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Dedicated to Professor L. C. Hsu on the occasion of his 95th birthday

Abstract

A Pascal matrix function is introduced by Call and Velleman in [3]. In this paper, we will use the function to give a unified approach in the study of Bernoulli numbers and Bernoulli polynomials. Many well-known and new properties of the Bernoulli numbers and polynomials can be established by using the Pascal matrix function. The approach is also applied to the study of Euler numbers and Euler polynomials.

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Key Words and Phrases: Pascal matrix, Pascal matrix function, Bernoulli number, Bernoulli polynomial, Euler number, Euler polynomial.

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1 Introduction

A large literature scatters widely in books and journals on Bernoulli numbers B_n, and Bernoulli polynomials $B_n(x)$. They can be studied by means of the binomial expression connecting them,

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \quad n \geq 0. \tag{1}$$

The study brings consistent attention of researchers working in combinatorics, number theory, etc. In this paper, we will establish a matrix form of the above binomial expression using the following generalized Pascal matrix $P[x]$ introduced by Call and Velleman in [3]:

$$P[x] := \left(\binom{n}{k} x^{n-k}\right)_{n,k\geq 0}, \tag{2}$$

which we also call the Pascal matrix function and $P[1]$ is clearly the classical Pascal matrix. Many well-known and new properties of Bernoulli numbers and Bernoulli polynomials can be obtained through this matrix approach. In fact, by denoting $B(x) = (B_0(x), B_1(x), \ldots)^T$ and $B = B(0) = (B_1, B_2, \ldots)^T$, we may write (1) as

$$B(x) = P[x]B(0). \tag{3}$$

$P[x]$ is a homomorphic mapping from $\mathbb{R}$ to the infinite lower triangular matrices with real entries due to Theorem 2 shown [3]:

Theorem 1.1 *[3] (Homomorphism Theorem) For any $x, y \in \mathbb{R}$,*

$$P[x + y] = P[x]P[y]. \tag{4}$$

By noting

$$\binom{i}{k}\binom{k}{j} = \binom{i}{j}\binom{i-j}{k-j}, \tag{5}$$

one immediately find the (i, j) entry of the right-hand side of (4) can be written as

$$\sum_{k=j}^{i} \binom{i}{k} x^{i-k} \binom{k}{j} y^{k-j}, \tag{6}$$

while the (i,j) entry of the left-hand side of (4) is

$$\binom{i}{j}(x+y)^{i-j} = \binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}y^k x^{i-j-k} = \sum_{k=j}^{i}\binom{i}{j}\binom{i-j}{k-j}x^{i-k}y^{k-j},$$

which is exactly (6) completing the proof of Theorem 1.1. In [4], identity (5) is used to rederive several known properties and relationships involving the Bernoulli and Euler polynomials. In this paper, a unified approach by means of $P[x]$ is presented in the study of Bernoulli polynomials and numbers and Euler polynomials and numbers.

For any square matrix A, the exponential of A is defined as the following matrix in a series form:

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{k=0} \frac{1}{k!}A^k.$$

We say the series converges for every A if each entry of e^A converges. From [13] and the definition of e^A, we have

$$\begin{aligned} &e^{(\alpha+\beta)A} = e^{\alpha A}e^{\beta A} \; for\ any\ \alpha,\beta \in \mathbb{R}, \\ &(e^A)^{-1} = e^{-A}, \\ &D_t e^{tA} = Ae^{tA} = e^{tA}A. \end{aligned} \tag{7}$$

For $n \in \mathbb{N} \cup 0$, denote $P_n[x] = (\binom{i}{j}x^{i-j})_{0\le i,j\le n}$. [3] shows there exists a unique $H_n = (h_{i,j})_{0\le i,j\le n}$ such that $P_n[x] = e^{xH_n}$. In fact, from (7), one may have

$$H_n = D_x e^{xH_n}\big|_{x=0} = D_x P_n[x]\big|_{x=0}. \tag{8}$$

Denote $H = (h_{i,j})_{0\le i,j}$. Then (4) of Theorem 1.1 can be proved in one line:

$$P[x+y] = e^{(x+y)H} = e^{xH}e^{yH} = P[x]P[y].$$

It is easy to see that the entry $h_{i,j}$ of H and H_n, $n = 0, 1, \ldots$, are

$$h_{i,j} = (H)_{i,j} = \begin{cases} i & if\ i = j+1, \\ 0 & otherwise. \end{cases}, \tag{9}$$

and the entry of H^k are

$$(H^k)_{i,j} = \begin{cases} \frac{i!}{j!} & if\ i = j + k, \\ 0 & otherwise. \end{cases} \tag{10}$$

Hence, $H_n^k = 0$ if $k \geq n$.

In next section, we will define new matrix functions from $P_n[x]$, which and the matrix relationship (3) between the Bernoulli numbers and the Bernoulli polynomials through $P[x]$ will be used to re-build some well-known properties of the Bernoulli numbers and the Bernoulli polynomials in Section 3. Finally, in Section 4, a similar approach is used to study the Euler numbers and Euler polynomials and their relationships with Bernoulli numbers and Bernoulli polynomials.

