $\mathcal{R}(x)$ is $(1,t)(x)$, that has Riordan representation $(1,t)$. The inverse of $(d(t), h(t))(x)$ has the Riordan representation $\left(\frac{1}{d(\bar{h}(xt))}, \frac{\bar{h}(xt)}{x}\right)$, where $\bar{h}$ is the compositional inverse of h. Hence, the collection of $(d(xt), h(xt)/x)$ forms a group which is isomorphic to $\mathcal{R}$.*

Proof. The results can be derived directly from the definition of Riordan group. The detailed proof is omitted here.

■

By means of Theorem 1.3 and the notation $P[x] = (1/(1-t), t/(1-t))(x)$, we may give an alternative expression of (6) as

$$\left(\frac{1}{1-xt}, \frac{t}{1-xt}\right)\left(\frac{1}{1-yt}, \frac{t}{1-yt}\right) = \left(\frac{1}{1-(x+y)t}, \frac{t}{1-(x+y)t}\right), \tag{8}$$

which is obviously true due to the multiplication rule of Riordan arrays. (8) also gives a much easier proof of (6).

It should be mentioned that $(d(xt), h(xt)/x)$ is not Sheffer matrix (see, for example, [12]). In other wards, $(d(xt),\ h(xt)/x) \cdot \mathbf{1}$, where vector $\mathbf{1} = (1, 1, \ldots)^T$, is not a Sheffer polynomial coefficient vector.

As noted in Shapiro [19], any element of Riordan group $\mathcal{R}$ with integer entries having finite order must have order 1 or 2. We call the element of order 2 in $\mathcal{R}$ a Riordan involution. In combinatorial situations, a Riordan array often has nonnegative integer entries and hence it can not have order 2. Therefore, we define (see, for example, Cheon and Kim [7] and Cheon, Kim, and Shapiro [8]) an element $R \in \mathcal{R}$ to have pseudo-order 2 if RM has order 2, where $M = (1, -t)$. Those R are called pseudo-Riordan involutions or briefly pseudo-involutions (see Cameron and Nkwanta [5] and [19]).

We say two matrices A and B are similar associated with a Riordan type array R if there holds the relationship $A = RBR^{-1}$. We now present an alternative expression of Riordan matrix functions. From the multiplication rule of Riordan arrays, we have

Proposition 1.4 *Every element $R(x) \equiv (d(t), h(t))(x)$ in $\mathcal{R}(x)$ is similar to $(d(t), h(t)) \in \mathcal{R}$ associated with $(1, xt)$, i.e.,*

$$(d(t), h(t))(x) = (1, xt)(d(t), h(t))(1, xt)^{-1}. \tag{9}$$

Furthermore, $(d(t), h(t))$ is a pseudo-involution in $\mathcal{R}$ if and only if

$$(d(t), h(t))^{-1} = (d(-t), -h(-t)). \tag{10}$$

Corollary 1.5 *$(d(t), h(t))$ is a pseudo-involution in $\mathcal{R}$ if and only if*

$$\bar{h}(t) = -h(-t) \quad \textit{and} \quad d(-t)d(\bar{h}(t)) = 1, \tag{11}$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$. The last equation is equivalent to

$$d(t)d(-h(t)) = 1. \tag{12}$$

Proof. Comparing (10) and $(d(t),h(t))^{-1}=(1/d(\bar{h}(t)),\bar{h}(t))$, where $\bar{h}(t)$ is the compositional inverse of $h(t)$, we obtain the sufficient and necessary conditions (11). Then, (12) follows. ■

If $R(x)\equiv(d(t),h(t))(x)\in\mathcal{R}(x)$ has order 2 for every real x, then, $R(x)$ is called an involution of $\mathcal{R}(x)$.

Several subgroups of $\mathcal{R}(x)$ are defined below based on Theorems 1.2 and 1.3.

- Appell subgroup of $\mathcal{R}(x)$ is the set of Riordan arrays $D=(d(xt),\, h(xt)/x)$ for which $h(xt)=xt$;
- the associated subgroup of $\mathcal{R}(x)$, that is the set of Riordan arrays $D=(d(xt),\, h(xt)/x)$ for which $d(xt)=1$;
- The Bell subgroup of of $\mathcal{R}(x)$ is the set of Riordan arrays $D=(d(xt),\, h(xt)/x)$ for which $h(xt)=xtd(xt)$, i.e., $D=(d(xt),\, td(xt))$;
- The checkerboard subgroup of $\mathcal{R}(x)$ is the set of Riordan arrays $D=(d(xt),\, h(xt)/x)$ for which $d(xt)$ is an even function and $h(xt)$ is an odd function in terms of t;
- The stochastic subgroup of $\mathcal{R}(x)$ is the set of Riordan arrays $D=(d(xt),\, h(xt)/x)$ whose row weighted sums with weight $(1,x,x^2,\ldots)^T$ are x^n;
- The hitting-time subgroup of $\mathcal{R}(x)$ is the set of Riordan arrays $D=(d(xt),\, h(xt)/x)$ for which $d(xt)=t(d/dt)h(xt)/h(xt)$, i.e., $D=(xth'(xt)/h(xt),\, h(xt)/x)$.
- The derivative subgroup of $\mathcal{R}(x)$ is the set of Riordan arrays $D=((d/dt)h(xt)/x,\, h(xt))=(h'(xt),\, h(xt)/x)$.

Similar to Theorems 2.1 and 2.2 of [14], we have the following sequence characterizations of Riordan matrix function $R(x)\in\mathcal{R}(x)$.

Theorem 1.6 *Let $R(x):=(d_{n,k}x^{n-k})_{n,k\in\mathbb{N}_0}$ be a lower triangular matrix function. Then $R(x)$ is a Riordan matrix function if and only if there exist an $A(x)-$ and a $Z(x)-$ sequences of functions, $A(x)=(a_0(x),a_1(x),\ldots,)$ and $Z(x)=(z_0(x),z_1(x),\ldots)$, such that*

$$d_{n+1,k+1}x^{n-k}=\sum_{j=0}^{n}a_j(x)d_{n,k+j}x^{n-k-j} \tag{13}$$

and

$$d_{n+1,0}=\sum_{j=0}^{n}z_j(x)d_{n,j}x^{n-j}, \tag{14}$$

or equivalently,

$$xt=\bar{h}(xt)A(xt) \quad \text{and} \quad Z(xt)=\frac{d(xt)-d_{0,0}}{td(xt)}. \tag{15}$$

In next section, we will demonstrate some applications of Pascal array functions to construct the duals of Bernoulli and Euler numbers and polynomials. In Section 3, we discuss the construction of conjugates of Bernoulli and Euler numbers and polynomials as well as their duals by using Riordan matrix functions.