2 Matrix functions $L_n[x]$ and $\bar{L}_n[x]$ related to $P_n[x]$

We now present the integrals of $P[x]$. From [1] (on Page 240), there holds expression of the linear mapping $L_n : P_n[x] \mapsto \int_0^1 P_n[x]dx$ from matrix functions to matrices as

$$L_n = \int_0^1 P_n[x]dx = \sum_{k=0}^{n-1} \frac{H_n^k}{(k+1)!}, \tag{11}$$

which is simply from the fact $P_n[x] = e^{xH_n}$ and the last formula in (7). Notice that L is a nonsingular lower triangular matrix with all main diagonal entries equal to 1. We now extend matrices (11) to matrix functions

$$L_n[x] := \int_0^x P_n[t]dt = \int_0^x e^{tH_n}dt = \sum_{k=0}^{n-1} \frac{H_n^k}{(k+1)!} x^{k+1}, \tag{12}$$

which can be considered a mapping $L_n[x] : P_n[t] \mapsto \int_0^x P_n[t]dt$ associated with any $x \in \mathbb{R}$. It is obvious $L_n[1] \equiv L_n$ defined in (11). From (10) and noting $k = i - j$, the entries of $L_n[x]$ can be given as

$$(L_n[x])_{ij} = \begin{cases} \frac{1}{i-j+1}\binom{i}{j} x^{i-j+1}, & if\ i \geq j \geq 0, \\ 0, & otherwise. \end{cases} \tag{13}$$

Particularly,

$$(L_n)_{ij} = \begin{cases} \frac{1}{i-j+1}\binom{i}{j}, & if\ i \geq j, \\ 0, & otherwise. \end{cases} \tag{14}$$

We now give some properties of $L_n[x]$ and its relationships with $P_n[x]$.

Proposition 2.1 *Let matrices H_n, $P_n[x]$, and $L_n[x]$ be defined as above. Then there holds*

$$H_n L_n[x] = L_n[x] H_n = P_n[x] - I_n. \tag{15}$$

Proof. By transferring the indexes, we have

$$H_n L_n[x] = L_n[x] H_n = \sum_{k=1}^{n} \frac{H_n^k}{k!} x^k$$
$$= \sum_{k=0}^{n} \frac{H_n^k}{k!} x^k - I_n = P_n[x] - I_n.$$

■

Inspired by [1], we define

$$\begin{aligned} \bar{L}_n &= D_n(-1) L_n D_n(-1)^{-1}, \quad and \\ \bar{L}_n[x] &= D_n(-1) L_n[x] D_n(-1)^{-1}, \end{aligned} \tag{16}$$

where $D_n(-1) = diag(1, -1, 1, -1, \dots, (-1)^{n-1})$. Hence, $\bar{L}_n = \bar{L}_n[1]$.

Note. Since $D_n(-1) H_n D_n(-1)^{-1} = -H_n$, we have

$$\begin{aligned} D_n(-1) P_n[x] D_n(-1) &= \sum_{k=0}^{n-1} \frac{D_n(-1) H_n^k D_n(-1)^{-1}}{k!} x^k \\ &= \sum_{k=0}^{n-1} \frac{(D_n(-1) H_n D_n(-1))^k}{k!} x^k \\ &= \sum_{k=0}^{n-1} \frac{(-H_n)^k}{k!} x^k = \sum_{k=0}^{n-1} \frac{(H_n)^k}{k!} (-x)^k = P_n[-x]. \end{aligned}$$

The relationship between $L_n[x]$ and $\bar{L}_n[x]$ can be shown below.

Proposition 2.2 *Let $L_n[x]$ and $\bar{L}_n[x]$ be defined as before. Then there hold*

$$\bar{L}_n[x] = -L_n[-x], \quad L_n[x] = -\bar{L}_n[-x], \quad \text{and} \tag{17}$$
$$\bar{L}_n = -L_n[-1], \quad L_n = -\bar{L}_n[-1]. \tag{18}$$

Proof. From (16),

$$\bar{L}_n[x] = D_n(-1)\left(\int_0^x P_n[t]dt\right)D_n(-1)^{-1} = \int_0^x P_n[-t]dt = -\int_0^{-x} P_n[t]dt,$$

which implies (17), and (18) follows as well.

■

Using the above relationship between $\bar{L}_n[x]$ and $L_n[x]$ and Proposition 2.1, we immediately have

Proposition 2.3 *Let H_n, $P_n[x]$, and $\bar{L}_n[x]$ be defined as before. Then*

$$\bar{L}_n[x] = \sum_{k=0}^{n-1} \frac{(-H_n)^k}{(k+1)!} x^{k+1}, \quad \text{and} \tag{19}$$
$$-H_n\bar{L}_n[x] = -\bar{L}_n[x]H_n = P[-x] - I_n, \tag{20}$$
$$\sum_{k=0}^{n-1} \frac{(-H_n)^k}{(k+1)!}(-x)^{k+1} = -\sum_{k=0}^{n-1} \frac{H_n^k}{(k+1)!} x^{k+1}. \tag{21}$$

Proof. From (16) and (17)

$$\begin{aligned} \bar{L}_n[x] &= D_n(-1)L_n[x]D_n(-1)^{-1} \\ &= \sum_{k=0}^{n-1} \frac{(D_n(-1)H_nD_n(-1))^k}{(k+1)!} x^{k+1} \\ &= \sum_{k=0}^{n-1} \frac{(-H_n)^k}{(k+1)!} x^{k+1}, \end{aligned}$$

i.e., formula (19). Again, (17) and (15) yield

$$-H_n\bar{L}_n[x] = H_nL_n[-x] = P_n[-x] - I_n$$

and

$$-\bar{L}_n[x]H_n = L_n[-x]H_n = P_n[-x] - I_n.$$

Substituting $x = -x$ into (12) and noting (17) and (19), we have

$$\sum_{k=0}^{n-1} \frac{(-H_n)^k}{(k+1)!}x^{k+1} = \bar{L}_n[x] = -L_n[-x] = -\sum_{k=0}^{n-1} \frac{H_n^k}{(k+1)!}(-x)^{k+1},$$

which implies (21).

■

Corollary 2.4 *Let H_n, $P_n[x]$, $L_n[x]$, and $\bar{L}_n[x]$ be defined as before. Then*

$$\begin{aligned} &L_n[x] = P_n[x]\bar{L}_n[x], \quad \textit{or equivalently} \\ &L_n[x]D_n(-1) = P_n[x]D_n(-1)L_n[x]. \end{aligned} \tag{22}$$

Proof. For $k = 1, 2, \ldots, n$, we have

$$P_n[x]\bar{L}_n[x]H_n^k = P_n[x](-P_n[-x]+I_n)H_n^{k-1} = (P_n[x]-I_n)H_n^{k-1} = L_n[x]H_n^k,$$

which imply (22) from the structure of H_n^k.

■

Proposition 2.5 *Let $P_n[x]$, $L_n[x]$, and $\bar{L}_n[x]$ be defined as before. Then*

$$\begin{aligned} L_n[y] - L_n[x] &= P_n[x]L_n[y-x] = L_n[y-x]P_n[x], \\ \bar{L}_n[y] - \bar{L}_n[x] &= P_n[-y]L_n[y-x] = L_n[y-x]P_n[-y]. \end{aligned} \tag{23}$$

Proof. The first formula of (8) can be proved as follows, and the second formula is from the first formula and the relationship between $L_n[x]$ and $\bar{L}_n[x]$ shown in (17). In fact,

$$L_n[y] - L_n[x] = \int_x^y P_n[t]dt = \int_0^{y-x} P_n[t+x]dt = P_n[x]\int_0^{y-x} P_n[t]dt, \tag{24}$$

which completes the proof.

■

Proposition 2.6 *Let H_n, $L_n[x]$, and $\bar{L}_n[x]$ be defined as before. Then for $a \neq 0$,*

$$\begin{aligned} D_n(a)L_n[x]D_n\left(\frac{1}{a}\right) &= \frac{1}{a}L_n[ax], \\ D_n(a)\bar{L}_n[x]\left(\frac{1}{a}\right) &= \frac{1}{a}\bar{L}_n[ax]. \end{aligned} \tag{25}$$

Proof. For $a \neq 0$, the left-hand side of the first equation of (25) can be changed to

$$D_n(a)L_n[x]D_n\left(\frac{1}{a}\right) = \sum_{k=0}^{n-1} \frac{(aH_n)^k}{(k+1)!}x^{k+1}$$

$$= \frac{1}{a}\sum_{k=0}^{n-1} \frac{H_n^k}{(k+1)!}(ax)^{k+1},$$

which proves the proposition.

■

As $n \to \infty$, matrix functions of nth order, $L_n[x]$ and $\bar{L}_n[x]$, can be extended to the infinite lower triangular matrix functions $L[x] : P[x] \mapsto \int_0^x P[t]dt$ and $\bar{L}[x] : P[x] \mapsto -\int_0^{-x} P[t]dt (= \int_0^x P[-t]dt)$, respectively. The properties shown in Propositions 2.1-2.6 can be extended to the case of infinite series associated with certain convergence conditions accordingly.

3 Pascal matrix function applied to Bernoulli numbers and Bernoulli polynomials

Since

$$P[-x]P[x] = P[x]P[-x] = P[0] = I, \tag{26}$$

there exists

$$P[x]^{-1} = P[-x].$$

Thus, (3) implies the following inverse relationship between $B(x)$ and B:

Theorem 3.1 *Let $B(x) = (B_0(x), B_1(x), \ldots)^T$ and $B = (B_0, B_1, \ldots)^T$, and let $P[x]$ be defined as (2). Then there holds a pair of inverse relationship*

$$B(x) = P[x]B(0) \qquad B = B(0) = P[-x]B(x). \tag{27}$$

And the latter can be presented as

$$\begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_n \\ \vdots \end{bmatrix} = \begin{bmatrix} \binom{0}{0} & 0 & \cdots & 0 & 0 & \cdots \\ \binom{1}{0}(-x) & \binom{1}{1} & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n}{0}(-x)^n & \binom{n}{1}(-x)^{n-1} & \cdots & \binom{n}{n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} B_0(x) \\ B_1(x) \\ \vdots \\ B_n(x) \\ \vdots \end{bmatrix}, \tag{28}$$

which implies

$$B_n = \sum_{k=0}^{n} \binom{n}{k} (-x)^{n-k} B_k(x) \tag{29}$$

for $n = 0, 1, \ldots$.

The proof of the theorem is straightforward from (26) and is omitted. Similarly, we have

Proposition 3.2 *Let $B(x)$ and $P[x]$ be defined as above. Then there hold*

$$B_n(x+y) = (P[x]B(y))_n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k(y),$$
$$B_n(y) = P_n[-x]B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} (-x)^{n-k} B_k(x+y). \quad (30)$$

We now consider recursive relations of $B_n(x)$.

Proposition 3.3 *Let $B(x)$, B, and $P[x]$ be defined as above, and let $\eta(x) = (0, 1, x, x^2, \ldots)^T$. Then there hold*

$$B(x+1) - B(x) = P[x](B(1) - B(0)) = P[x]\eta(0), \quad (31)$$
$$(P[1] - I)B(x) = P[x]\eta(0), \quad (32)$$

which imply

$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad \textit{and} \quad (33)$$
$$\sum_{k=0}^{n-1} \binom{n}{k} B_k(x) = nx^{n-1}, \quad (34)$$

respectively.

Proof. It is well-known that $B_n(1) - B_n(0) = (\delta_{n,1})^T$ for $n = 0, 1, \ldots,$ where δ is the Kronecker symbol. Hence, by using $\eta(x) = (0, 1, x, \ldots)^T$, we may write

$$B(1) - B(0) = \eta(0).$$

Hence,

$$B(x+1) - B(x) = P[x](B(1) - B(0)) = P[x]\eta(0) = (0, 1, 0, \ldots)^T, \quad (35)$$

which implies (33):

$$B_n(x+1) - B_n(x) = nx^{n-1}.$$

On the other hand, we may write $B_n(x+1) - B_n(x)$ as

$$B(x+1) - B(x) = P[x+1]B - P[x]B = P[1]P[x]B - P[x]B = (P[1]-I)B(x).$$

Thus (32) follows from (31). Since the $n+1$st row of the matrix on the rightmost side is

$$\left(\binom{n}{0}B_0(x), \binom{n}{1}B_1(x), \ldots, \binom{n}{n-1}B_{n-1}(x), 0, \ldots\right)^T,$$

by comparing the components on the two sides of (32), we may have (34).