2 Applications to Bernoulli and Euler polynomials

Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ for $n = 0, 1, \ldots$ are defined by (see, for example, Apostol [2] and Olver, Lozier, Boisvert, and Clark [16])

$$\frac{te^{xt}}{e^t - 1} = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!} \text{ and } \frac{2e^{xt}}{e^t + 1} = \sum_{n\geq 0} E_n(x)\frac{t^n}{n!}. \tag{16}$$

Bernoulli numbers B_n and Euler numbers E_n for $n = 0, 1, \ldots$ are defined by

$$B_n = B_n(0) \quad \text{and} \quad E_n = 2^n E_n\left(\frac{1}{2}\right). \tag{17}$$

A large literature scatters widely in books and journals on Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ and Euler numbers E_n and Euler polynomials $E_n(x)$. They can be studied by means of the binomial expressions connecting them:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \quad n \geq 0, \tag{18}$$

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \left(x - \frac{1}{2}\right)^{n-k} \frac{E_k}{2^k}, \quad n \geq 0, \tag{19}$$

where $E_k = 2^k E_k(1/2)$. The study brings consistent attention of researchers working in combinatorics, number theory, applied analysis, etc.

We now give some applications of a typically useful Riordan array function, the Pascal matrix function $P[x] = (1/(1 - xt), t/(1 - xt))$ (see [4]), to the Bernoulli and Euler polynomials. Since

$$P[-x]P[x] = P[x]P[-x] = P[0] = I, \tag{20}$$

there exists

$$P[x]^{-1} = P[-x].$$

Thus, the well-known formula about Bernoulli polynomials $B_n(x)$ and Bernoulli numbers $B_n = B_n(0)$ shown in (18) implies the following inverse relationship between $B(x) = (B_0(x), B_1(x), \ldots)^T$ and $B = (B_1, B_2, \ldots)^T$ (see, for example, [13]):

$$B(x) = P[x]B. \tag{21}$$

More precisely, we have

Theorem 2.1 *[13] Let $B(x) = (B_0(x), B_1(x), \ldots)^T$ and $B = (B_0, B_1, \ldots)^T$, and let $P[x]$ be defined as above. Then there exists a pair of inverse relationships*

$$B(x) = P[x]B \qquad B = P[-x]B(x). \tag{22}$$

And the latter can be presented as

$$\begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_n \\ \vdots \end{bmatrix} = \begin{bmatrix} \binom{0}{0} & 0 & \cdots & 0 & 0 & \cdots \\ \binom{1}{0}(-x) & \binom{1}{1} & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n}{0}(-x)^n & \binom{n}{1}(-x)^{n-1} & \cdots & \binom{n}{n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} B_0(x) \\ B_1(x) \\ \vdots \\ B_n(x) \\ \vdots \end{bmatrix}, \tag{23}$$

which implies

$$B_n = \sum_{k=0}^{n} \binom{n}{k} (-x)^{n-k} B_k(x) \tag{24}$$

for $n = 0, 1, \ldots$.

Similarly, Euler polynomials $E_n(x)$ *can be presented in terms of Euler numbers shown in* (19)*. Denote* $E(x) = (E_0(x), E_1(x), \ldots)^T$ *and* $E = (E_0, E_1, \ldots)^T$*. By making use of the Pascal matrix, we may write* (19) *as a matrix form*

$$E(x) = P\left[x - \frac{1}{2}\right] D\left[\frac{1}{2}\right] E = P\left[x - \frac{1}{2}\right] E\left(\frac{1}{2}\right), \tag{25}$$

where

$$D[t] = diag[1, t, t^2, \ldots].$$

A numerous properties of Bernoulli and Euler polynomials are studied by using a unified approach in [13] based on the matrix representation. In this section, we will use Pascal matrix function to study the duals of Bernoulli and Euler numbers.

Let $\{a_n\}_{n\geq 0}$ be a sequence of complex numbers. We refer to $\{a^*_n\}_{n\geq 0}$ defined by

$$a^*_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k \tag{26}$$

as the dual sequence of $\{a_n\}_{n\geq 0}$ (see, for example, Graham, Knuth, and Patashnik [10]). If $a^*_n = a_n$, then $\{a_n\}_{n\geq 0}$ is called the self-dual sequence (see Sun [22]). It is easy to see that

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a^*_k. \tag{27}$$

Because

$$\sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k+j} = \delta_{n,j},$$

where $\delta_{n,j}$ is a Kronecker symbol, which is 1 when $j = n$ and zero otherwise, the pair of $\{a_n\}_{n\geq 0}$ and $\{a^*_n\}_{n\geq 0}$ is a pair of inverse sequences. Hence, $(a^*)^*_n = a_n$.

It is well known that $\{(-1)^n B_n\}_{n\geq 0}$ is a self-dual sequence (see Sun [23]). We now study the dual sequences of $\{B_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$, denoted by $\{B^*_n\}_{n\geq 0}$ and $\{E^*_n\}_{n\geq 0}$, respectively, and dual Bernoulli polynomials and dual Euler polynomials, denoted by $\{B^*_n(x)\}_{n\geq 0}$ and $\{E^*_n(x)\}_{n\geq 0}$, respectively. Here,

$$B_n^* = \sum_{k=0}^{n} \binom{n}{k} (-1)^k B_k \quad \text{and} \quad E_n^* = \sum_{k=0}^{n} \binom{n}{k} (-1)^k E_k. \tag{28}$$

Inspired by (18) and (19) we define the dual Bernoulli polynomials and the dual Euler polynomials below:

Definition 2.2 *Denote* $B^*(x) := (B_0^*(x), B_1^*(x), \ldots)$*,* $B^* := (B_0^*, B_1^*, \ldots)$*,* $E^*(x) := (E_0^*(x), E_1^*(x), \ldots)$*, and* $E^* := (E_0^*, E_1^*, \ldots)$*. Let* $D[t] := diag(1, t, t^2, \ldots)$*. Then*

$$B^*(x) = P[x]B^* \quad \textit{and} \quad E^*(x) = P\left[x - \frac{1}{2}\right] D \left[\frac{1}{2}\right] E^*, \tag{29}$$

which imply

$$B_n^*(x) = \sum_{k=0}^{n} \binom{n}{k} B_k^* x^{n-k}, \quad n \geq 0, \quad \textit{and} \tag{30}$$

$$E_n^*(x) = \sum_{k=0}^{n} \binom{n}{k} \left(x - \frac{1}{2}\right)^{n-k} \frac{E_k^*}{2^k}, \quad n \geq 0, \tag{31}$$

where $E_k^* = 2^k E_k^*(1/2)$*.*

The relationships between Bernoulli polynomials, Euler polynomials and their duals can be presented below.