■

Substituting $x = 0$ into (32) and (34), we immediately have

$$\begin{aligned} &(P[1] - I)B = \eta(0), \\ &\sum_{k=0}^{n-1} \binom{n}{k} B_k = \delta_{1,n}, \quad n = 0, 1, 2, \ldots. \end{aligned}$$

From expressions (9) and (10), one may obtain the differential formulas for $B_n(x)$ readily.

Proposition 3.4 *Let $B(x)$ and $P[x]$ be defined as above. Then there hold*

$$B_n^{(k)}(x) = k!\binom{n}{k}B_{n-k}(x). \tag{36}$$

Particularly, for $k = 1$

$$B_n'(x) = nB_{n-1}(x).$$

Proof. From (10), we have

$$D_x^k B(x) = (D_x^k P[x])B = (D_x^k e^{xH})B = H^k P[x]B = H^k B(x).$$

Comparing the nth components on both side vectors of the above equation, we obtain (36).

■

From the previous section, we have seen that the operator $L_n : P_n \mapsto \int_0^1 P_n[t]dt$ and its extension $L_n[x] : P_n \mapsto \int_0^x P_n[t]dt$ as well as $\bar{L}_n$ and $\bar{L}_n[x]$ have closed relationships with $P_n[x]$, and, hence, they can be used to derive integral properties of Bernoulli numbers and Bernoulli polynomials accordingly. Denote $b_n(x) = (B_0(x), B_1(x), \ldots, B_{n-1}(x))$ and $b_n = (B_0, B_1, \ldots, B_{n-1})$. From the definition of L_n shown in (11) and Theorem 3.1 (see also [1]),

$$L_n b_n = \int_0^1 P_n[t]dt b_n = \int_0^1 b_n(t)dt = e_0, \tag{37}$$

where $e_0 = (1, 0, \ldots, 0)$, the first element of the standard basis of $\mathbb{R}^n$, and the last step comes from $\int_0^1 B_k(x)dx = \delta_{k,0}$. In [1], a sequence of mappings, $\{\hat{L}_n\}_{n\geq 0}$, from e_0 to b_n for any $n \geq 0$ is defined as

$$\hat{L}_n = \sum_{k=0}^{n-1} \frac{B_k}{k!} H_n^k. \tag{38}$$

Thus, by using $k!e_k = H_n^k e_0$, we have (see also [1])

$$\hat{L}_n e_0 = \sum_{k=0}^{n-1} B_k e_k = b_n, \tag{39}$$

where e_k is the kth element of the standard basis of $\mathbb{R}^n$. From the pair of relations (37) and (39), we may say Ł$_n$ and $\hat{L}_n$ are inverse each other in the sense of $(L_n\hat{L}_n)e_0 = e_0$ and $(\hat{L}_n L_n)b_n = b_n$. In addition, denote $\xi_n(x) = (1, x, \ldots, x^{n-1})^T$, there hold

$$\begin{aligned} & b_n(x) = P_n[x]b_n = P[x]\hat{L}_n e_0 = \hat{L}_n P_n[x]e_o \\ = \quad & \hat{L}_n \sum_{k=0}^{n-1} \frac{x^k}{k!} H^k e_0 = \hat{L}_n \sum_{k=0}^{n-1} x^k e_k = \hat{L}_n \xi_n(x) \end{aligned} \tag{40}$$

and

$$\begin{aligned} L_n b_n(x) &= L_n P_n[x] b_n = P_n[x] L_n b_n \\ &= \sum_{k=0}^{n-1} \frac{x^k}{k!} H_n^k e_0 = \sum_{k=0}^{n-1} x^k e_k = \xi_n(x). \end{aligned} \tag{41}$$

(40) is given in [1] as (42). From the pair relations of (40) and (41), we may also say Ł$_n$ and $\hat{L}_n$ are inverse each other in the sense of $(L_n \hat{L}_n)\xi_n(x) = \xi_n(x)$ and $(\hat{L}_n L_n) b_n(x) = b_n(x)$. By using (10), we also have that the entries of $\hat{L}_n$ are

$$\left(\hat{L}_n\right)_{i,j} = \begin{cases} B_{i-j}\binom{i}{j} & if\ n-1 \geq i \geq j \geq 0, \\ 0 & otherwise. \end{cases} \tag{42}$$

From (40) and (14) we immediately obtain

Proposition 3.5 *Let $B_n(x)$ be the nth Bernoulli polynomial. Then*

$$x^n = \sum_{k=0}^{n} \frac{1}{n-k+1} \binom{n}{k} B_k(x).$$

Theorem 3.6 *For any $n \geq 1$, there holds*

$$\begin{aligned} b_n(x) &= H_n L_n[x] b_n(t) + b_n = -H_n \bar{L}_n[-x] b_n(t) + b_n \\ &= H_n \int_0^x b_n(t) dt + b_n \end{aligned} \tag{43}$$

and then

$$B_n(x) = n \int_0^x B_{n-1}(t) dt + B_n. \tag{44}$$

Proof. From (27) of Theorem 3.1 and (15) of Proposition 2.1, we have

$$b_n(x) - b_n = (P_n[x] - I_n) b_n = H_n L_n[x] b_n = H_n \int_0^x P_n[t] b_n dt,$$

which implies (43) and (44).

■

Theorem 3.7 *For any $n \geq 0$, there holds*

$$\int_x^{x+1} b_n(t)dt = \xi_n(x) \tag{45}$$

and then

$$\int_x^{x+1} B_n(t)dt = x^n. \tag{46}$$

Proof. From Proposition 2.5, we may have

$$\begin{aligned} & \int_x^{x+1} b_n(t)dt = \int_0^{x+1} b_n(t)dt - \int_0^x b_n(t)dt \\ = \ & (L_n[x+1] - L_n[x])b_n = P_n[x]L_nb_n \\ = \ & L_nP_n[x]b_n = L_nb_n(x) = \xi_n(x), \end{aligned}$$

where the last step is from (41), which completes the proof.

■

Theorem 3.8 *For any $n \geq 0$, there holds*

$$b_n(y) - b_n(x) = H\int_x^y b_n(t)dt \tag{47}$$

and then

$$\int_x^y B_n(t)dt = \frac{1}{n+1}(B_{n+1}(y) - B_{n+1}(x)). \tag{48}$$

Proof. From (27) of Theorem 3.1 and (15) of Proposition 2.1, we have

$$\begin{aligned} & b_n(y) - b_n(x) = (P_n[y] - P_n[x])b_n \\ = \ & [(P_n[y] - I_n) - (P_n[x] - I_n)]\, b_n \\ = \ & (H_nL_n[y] - H_nL_n[x])\, b_n \\ = \ & H\left(\int_0^y P_n[t]dt - \int_0^x P_n[t]dt\right) b_n \\ = \ & H_n\int_x^y P_n[t]dtb_n = H_n\int_x^y b_n(t)dt, \end{aligned}$$

completing the proof of the theorem.

■

Theorem 3.6 can be considered as a special case of Theorem 3.8.

Remark 3.1 It should be noted that the Pascal matrix function $P[x]$ we used in the study of properties of Bernoulli number set B and Bernoulli polynomial set $B(x)$ is a type of globe approach. Some properties based on some internal relationships of B and $B(x)$ may not be obtained by using our approach. For instance, the matrix form of the well-known property

$$B_n(x) = 2^{n-1}\left(B_n\left(\frac{x}{2}\right) + B_n\left(\frac{x+1}{2}\right)\right) \tag{49}$$

is

$$B(x) = \frac{1}{2}D(2)\left(B\left(\frac{x}{2}\right) + B\left(\frac{x+1}{2}\right)\right),$$

or equivalently,

$$P[x]B = \frac{1}{2}P[x](I+P)D(2)B.$$

However, $P[x]$ and $(1/2)P[x](I+P)D(2)$ are different because $(1/2)(I+P)D(2) \neq I$.

4 Pascal matrix function and Euler numbers and Euler polynomials

Euler polynomials $E_n(x)$ can be presented in terms of Euler numbers as

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \left(x - \frac{1}{2}\right)^{n-k} \frac{E_k}{2^k}, \quad n = 0, 1, \ldots, \tag{50}$$

where $E_n = 2^n E_n(1/2)$. Denote $E(x) = (E_0(x), E_1(x), \ldots)^T$ and $E = (E_0, E_1, \ldots)^T$. By making use of the Pascal matrix, we may write (50) as a matrix form

$$E(x) = P\left[x - \frac{1}{2}\right] D\left[\frac{1}{2}\right] E = P\left[x - \frac{1}{2}\right] E\left(\frac{1}{2}\right), \tag{51}$$

where

$$D[t] = diag[1, t, t^2, \ldots].$$

Since $E_n(1) + E_n(0) = 2\delta_{0,n}$, by denoting $\xi(x) = (1, x, x^2, \ldots)^T$ and using the homomorphism of mapping P , we have

Proposition 4.1 *Let $E_n(x)$, E, and $P[x]$ be defined as before. Then there hold*

$$E(x) = P[x]E(0), \tag{52}$$
$$E[x+1] + E[x] = 2P[x]\xi(0), \tag{53}$$
$$(P[1] + I)E(x) = 2P[x]\xi(0), \tag{54}$$
$$E[x+y] = P[x]E(y). \tag{55}$$

In addition, (53), (54), *and* (55) *imply*

$$E_n(x+1) + E_n(x) = 2x^n, \tag{56}$$
$$\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) = 2x^n, \quad \text{and} \tag{57}$$
$$E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} E_k(y) \tag{58}$$

respectively.

Proof. (52) is from (51) due to

$$E(x) = P\left[x - \frac{1}{2}\right] E\left(\frac{1}{2}\right) = P[x]P\left[-\frac{1}{2}\right] E\left(\frac{1}{2}\right) = P[x]E(0).$$

Because of

$$E(x+1) + E(x) = P[x](E(1) + E(0)),$$

we obtain (53). On the other hand,

$$E(x+1) + E(x) = P[x+1]E(0) + P[x]E(0) = (P[1] + I)P[x]E(0).$$

Thus, (54) follows from (53). From (52), we can find (55)

$$E(x+y) = P[x+y]E(0) = P[x]P[y]E(0) = P[x]E(y).$$

(56), (57), and (58) are from the comparison of components of two side vectors of (53), (54), and (55), respectively.

■

We now discuss the relationships between Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$. Denote $e_n(x) = (E_0(x), E_1(x), \ldots, E_{n-1}(x))^T$. From (54), we have

$$(P_n[1] + I_n)e_n(x) = 2P_n\xi(0) = 2\xi_n(x),$$

or equivalently,

$$e_n(x) = 2(P_n + I_n)^{-1}\xi_n(x), \tag{59}$$

which and formulas (38) and (40) will be used rederive some relationships between Bernoulli polynomials and Euler polynomials shown in [4] and [11].