Theorem 2.3 *Let* $B^*(x)$*,* B^**,* $E^*(x)$*,* E^**, and* $D[t]$ *be defined before. Then*

$$B^*(x) = D[-1]B(-x-1) \quad \textit{and} \quad E^*(x) = D[-1]E\left(-x + \frac{1}{2}\right), \tag{32}$$

which imply

$$B_n^*(x) = (-1)^n B_n(-x-1) \quad \textit{and} \quad E_n^*(x) = (-1)^n E_n\left(-x + \frac{1}{2}\right). \tag{33}$$

Furthermore,

$$B^* = D[-1]B(-1) \quad \textit{and} \quad E^* = D[-2]E\left(0\right), \tag{34}$$

which imply

$$B_n^* = (-1)^n B_n + n \quad \textit{and} \quad E_n^* = (-2)^n E_n(0) = 2^n E_n(1). \tag{35}$$

Proof. From (19) we have

$$B^* = D[-1]P[-1]B \quad \text{and} \quad E^* = D[-1]P[-1]E. \tag{36}$$

Using (36) and (29), we obtain

$$B^*(x) = P[x]D[-1]P[-1]B = D[-1]P[-x]P[-1]B = D[-1]P[-x-1]B. \tag{37}$$

Then the first formula of (32) follows from the rightmost expression and (21). The second formulas of (29) and (36) can be applied to derive

$$
\begin{aligned}
E^*(x) = P\left[x-\frac{1}{2}\right] D\left[\frac{1}{2}\right] D[-1]P[-1]E \\
= \quad P\left[x-\frac{1}{2}\right] P\left[\frac{1}{2}\right] D[-1]D\left[\frac{1}{2}\right] E \\
= \quad D[-1]P\left[-x+\frac{1}{2}-\frac{1}{2}\right] D\left[\frac{1}{2}\right] E = D[-1]E\left(-x+\frac{1}{2}\right),
\end{aligned}
$$

completing the proof of the second formula of (32). Thus, (33) follows immediately.

Substituting $x=0$ and $x=1/2$ into the first and the second formulas of (32), respectively, yield (34). Hence,

$$B_n^* \equiv B_n^*(0) = (-1)^n B_n(-1) = B_n(1)+n = (-1)^n B_n + n$$

and

$$E_n^* \equiv 2^n E_n^*\left(\frac{1}{2}\right) = (-2)^n E_n\left(0\right)$$

give (35).

■

The first few elements of dual Bernoulli numbers and Euler numbers are

$$
\begin{aligned}
\{B_n^*\}_{n\geq 0} &= \left\{1, \frac{3}{2}, \frac{13}{6}, 3, \frac{119}{30}, 5, \frac{253}{42}, \ldots\right\}, \text{ and} \\
\{E_n^*\}_{n\geq 0} &= \{1, 1, 0, -2, 0, 16, 0, \ldots\},
\end{aligned}
$$

respectively, where we notice that starting from the fourth term, the even-indexed dual Euler numbers are zero while the odd-indexed (classical) Euler numbers are zero.

The first few terms of the dual Bernoulli polynomial sequence $\{B_n^*\}_{n\geq 0}$ are

$$
\begin{aligned}
B_0^*(x) &= 1, \\
B_1^*(x) &= x + \frac{3}{2}, \\
B_2^*(x) &= x^2 + 3x + \frac{13}{6}, \\
B_3^*(x) &= x^3 + \frac{9}{2}x^2 + \frac{13}{2}x + 3, \\
B_4^*(x) &= x^4 + 6x^3 + 13x^2 + 12x + \frac{191}{30}, \\
B_5^*(x) &= x^5 + \frac{15}{2}x^4 + \frac{65}{3}x^3 + 30x^2 + \frac{191}{6}x + 5, \textit{etc.},
\end{aligned}
$$

while the first few terms of the dual Euler polynomial sequence $\{E_n^*\}_{n\geq 0}$ are

$$\begin{aligned}
&E_0^*(x) = 1,\\
&E_1^*(x) = x,\\
&E_2^*(x) = x^2 - \frac{1}{4},\\
&E_3^*(x) = x^3 - \frac{3}{4}x,\\
&E_4^*(x) = x^4 - \frac{3}{2}x^2 + \frac{5}{16},\\
&E_5^*(x) = x^5 - \frac{5}{2}x^3 + \frac{25}{16}x, etc.
\end{aligned}$$

The generating function of dual Bernoulli polynomial sequence $B_n^*(x)$ is

$$\begin{aligned}
&\sum_{n\geq 0} B_n^*(x)\frac{t^n}{n!} = \sum_{n\geq 0}(-1)B_n(-x-1)\frac{t^n}{n!}\\
= \ &\frac{-te^{(x+1)t}}{e^{-t}-1} = \frac{e^{(x+1)t}}{1+\frac{1-t-e^{-t}}{t}},
\end{aligned}$$

where the rightmost expression will be used to define the conjugate polynomial sequence of $\{B_n^*(x)\}_{n\geq 0}$ in next section. Similarly, the generating function of dual Euler polynomial sequence $E_n^*(x)$ is

$$\begin{aligned}
&\sum_{n\geq 0} E_n^*(x)\frac{t^n}{n!} = \sum_{n\geq 0}(-1)^n E_n\left(-x+\frac{1}{2}\right)\frac{t^n}{n!}\\
= \ &\frac{2e^{(x-1/2)t}}{e^{-t}+1} = \frac{e^{(x-1/2)t}}{1+\frac{e^{-t}-1}{2}},
\end{aligned}$$

where the rightmost expression will be used to define the conjugate polynomial sequence of $\{E_n^*(x)\}_{n\geq 0}$ in next section. We survey those results below and leave more discussion about conjugate polynomial sequences in Section 3.