Theorem 4.2 *Let $b_n(x)$ and $e_n(x)$ be defined as above. Then*

$$D_n(2)b_n(x) = \hat{L}_n e_n(2x). \tag{60}$$

Hence,

$$B_n(x) = 2^{-n}\sum_{k=0}^{n} B_{n-k}E_k(2x) \tag{61}$$

for all $n \geq 0$.

Proof. From the definition of $P_n[x]$, we have

$$D_n(2)P_n[x] = P_n[2x]D_n(2),$$

where $D_n(2)$ is the $n \times n$ diagonal matrix $diag(1, 2, 2^2, \ldots, 2^{n-1})$. Thus, using (27) yields

$$\begin{aligned} & D_n(2)b_n(x) = D_n(2)P_n[x]b_n = P_n[2x]D_n(2)b_n \\ = \; & P_n[2x]D_n(2)\hat{L}_n e_0 \end{aligned} \tag{62}$$

Substituting $y = 2$ and $x = 1$ into (23) of Proposition 2.5, we obtain

$$L_n[2] = P_n[1]L_n[2-1] + L_n[1] = (P_n + I_n)L_n,$$

where $L_n[2]$ can be written as $2D(2)L_nD(1/2)$ due to (25) of Proposition 2.6. Thus,

$$2D(2)L_nD(1/2) = (P_n + I_n)L_n,$$

or equivalently,

$$2D(2)L_n = (P_n + I_n)L_nD(2).$$

Applying the both sides operators of the last equation to $\hat{L}_n e_0$ yields

$$2D(2)(L_n\hat{L}_n)e_0 = (P_n + I_n)L_nD(2)\hat{L}_n e_0,$$

which implies

$$\begin{aligned} & D(2)\hat{L}_n e_0 = 2\hat{L}_n(P_n + I_n)^{-1}D(2)e_0 \\ = \;& 2\hat{L}_n(P_n + I_n)^{-1}e_0 \\ = \;& 2\hat{L}_n(P_n + I_n)^{-1}\xi_n(0) \\ = \;& \hat{L}_n e_n(0), \end{aligned}$$

where the last step is due to (59). By combining the above equation and (62), we finally have

$$D(2)b_n(x) = P_n[2x]D_n(2)\hat{L}_n e_0 = \hat{L}_n P_n[2x]e_n(0) = \hat{L}_n e_n(2x),$$

where (52) is applied in the last step, which completes the proof of the theorem with using (42).

■

Theorem 4.2 can be used to derive many relationships between Bernoulli polynomials and Euler polynomials. For instance, we have

Corollary 4.3 *Let $b_n(x)$ and $e_n(x)$ be defined as above. Then*

$$H_n e_n(x) = D_n(2)\left(b_n\left(\frac{x+1}{2}\right) - B\left(\frac{x}{2}\right)\right). \tag{63}$$

Hence,

$$E_n(x) = \frac{2^{n+1}}{n+1}\left(B_{n+1}\left(\frac{x+1}{2}\right) - B_{n+1}\left(\frac{x}{2}\right)\right) \tag{64}$$

for $n \geq 0$. *Furthermore,*

$$E_n(x) = \frac{2}{n+1}\left(B_{n+1}(x) - 2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right). \tag{65}$$

Proof. From (60), there holds

$$e_n(x) = L_n D_n(2) b_n\left(\frac{x}{2}\right).$$

Hence, noting (15) of Proposition 2.1 we have

$$\begin{aligned} H_n e_n(x) &= H_n L_n D_n(2) b_n\left(\frac{x}{2}\right) \\ &= (P_n - I_n) D_n(2) b_n\left(\frac{x}{2}\right) \\ &= D_n(2) P_n\left[\frac{1}{2}\right] b_n\left(\frac{x}{2}\right) - D_n(2) b_n\left(\frac{x}{2}\right) \\ &= D_n(2)\left(b_n\left(\frac{x+1}{2}\right) - b_n\left(\frac{x}{2}\right)\right). \end{aligned}$$

Combining (49) and (64), we may prove (65).

■

We can also derive the following known results (see, for example, [9]) by using our simple unified approach.

Corollary 4.4 *Let* $b_n(x)$ *and* $e_n(x)$ *be defined as above. Then*

$$e_n(2x) = 2D_n(2)\int_x^{x+1/2} b_n(t)dt. \tag{66}$$

Hence,

$$E_n(2x) = 2^{n+1}\int_x^{x+1/2} B_n(t)dt \tag{67}$$

for $n \geq 0$. *Particularly, Euler numbers*

$$E_n = 2^{2n+1} \int_{1/4}^{3/4} B_n(t)dt \tag{68}$$

for $n \geq 0$.

Proof. From (25) of Proposition 2.6, we have

$$L_n = 2D_n(2)L_n \left[\frac{1}{2}\right] D_n\left(\frac{1}{2}\right).$$

Thus, by using (60) and $L_n[x+1/2] - L_n[x] = P_n[x]L_n[1/2]$, we have

$$\begin{aligned} e_n(2x) &= L_n D_n(2) b_n(x) = 2D_n(2)L_n\left[\frac{1}{2}\right] P[x] b_n \\ &= 2D_n(2)\left(L_n\left[x+\frac{1}{2}\right] - L_n[x]\right) b_n \\ &= 2D_n(2)\int_x^{x+1/2} P_n[t]dt b_n \\ &= 2D_n(2)\int_x^{x+1/2} b_n(t)dt, \end{aligned}$$

which implies (67) and (68) after inputing $E_n = 2^n E_n(1/2)$.

■

Note. In Sun and Pan's work [10, 12], they have presented numerous new identities by using the finite difference calculus and differentiation. But we do not know whether our matrix method can be applied or not. Further investigations are needed.