Theorem 2.4 *Let $B_n^*(x)$ and $E_n^*(x)$ be dulas of Bernoulli and Euler polynomials defined before. Then their generating functions are respectively*

$$\frac{e^{(x+1)t}}{1+\frac{1-t-e^{-t}}{t}} = \sum_{n\geq 0} B_n^*(x)\frac{t^n}{n!} \tag{38}$$

$$\frac{e^{(x-1/2)t}}{1+\frac{e^{-t}-1}{2}} = \sum_{n\geq 0} E_n^*(x)\frac{t^n}{n!}. \tag{39}$$

Remark 3.1 By using the same argument, the self dual sequence $\{(-1)^n B_n\}_{n\geq 0}$ induces the corresponding dual polynomial sequence $\{B_n(x+1)\}_{n\geq 0}$, which has the generating function

$$\frac{te^{(x+1)t}}{e^t-1}=\sum_{n\geq 0} B_n(x+1)\frac{t^n}{n!}.$$

3 Conjugate polynomials and their duals

Let $(d(xt),h(t))$ be a Riordan arrays in terms of real variable t. Then we will define a pair of polynomial sequences, denoted by $\{F_n(x)\}_{n\geq 0}$ and $\{\tilde{F}_n(x)\}_{n\geq 0}$, respectively, where $\{\tilde{F}_n(x)\}_{n\geq 0}$ is called the conjugate polynomial sequence of $\{F_n(x)\}_{n\geq 0}$, which will be also defined as follows:

Definition 3.1 *Let $(d(xt),h(t))$ be a Riordan arrays in terms of real variable t. Then the polynomial sequence $\{F_n(x)\}_{n\geq 0}$ and its conjugate polynomial sequence $\{\tilde{F}_n(x)\}_{n\geq 0}$ are defined by their exponential generating functions (see, for example, Wilf [24]) as*

$$\frac{d(xt)}{1-h(t)}=\sum_{n\geq 0} F_n(x)\frac{t^n}{n!} \quad \text{and} \quad \frac{d(xt)}{1+h(t)}=\sum_{n\geq 0} \tilde{F}_n(x)\frac{t^n}{n!}. \tag{40}$$

Here, $d(xt)/(1-h(t))$ and $d(xt)/(1+h(t))$ are called the sum (generating) function and alternating sum (generating) function of the Riordan array $(d(xt),h(t))$, which are defined in Shapiro and the author [15].

Particularly, if $d(xt)=e^{xt}$, then we have

Proposition 3.2 *Let $(d(xt),h(t))$ be a Riordan arrays in terms of variable t with $d(xt)=e^{xt}$, and let $\{F_n(x)\}_{n\geq 0}$ and $\{\tilde{F}_n(x)\}_{n\geq 0}$ be the polynomial sequence and its conjugate defined in Definition 3.1. Then for a real α there hold*

$$F(x)=P[x-\alpha]F(\alpha) \quad \text{and} \quad \tilde{F}(x)=P[x-\alpha]\tilde{F}(\alpha) \tag{41}$$

and

$$F(\alpha)=P[-x+\alpha]F(x) \quad \text{and} \quad \tilde{F}(\alpha)=P[-x+\alpha]\tilde{F}(x), \tag{42}$$

where $F(x):=(F_0(x),F_1(x),F_2(x),\ldots)^T$, $F(\alpha):=(F_0(\alpha),F_1(\alpha),F_2(\alpha),\ldots)^T$, $\tilde{F}(x):=(\tilde{F}_0(x),\tilde{F}_1(x),\tilde{F}_2(x),\ldots)^T$, and $\tilde{F}(\alpha):=(\tilde{F}_0(\alpha),\tilde{F}_1(\alpha),\tilde{F}_2(\alpha),\ldots)^T$.

Proof. From the definitions of $\tilde{F}_n(x)$ and $\tilde{F}_n(x)$, we have

$$\frac{e^{\alpha t}}{1-h(t)}=\sum_{n=0}^{\infty} F_n(\alpha)\frac{t^n}{n!} \quad \text{and} \quad \frac{e^{\alpha t}}{1+h(t)}=\sum_{n=0}^{\infty} \tilde{F}_n(\alpha)\frac{t^n}{n!}$$

Hence, denoted by $f_n(x)$ either $F_n(x)$ and $\tilde{F}_n(x)$, we obtain

$$\sum_{n=0}^{\infty} f_n(x)\frac{t^n}{n!} = e^{(x-\alpha)t}\frac{e^{\alpha t}}{1 \pm h(t)} = \sum_{n\geq 0}\sum_{k\geq 0}(x-\alpha)^n f_k(\alpha)\frac{t^{n+k}}{n!k!}$$

$$= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}(x-\alpha)^{n-k}f_k(\alpha)\right)\frac{t^n}{n!}.$$

Comparing the leftmost and the rightmost sides, we obtain (41). (42) follows from (41).

■

The Bernoulli polynomial sequence defined by (16) can be considered as the sum generating function of Riordan matrix function $(e^{xt}, (1+t-e^t)/t)$:

$$\frac{te^{xt}}{e^t-1} \equiv \frac{e^{xt}}{1-\frac{1+t-e^t}{t}} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

Inspired by the work shown in Shapiro and the author [15], we define the conjugate Bernoulli polynomial sequence below by using the alternating sum (generating) function of $(e^{xt}, (1+t-e^t)/t)$:

$$\frac{e^{xt}}{1+\frac{1+t-e^t}{t}} = \sum_{n=0}^{\infty} \tilde{B}_n(x)\frac{t^n}{n!}. \tag{43}$$

The coefficient polynomials of the above expansion form a sequence denoted by $\{\tilde{B}_n\}_{n\geq 0}$ and referred to as the conjugate Bernoulli polynomial sequence, which first few terms are

$$\begin{aligned}
\tilde{B}_0(x) &= 1,\\
\tilde{B}_1(x) &= x + \frac{1}{2},\\
\tilde{B}_2(x) &= x^2 + x + \frac{5}{6},\\
\tilde{B}_3(x) &= x^3 + \frac{3}{2}x^2 + \frac{5}{2}x + 2,\\
\tilde{B}_4(x) &= x^4 + 2x^3 + 5x^2 + 8x + \frac{191}{30},\\
\tilde{B}_5(x) &= x^5 + \frac{15}{6}x^4 + \frac{25}{3}x^3 + 20x^2 + \frac{191}{6}x + \frac{76}{3}, etc.
\end{aligned}$$