5 An extension of Bernoulli polynomials and their application in numerical analysis

The properties of $B_n'(x) = nB_{n-1}(x)$ and $B_n(0) = B_n(1)$ for $n \geq 2$ make important rule in the applications of Bernoulli polynomials. For instance, from those properties we immediately have $\int_0^1 B_n(x)dx = 0$ for every $n \geq 1$, which can be used to construct numerical integration formula as follows (see a survey in [6]). Due to the property $B_n'(x) = nB_{n-1}(x)$, we may say that Bernoulli polynomials $B_n(x)$ are "deformations" of standard polynomials x^n.

We start from the *Daroux formula*. The Darboux formula, first given in 1876, is an expansion formula for an analytic function with Taylor formula as one of its special cases. Let $f(z)$ be normal analytic on the line connecting points a and z, and let $\phi(x)$ be a polynomial of degree n. Then there holds

$$\begin{aligned} & \tfrac{d}{dx}\textstyle\sum_{k=1}^n (-1)^k (z-a)^k \phi^{(n-k)}(x) f^{(k)}(a+x(z-a)) \\ = \; & -(z-a)\phi^{(n)}(x) f'(a+x(z-a)) \\ & +(-1)^n (z-a)^{n+1} \phi(x) f^{(n+1)}(a+x(z-a)), \end{aligned}$$

which can be proved using the product rule. By taking the integral in terms of x from 0 to 1 and noting that $\phi^{(n)}(x) = \phi^{(n)}(0)$, we obtain

$$\begin{aligned} & \phi^{(n)}(0)[f(z)-f(a)] \\ = \; & \sum_{k=1}^n (-1)^{k-1}(z-a)^k \left[\phi^{(n-k)}(1) f^{(k)}(z) - \phi^{(n-k)}(0) f^{(k)}(a)\right] \\ & +(-1)^n (z-a)^{n+1} \int_0^1 \phi(x) f^{(n+1)}(a+x(z-a))dx. \end{aligned} \tag{69}$$

Equation (69) is the Darboux formula. Now we will consider several of its special cases.

Let $\phi(x) = (x-1)^n$ in equation (69). Then $\phi^{(n)}(0) = n!$, $\phi^{(n-k)}(1) = 0$, and $\phi^{(n-k)}(0) = (-1)^k n!/k!$, for $1 \leq k \leq n$. This is the Taylor formula with a Cauchy integral form remainder.

Without a loss of generality, let the highest power term of $\phi(x)$ be x^n and $F(x)$ be the derivative of $f(x)$ ($F(x)$ is assumed to be n order continuously differentiable; then the following integral quadrature formula will be obtained by setting $a = 0$ and $z = 1$ in equation (69)

$$\begin{aligned} \int_0^1 F(x)dx &= \sum_{k=1}^n \frac{(-1)^{k-1}}{n!} \left[\phi^{(n-k)}(x) F^{(k-1)}(x)\right]_{x=0}^{x=1} \\ &+ \frac{(-1)^n}{n!} \int_0^1 \phi(x) F^{(n)}(x)dx. \end{aligned} \tag{70}$$

Particularly, let $\phi(x) = B_n(x)$, the nth Bernoulli polynomial, then (70) becomes

$$\begin{aligned}&\int_0^1 F(x)dx\\ = \quad &\frac{F(1)+F(0)}{2} + \sum_{k=2}^{n} \frac{(-1)^{k-1}}{k!} B_k(0)\left(F^{(k-1)}(1) - F^{(k-1)}(0)\right)\\ &+\frac{(-1)^n}{n!}\int_0^1 B_n(x)F^{(n)}(x)dx \end{aligned} \tag{71}$$

for $n \geq 2$, which is a numerical integration formula with the remainder associated with Bernoulli polynomials. For $n = 1$, the summation term in (71) vanishes, and its right-hand side can be shown as identical as its left-hand side by using integration by parts for the integral term. For $n = 0$, the right-hand side of (71) reduces to the same integral as its left-hand side. In this sense, (71) holds for all integers $n \geq 0$. To prove (71) from (70), we need use the fact $B_n(1) = B_n(0)$ for $n \geq 2$ and (36), i.e.,

$$B_n^{n-k} = \frac{n!}{k!} B_k(x).$$

In fact, equation (70) can easily be verified directly by applying n times integral by parts to its integral form remainder.

One may rewrite the above formula into a more general form (a suitable transformation of variable is needed).

$$\int_a^b F(x)dx = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{n!}\left[\phi^{(n-k)}(x)F^{(k-1)}(x)\right]_{x=a}^{x=b} + R_n \tag{72}$$

where remainder R_n is

$$R_n = \frac{(-1)^n}{n!}\int_a^b \phi(x)F^{(n)}(x)dx. \tag{73}$$

Equations (72) and (73) can be understood as the integral form of the Darboux formula. However, the applications of the Darboux formula to numerical integration were only given attention to after 1940 (see [8]).

The formula shown in equation (72) has two features: (1) Except for the remainder, the right-hand side of the equation consists of the values of the integrand and its derivatives at end points of the integral interval. Hence, the formula is a boundary type quadrature formula (BTQF). (2) By choosing a suitable weight function $\phi(t)$ in the remainder (73), we can make $R_n \equiv R_n(F)$ have the smallest possible estimate in various norms.