Similarly, $\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$ is the sum (generating) function of exponential Riordan array $(e^{xt}, (1-e^t)/2)$. The corresponding alternating sum (generating) function is

$$\frac{e^{xt}}{1+\frac{1-e^t}{2}} = \sum_{n=0}^{\infty} \tilde{E}_n(x)\frac{t^n}{n!}, \tag{44}$$

where the first few terms of the polynomial sequence $\{\tilde{E}_n\}_{n\geq 0}$, called the conjugate Euler polynomials, are

$$\begin{aligned}
\tilde{E}_0(x) &= 1,\\
\tilde{E}_1(x) &= x + \frac{1}{2},\\
\tilde{E}_2(x) &= x^2 + x + 1,\\
\tilde{E}_3(x) &= x^3 + \frac{3}{2}x^2 + 3x + \frac{11}{4},\\
\tilde{E}_4(x) &= x^4 + 2x^3 + 6x^2 + 11x + 10,\\
\tilde{E}_5(x) &= x^5 + \frac{5}{2}x^4 + 10x^3 + \frac{55}{2}x^2 + 50x + \frac{91}{2}, etc.
\end{aligned}$$

From Proposition 3.2 for $\alpha = 0$, we may know that the conjugate Bernoulli polynomials $\{\tilde{B}_n(x)\}_{n\geq 0}$ have expression

$$\tilde{B}_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \tilde{B}_k, \tag{45}$$

where

$$\tilde{B}_n := \tilde{B}_n(0), \tag{46}$$

called the conjugate Bernoulli numbers. Hence, by denoting

$$\tilde{B}(x) = (\tilde{B}_0(x), \tilde{B}_1(x), \ldots, \tilde{B}_n(x), \ldots)^T \text{ and } \tilde{B} = (\tilde{B}_0, \tilde{B}_1, \ldots, \tilde{B}_n, \ldots)^T,$$

one may write

$$\tilde{B}(x) = P[x]\tilde{B} \text{ and } \tilde{B} = P[-x]\tilde{B}(x). \tag{47}$$

Similarly, from Proposition 3.2 for $\alpha = 1/2$, we learn that the conjugate Euler polynomials $\{\tilde{E}_n(x)\}_{n\geq 0}$ satisfy

$$\tilde{E}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \left(x - \frac{1}{2}\right)^{n-k} \frac{\tilde{E}_k}{2^k}, \tag{48}$$

where

$$\tilde{E}_k := 2^k \tilde{E}_k \left(\frac{1}{2}\right), \tag{49}$$

called the conjugate Euler numbers. Hence, by denoting

$$\tilde{E}(x) = (\tilde{E}_0(x), \tilde{E}_1(x), \ldots, \tilde{E}_n(x), \ldots)^T \text{ and } \tilde{E} = (\tilde{E}_0, \tilde{E}_1, \ldots, \tilde{E}_n, \ldots)^T,$$

one may write

$$\tilde{E}(x) = P\left[x - \frac{1}{2}\right] D\left[\frac{1}{2}\right] \tilde{E} = P\left[x - \frac{1}{2}\right] \tilde{E}\left(\frac{1}{2}\right) \text{ and} \tag{50}$$

$$\tilde{E} = P[-2x+1]D[2]\tilde{E}(x). \tag{51}$$

The first few terms of conjugate Bernoulli numbers are

$$\left\{\tilde{B}_n\right\} = \left\{1, \frac{1}{2}, \frac{5}{6}, 2, \frac{191}{3}0, \frac{76}{3}, \ldots\right\}.$$

The first few terms of conjugate Euler numbers are

$$\left\{\tilde{E}_n\right\} = \{1, 2, 7, 38, 277, 2522, \ldots\}.$$

We now define duals of conjugate Bernoulli polynomial sequence and conjugate Euler number sequence similar to (26) and (29), respectively.

Definition 3.3 *Let $\{\tilde{B}_n(x)\}_{n\geq 0}$ be the conjugate Bernoulli polynomial sequence defined by* (43)*, and let the conjugate Bernoulli numbers be defined by* (46)*. Then the duals of conjugate Bernoulli numbers are defined by*

$$\tilde{B}^*_n := \sum_{k=0}^{n} \binom{n}{k} (-1)^k \tilde{B}_k \tag{52}$$

and the duals of conjugate Bernoulli polynomials are defined by

$$\tilde{B}^*(x) := \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \tilde{B}^*_k. \tag{53}$$

Denote $\tilde{B}^* := (\tilde{B}^*_0, \tilde{B}^*_1, \ldots)^T$ and $\tilde{B}^*(x) := (\tilde{B}^*_0(x), \tilde{B}^*_1(x), \ldots)^T$. Then $\tilde{B}^*$ and $\tilde{B}^*(x)$ satisfy

$$\tilde{B}^* = D[-1]P[-1]\tilde{B} \quad \text{and} \quad \tilde{B}^*(x) = P[x]\tilde{B}^*, \tag{54}$$

where $D[t] := diag(1, t, t^2, \ldots)$.

Similarly, let $\{\tilde{E}_n(x)\}_{n\geq 0}$ be the conjugate Euler polynomial sequence defined by (44)*, and let the conjugate Euler numbers be defined by* (46)*. Then the duals of conjugate Euler numbers are defined by*

$$\tilde{E}^*_n := \sum_{k=0}^{n} \binom{n}{k} (-1)^k \tilde{E}_k \tag{55}$$

and the duals of conjugate Euler polynomials are defined by

$$\tilde{E}^*(x) := \sum_{k=0}^{n} \binom{n}{k} \left(x - \frac{1}{2}\right)^{n-k} \frac{\tilde{E}^*_k}{2^k} \tag{56}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(x - \frac{1}{2}\right)^{n-k} \tilde{E}^*_k \left(\frac{1}{2}\right). \tag{57}$$

Denote $\tilde{E}^* := (\tilde{E}^*_0, \tilde{E}^*_1, \ldots)^T$ and $\tilde{E}^*(x) := (\tilde{E}^*_0(x), \tilde{E}^*_1(x), \ldots)^T$. Then $\tilde{E}^*$ and $\tilde{E}^*(x)$ satisfy

$$\tilde{E}^* = D[-1]P[-1]\tilde{E} \quad \text{and} \quad \tilde{E}^*(x) = P\left[x - \frac{1}{2}\right] D\left[\frac{1}{2}\right] \tilde{E}^*, \tag{58}$$

where $D[t] := diag(1, t, t^2, \ldots)$.