In addition, setting $\phi(t)$ as the nth order generalized Bernoulli polynomial $\tilde{B}_n(x)$ defined on interval $[a,b]$ in equation (72), we obtain the *Euler-Maclaurin type formula.* Here, $\tilde{B}_n(x)$, $x \in [a,b]$, $n \in \mathbb{N}_0$, are defined as

$$\tilde{B}_n(x) = (b-a)^n B_n\left(\frac{x-a}{b-a}\right). \tag{74}$$

Hence, for $[a,b] = [0,1]$ $\tilde{B}_n(x) = B_n(x)$. Denote $\tilde{B}(x) := (\tilde{B}_0(x), \tilde{B}_1(x), \ldots)^T$ for $x \in [a,b]$. Particularly, for $[a,b] = [0,1]$ $\tilde{B}(x) = B(x)$ defined before. From (3), we have

$$\tilde{B}(x) = P\left[\frac{x-a}{b-a}\right] B. \tag{75}$$

Using (75), we may transfer the properties of $B_n(x)$ to their generalization $\tilde{B}_n(x)$. For instance,

$$\tilde{B}_n'(x) = n(b-a)^{n-1} B_{n-1}\left(\frac{x-a}{b-a}\right) = n\tilde{B}_{n-1}(x), \quad n \geq 0, \tag{76}$$

and

$$\tilde{B}_n(b) = \tilde{B}_n(a), \quad n \geq 2. \tag{77}$$

From Theorem 3.1 and Proposition 3.3, we have

Proposition 5.1 *Let $\tilde{B}_n(x)$, $\tilde{B}(x)$, $B(x)$, and B be defined as before. Then there hold*

$$\tilde{B}(x) = P\left[\frac{x-a}{b-a}\right] B, \quad B = P\left[\frac{a-x}{b-a}\right]\tilde{B}(x), \tag{78}$$

$$\tilde{B}_n(x+b-a) - \tilde{B}_n(x) = n(b-a)(x-a)^{n-1}, \tag{79}$$

$$\sum_{k=0}^{n-1}\binom{n}{k}(b-a)^{n-k}\tilde{B}_n(x) = n(b-a)(x-a)^{n-1}. \tag{80}$$

Proof. The correction of the formulas (78)-(80) may be proved from (75) and (76) straightforwardly.

■

The analogies of some other properties of $B_n(x)$ for $\tilde{B}_n(x)$ can be derived similarly. Substituting $\phi(x) = \tilde{B}_n(x)$ into (72) and (73) and applying (75) and (76) yield the following Euler-Maclaurin type formula.

Proposition 5.2 *Let $\tilde{B}_n(x)$ be defined as before. Then for any analytic function $F(x)$ defined on $[a, b]$ there holds*

$$\begin{aligned}
&\int_a^b F(x)dx \\
= \quad & (b-a)\frac{F(1)+F(0)}{2} + \sum_{k=2}^{n}\frac{(-1)^{k-1}}{k!}\tilde{B}_k(a)\left(F^{(k-1)}(b) - F^{(k-1)}(a)\right) \\
& + \frac{(-1)^n}{n!}\int_a^b \tilde{B}_n(x)F^{(n)}(x)dx
\end{aligned} \tag{81}$$

for all $n \geq 2$.

Proof. Denote $g(x) = F(a + x(b-a)$. Then from (71) and noting $g^{(k)}(x) = (b-a)^k F^{(k)}(a + x(b-a))$ and $\tilde{B}_k(a) = \tilde{B}_k(b)$ for $n \geq 2$, we have

$$\begin{aligned}
\int_a^b F(x)dx &= (b-a)\int_0^1 g(x)dx \\
&= (b-a)\left[\frac{g(1)+g(0)}{2} + \sum_{k=2}^{n} \frac{(-1)^{k-1}}{k!} B_k(0)\left(g^{(k-1)}(1) - g^{(k-1)}(0)\right)\right. \\
&\quad \left. + \frac{(-1)^n}{n!}\int_0^1 B_n(x) g^{(n)}(x)dx\right] \\
&= (b-a)\frac{F(b)+F(a)}{2} + \sum_{k=2}^{n} \frac{(-1)^{k-1}}{k!} \tilde{B}_k(a)\left(F^{(k-1)}(b) - F^{(k-1)}(a)\right) \\
&\quad + (b-a)^{n+1}\frac{(-1)^n}{n!}\int_0^1 B_n(x) F^{(n)}(a + x(b-a))dx,
\end{aligned}$$

which implies (81).

■

Proposition 5.3 *Let $\tilde{B}_n(x)$ be defined as before. Denote the remainder of Euler-Maclaurin formula (81) by $R_n(F)$, i.e.,*

$$R_n(F) := \frac{(-1)^n}{n!}\int_a^b \tilde{B}_n(x) F^{(n)}(x)dx.$$

Then for any analytic function $F(x)$ defined on $[a, b]$ with $L_2([a, b])$ norm $\|F^{(n)}(x)\|_2 := \left(\int_a^b \left|F^{(n)}(x)\right|^2 dx\right)^{1/2} =: M_n$, Euler-Maclaurin formula (81) has error bound

$$|R_n(F)| \le M_n (b-a)^n \sqrt{\frac{(b-a)(-1)^{n-1}B_{2n}}{(2n)!}}, \tag{82}$$

where B_{2n} is the $2n$th Bernoulli number.

Proof. Using the Cauchy-Schwarz's inequality and formula (see [7])

$$\int_0^1 B_n(x) B_m(x) dx = (-1)^{n-1}\frac{m!n!}{(m+n)!}B_{n+m},$$

we have

$$
\begin{aligned}
|R_n(F)| &\le \frac{1}{n!}\int_a^b \left|\tilde{B}_n(x)F^{(n)}(x)\right| dx \\
&= \frac{M_n}{n!}\left(\int_a^b (b-a)^{2n}\left|B_n\left(\frac{x-a}{b-a}\right)\right|^2 dx\right)^{1/2} \\
&= \frac{M_n}{n!}\left(\int_0^1 (b-a)^{2n+1}\left|B_n(x)\right|^2 dx\right)^{1/2} \\
&= \frac{M_n\sqrt{b-a}}{n!}(b-a)^n\left(\frac{(n!)^2}{(2n)!}(-1)^{n-1}B_{2n}\right)^{1/2},
\end{aligned}
$$

which implies (82). Here, $(-1)^{n-1}B_{2n} > 0$ for $n = 1, 2, \ldots$.

■

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