Clearly, the first formula of (52) implies

$$\tilde{B} = P[1]D[-1]\tilde{B}^* = D[-1]P[-1]\tilde{B}^*.$$

From Definition 3.3, we immediately have

Theorem 3.4 *Let $\{\tilde{B}_n(x)\}_{n\geq 0}$ and $\{\tilde{B}^*(x)\}_{n\geq 0}$ be the conjugate Bernoulli polynomial sequence and its dual sequence defined by (43) and (52), respectively, and let $\tilde{B}(x)$ and $\tilde{B}^*(x)$ be the vectors of the conjugate Bernoulli polynomial sequence and its dual sequence defined before. Then there hold*

$$\begin{aligned} &\tilde{B}^*(x) = D[-1]P[-x-1]\tilde{B} = D[-1]\tilde{B}(-x-1) \text{ and} \\ &\tilde{B}(x) = D[-1]P[-x-1]\tilde{B}^* = D[-1]\tilde{B}^*(-x-1), \end{aligned} \tag{59}$$

which implies

$$\tilde{B}^*_n(x) = (-1)^n\tilde{B}_n(-x-1) \text{ and } \tilde{B}_n(x) = (-1)^n\tilde{B}^*_n(-x-1). \tag{60}$$

Proof. From the corresponding definitions, we have

$$\begin{aligned} &\tilde{B}^*(x) = P[x]\tilde{B}^* = P[x]D[-1]P[-1]\tilde{B} \\ = \quad &D[-1]P[-x]P[-1]\tilde{B} = D[-1]P[-x-1]\tilde{B}, \end{aligned}$$

which implies the first formulas of (53) and (54), where we used the identity

$$D[\alpha]P[x] = P[\alpha x]D[\alpha].$$

Using a similar argument, we may prove the second formulas of (53) and (54).

■

Furthermore, we obtain

Theorem 3.5 *Let $\{\tilde{E}_n(x)\}_{n\geq 0}$ and $\{\tilde{E}^*(x)\}_{n\geq 0}$ be the conjugate Euler polynomial sequence and its dual sequence defined by (44) and (55), respectively, and let $\tilde{E}(x)$ and $\tilde{E}^*(x)$ be the vectors of the conjugate Euler polynomial sequence and its dual sequence defined before. Then there hold*

$$\begin{aligned} &\tilde{E}^*(x) = D[-1]P[-x]D\left[\frac{1}{2}\right]\tilde{E} = D[-1]\tilde{E}\left(-x+\frac{1}{2}\right) \text{ and} \\ &\tilde{E}(x) = D[-1]P[-x]D\left[\frac{1}{2}\right]\tilde{E}^* = D[-1]\tilde{E}^*\left(-x+\frac{1}{2}\right), \end{aligned} \tag{61}$$

which implies

$$\tilde{E}^*_n(x) = (-1)^n\tilde{E}_n\left(-x+\frac{1}{2}\right) \text{ and } \tilde{E}_n(x) = (-1)^n\tilde{E}^*_n\left(-x+\frac{1}{2}\right). \tag{62}$$

Proof. From the corresponding definitions, we have

$$\begin{aligned}
\tilde{E}^*(x) &= P\left[x-\frac{1}{2}\right] D\left[\frac{1}{2}\right] \tilde{E}^* \\
&= P\left[x-\frac{1}{2}\right] D\left[\frac{1}{2}\right] D[-1]P[-1]\tilde{E} \\
&= D[-1]P[-x]D\left[\frac{1}{2}\right] \tilde{E},
\end{aligned}$$

which yields the first formulas of (61) and (62), and the second formulas of (61) and (62) follow immediately.

■

The first few terms of the dual of conjugate Bernoulli polynomial sequence are

$$\begin{aligned}
&\tilde{B}_0^*(x) = 1, \\
&\tilde{B}_1^*(x) = x + \frac{1}{2}, \\
&\tilde{B}_2^*(x) = x^2 + x + \frac{5}{6}, \\
&\tilde{B}_3^*(x) = x^3 + \frac{3}{2}x^2 + \frac{5}{2}x, \\
&\tilde{B}_4^*(x) = x^4 + 2x^3 + 5x^2 + \frac{71}{30}, \\
&\tilde{B}_5^*(x) = x^5 + \frac{5}{2}x^4 + \frac{25}{3}x^3 + \frac{71}{6}x - \frac{20}{3}, \textit{etc.}
\end{aligned}$$

Hence, the first few terms of dual conjugate Bernoulli numbers are

$$\left\{\tilde{B}_n^*\right\} = \left\{1, \frac{1}{2}, \frac{5}{6}, 0, \frac{71}{30}, -\frac{20}{3}, \ldots\right\}.$$

The first few terms of the dual of conjugate Euler polynomial sequence are

$$\begin{aligned}
&\tilde{E}_0^*(x) = 1, \\
&\tilde{E}_1^*(x) = x - 1, \\
&\tilde{E}_2^*(x) = x^2 - 2x + \frac{7}{4}, \\
&\tilde{E}_3^*(x) = x^3 - 3x^2 + \frac{21}{4}x - \frac{19}{4}, \\
&\tilde{E}_4^*(x) = x^4 - 4x^3 + \frac{21}{2}x^2 - 19x + \frac{277}{16}, \\
&\tilde{E}_5^*(x) = x^5 - 5x^4 + \frac{35}{2}x^3 - \frac{95}{2}x^2 + \frac{1385}{16}x - \frac{1261}{16}, \textit{etc.}
\end{aligned}$$

Hence, the first few terms of dual conjugate Euler numbers are

$$\left\{\tilde{E}_n^*\right\} = \{1, -1, 4, -22, 160, -1456, \ldots\}.$$

We now establish a relationship between conjugate Bernoulli polynomials and (classical) Bernoulli polynomials through the their duals.

Theorem 3.6 *Let $\{\tilde{B}_n(x)\}_{n\geq 0}$ be the conjugate Bernoulli polynomial sequence defined by (43), and let its dual sequence, $\{\tilde{B}_n^*(x)\}_{n\geq 0}$, be defined by (30). Denote by $\tilde{B}(x)$ and $\tilde{B}^*(x)$ the vectors of the conjugate Bernoulli polynomial sequence and its dual sequence defined before. Then the generating function of $\{\tilde{B}_n(x)\}_{n\geq 0}$ is*

$$\frac{e^{(x+1)t}}{1-\frac{1-t-e^{-t}}{t}} \equiv \frac{te^{(x+2)t}}{1-(1-2t)e^t} = \sum_{n=0}^{\infty} \tilde{B}_n^*(x)\frac{t^n}{n!}. \tag{63}$$

Furthermore, the conjugates of the duals of the conjugate of Bernoulli polynomials are Bernoulli polynomials. More precisely, we have

$$\frac{e^{(x+1)t}}{1+\frac{1-t-e^{-t}}{t}} \equiv \frac{-te^{(x+1)t}}{e^{-t}-1} = \sum_{n=0}^{\infty} B_n(-x-1)\frac{(-t)^n}{n!}, \tag{64}$$

or equivalently,

$$\widetilde{\tilde{B}^*}_n(x) = (-1)^n B_n(-x-1). \tag{65}$$

Proof. (63) is from (43) and (54), while (64) follows the definition of conjugate polynomials shown in Definition 3.1 and the generating function of Bernoulli polynomials.

∎

Similarly, we have

Theorem 3.7 *Let $\{\tilde{E}_n(x)\}_{n\geq 0}$ be the conjugate Euler polynomial sequence defined by (44), and let its dual sequence, $\{\tilde{E}_n^*(x)\}_{n\geq 0}$, be defined by (31). Denote by $\tilde{E}(x)$ and $\tilde{E}^*(x)$ the vectors of the conjugate Bernoulli polynomial sequence and its dual sequence defined above. Then the generating function of $\{\tilde{E}_n(x)\}_{n\geq 0}$ is*

$$\frac{e^{(x-1/2)t}}{1-\frac{e^{-t}-1}{2}} \equiv \frac{2e^{(x+1/2)t}}{3e^t-1} = \sum_{n=0}^{\infty} \tilde{E}_n^*(x)\frac{t^n}{n!}. \tag{66}$$

Furthermore, the conjugates of the duals of the conjugate of Euler polynomials are Euler polynomials. More precisely, we have

$$\frac{e^{(x-1/2)t}}{1+\frac{e^{-t}-1}{2}} \equiv \frac{2e^{(x-1/2)t}}{e^{-t}+1} = \sum_{n=0}^{\infty} E_n(-x+1/2)\frac{(-t)^n}{n!}, \tag{67}$$

or equivalently,

$$\widetilde{\tilde{E}^*}_n(x) = (-1)^n E_n\left(-x+\frac{1}{2}\right). \tag{68}$$

Comparing the results of Theorems 2.3 and 3.6 and 3.7, we finally obtain

Theorem 3.8 *Let F^*, $\tilde{F}$, $\tilde{F}^*$ and $\widetilde{\widetilde{F^*}}$, where $F = B$ or E, be defined as before. Then*

$$F^* = \widetilde{\widetilde{F^*}} \quad \text{or equivalently} \quad \widetilde{F^*} = (\tilde{F})^*.$$

Hence, the operators $\tilde{F}$ and F^* are commutative.

The matrix form of conjugate Bernoulli polynomials and Euler polynomials provides a unified approach in the construction of Bernoulli and Euler polynomials' identities. As an example, we present the following extension work of [14] for the conjugate Bernoulli polynomials. First, we need a notation of direct sum of matrices. Let A and B be any $m \times m$ and $n \times n$ square matrices, respectively. [7] use the notation $\oplus$ for the direct sum of matrices A and B:

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}. \tag{69}$$

Hence, we may obtain an extension to the conjugate Bernoulli polynomials in a matrix form of Lemma 3.2 of [17]

Theorem 3.9 *Let n be any positive integer, and let $\tilde{B}(x)$ and $P[x]$ be defined as before. Denote $(x) = (1, x, x^2, \ldots)^T$, $D = diag(0, 1, 1/2, \ldots)$, and $[X] = [0] \oplus \left[\frac{x^{n-k}}{k}\right]_{1 \le k \le n}$, i.e.,*

$$[X] = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & x & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x^{n-1} & \frac{x^{n-2}}{2} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}.$$

Then

$$[X]\tilde{B}(x+y) = P[x]D\tilde{B}(y) + [X](x), \tag{70}$$

which implies

$$\sum_{k=1}^{n} \frac{\tilde{B}_k(x+y)}{k} x^{n-k} = \sum_{l=1}^{n} \binom{n}{l} \frac{\tilde{B}_l(y)}{l} x^{n-l} + H_n x^n \tag{71}$$

for all $n \ge 0$.

Proof. Noting that $\tilde{B}_0(y) = 1$, the left-hand side of (70) can be written as

$$
\begin{aligned}
&LHS\ of\ (70)\\
&=[X]P[x]\tilde{B}(y)\\
&=[X]\begin{bmatrix} 1\\ x+\binom{1}{1}\tilde{B}_1(y)\\ x^2+\binom{2}{1}x\tilde{B}_1(y)+\binom{2}{2}\tilde{B}_2(y)\\ \vdots\\ x^n+\binom{n}{1}x^{n-1}\tilde{B}_1(y)+\cdots+\binom{n}{n}\tilde{B}_n(y)\\ \vdots \end{bmatrix}\\
&=[X]\left(\begin{bmatrix}1\\ x\\ x^2\\ \vdots\\ x^n\\ \vdots\end{bmatrix}+\begin{bmatrix} 0\\ \binom{1}{1}\tilde{B}_1(y)\\ \binom{2}{1}x\tilde{B}_1(y)+\binom{2}{2}\tilde{B}_2(y)\\ \vdots\\ \binom{n}{1}x^{n-1}\tilde{B}_1(y)+\cdots+\binom{n}{n}\tilde{B}_n(y)\\ \vdots \end{bmatrix}\right)\\
&=\begin{bmatrix}0\\ x\\ (1+\frac{1}{2})x^2\\ \vdots\\ (1+\frac{1}{2}+\cdots+\frac{1}{n})x^n\\ \vdots\end{bmatrix}+[X]\begin{bmatrix} 0\\ \binom{1}{1}\tilde{B}_1(y)\\ \binom{2}{1}x\tilde{B}_1(y)+\binom{2}{2}\tilde{B}_2(y)\\ \vdots\\ \binom{n}{1}x^{n-1}\tilde{B}_1(y)+\cdots+\binom{n}{n}\tilde{B}_n(y)\\ \vdots \end{bmatrix},
\end{aligned}
$$

where the second term can be written as

$$
\begin{aligned}
&\begin{bmatrix} 0\\ \left(\binom{1}{1}+\frac{1}{2}\binom{2}{1}\right)\tilde{B}_1(y)x+\frac{1}{2}\binom{2}{2}\tilde{B}_2(y)\\ \vdots\\ \left(\binom{1}{1}+\frac{1}{2}\binom{2}{1}+\cdots+\frac{1}{n}\binom{n}{1}\right)\tilde{B}_1(y)x^{n-1}+\left(\frac{1}{2}\binom{2}{2}+\frac{1}{3}\binom{3}{2}+\cdots+\frac{1}{n}\binom{n}{2}\right)\tilde{B}_2(y)x^{n-2}+\cdots+\frac{1}{n}\binom{n}{n}\tilde{B}_n(y)\\ \vdots \end{bmatrix}\\
&=\begin{bmatrix} 0\\ \binom{2}{1}\tilde{B}_1(y)x+\frac{1}{2}\binom{2}{2}\tilde{B}_2(y)\\ \vdots\\ \binom{n}{1}\tilde{B}_1(y)x^{n-1}+\frac{1}{2}\binom{n}{2}\tilde{B}_2(y)x^{n-2}+\cdots\frac{1}{n}\binom{n}{n}\tilde{B}_n(y)\\ \vdots \end{bmatrix}.
\end{aligned}
$$

In the last step we use the identities $\frac{1}{n}\binom{n}{k}=\frac{1}{k}\binom{n-1}{k-1}$ and

$$
\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-2}{k-1}+\cdots+\binom{k}{k-1}+\binom{k}{k}
$$

for $k, n \in \mathbb{N}$ with $1 \le k \le n$. Since the last matrix can be written as $P[x]DB(y)$, we obtain (70). By multiplying the matrices of (70),

$$LHS\ of\ (71) = H_n x^n + \sum_{l=1}^{n} \binom{n}{l} \frac{\tilde{B}_l(y)}{l} x^{n-l} = \sum_{l=1}^{n} \binom{n}{l} \frac{\tilde{B}_l(y)}{l} x^{n-l} + H_n x^n,$$

which is the RHS of (71).

■

Remark 3.1 Some of the results, for instance (41), in this section can be proved by using the multiplication rule of the exponential Riordan arrays. Here, we present a unified elementary approach in our proofs by suing matrix functions. Worth noting a recent paper [11] on a generalized Riordan group, to which our main results of this paper can be extended. Due to the limit of space, more properties and applications of conjugate and dual Bernoulli polynomials and Euler polynomials will presented in a subsequent paper.

4 Acknowledgements

We would like to express our thankfulness to the anonymous referees for their helpful comments and remarks that led to an improved/revised version of the original manuscript.

References

[1] L. Aceto and D. Trigiante, The matrices of Pascal and other greats, *American Math. Monthly*, 108 (2001), No. 3, 232–245.

[2] T. M. Apostol, A primer on Bernoulli numbers and polynomials, *Math. Magazine*, 81 (2008), No. 3, 178–190.

[3] P. Barry, On integer-sequence-based constructions of generalized Pascal triangles, *J. Int. Seq.*, 9 (2006), Article 06.2.4.

[4] G. S. Call and D. J. Velleman, Pascal's matrices, *American Math. Monthly*, 100 (1993), No. 4, 372–376.

[5] N. T. Cameron and A. Nkwanta, On some (pseudo) involutions in the Riordan group, *J. Integer Seq.*, 8 (2005), 1-16.

[6] G.-S. Cheon, A note on the Bernoulli and Euler polynomials, *Appl. Math. Letters*, 16(2003), 365-368.

[7] G.-S. Cheon and H. Kim, Simple proofs of open problems about the structure of involutions in the Riordan group. *Linear Algebra Appl.*, 428 (2008), 930–940.

[8] G.-S. Cheon, H. Kim, L. W. Shapiro, Riordan group involutions. *Linear Algebra Appl.*, 428 (2008), 941–952.

[9] A. Edelman and G. Strang, Pascal matrices, *American Math. Monthly*, 111 (2004), No. 3, 189–197.

[10] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, New York, 1989.

[11] T.-X. He, L. C. Hsu, and X. Ma, On an extension of Riordan arrays with an application to obtaining convolution-type identities, *European J. of Combin.*, 42 (2014), 112–134.

[12] T.-X. He, L. C. Hsu, P.J.-S. Shiue, The Sheffer group and the Riordan group, *Discrete Appl. Math.*, 155(2007), pp.1895–1909.

[13] T.-X. He, J. H.-C. Liao, and P. J.-S. Shiue, The Pascal matrix function and its applications to Bernoulli numbers and Bernoulli polynomials and Euler numbers and Euler polynomials, *J. Combin. Number Theory*, 6 (2015), No. 3, 189-207.

[14] T.-X. He and R. Sprugnoli. Sequence Characterization of Riordan Arrays, *Discrete Math.*, 309 (2009), 3962–3974.

[15] T. X. He and L. Shapiro, Row sums and alternating sums of Riordan arrays, manuscript, 2015.

[16] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (Eds), *NIST Handbook of Mathematical Functions*, U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010.

[17] H. Pan and Z.-W. Sun, New identities involving Bernoulli and Euler polynomials. *J. Combin. Theory Ser. A*, 113 (2006), no. 1, 156–175.

[18] L. W. Shapiro, A survey of the Riordan group, Talk at a meeting of the American Mathematical Society, Richmond, Virginia, 1994.

[19] L. W. Shapiro, Some open questions about random walks, involutions, limiting distributions and generating functions, *Advances in Applied Math.*, 27 (2001), 585–596.

[20] L. W. Shapiro, Bijections and the Riordan group, *Theoretical Computer Science*, 307 (2003), 403–413.

[21] L. V. Shapiro, S. Getu, W. J. Woan and L. Woodson, The Riordan group, *Discrete Appl. Math.* 34(1991) 229–239.

[22] Z.-W. Sun, Invariant sequences under binomial transformation, *Fibonacci Quart.* 39 (2001), 324-333.

[23] Z.- W. Sun, Combinatorial identities in dual sequences. *European J. Combin.* 24 (2003), no. 6, 709–718.

[24] H. S. Wilf, *Generatingfunctionology*, Acad. Press, New York, 1990.

[25] Z. Zhang and M. Liu, An extension of the generalized Pascal matrix and its algebraic properties. *Linear Algebra Appl.* 271 (1998), 169–177.

[26] X. Zhao and T. Wang, Some identities related to reciprocal functions. *Discrete Math.* 265 (2003), no. 1-3, 323–335.