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Abstract

In this paper we study a refinement equation of the form $\phi(x) = \sum_{n\in Z} h_n \phi(2x-n)$, where $\{h_n\}$ is a finitely supported sequence. Let the symbol of the sequnce $m(z) := \frac{1}{2}\sum_n h_n z^n$ take the form $m(z) = (\frac{1+z}{2})^N z^{k'} \sum_{j=0}^{k} a_j z^j$ where $\sum_j a_j = 1$ and $\sum_j (-1)^j a_j \neq 0$. Under the assumption $\left(\sum_{0\leq odd\, j\leq k} |a_j|\right)^2 + \left(\sum_{0\leq even\, j\leq k} |a_j|\right)^2 < 2^{2N-1}$ we show that the corresponding ϕ is in L_2. Then the B-spline type wavelet sequences that possess the largest possible regularities and required vanishing moments are characterized.

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1 Introduction

We start by setting some notations. We define a *low-pass filter* as

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$$m_0(\xi) = 2^{-1} \sum_n c_n e^{-in\xi}. \tag{1.1}$$

Here, we assume that only finitely many c_n are nonzero. However, some of our results can be extended to infinite sequences that have sufficient decay for $|n| \to \infty$. Next, we define ϕ by

$$\hat{\phi}(\xi) = \Pi_{j=1}^{\infty} m_0(2^{-j}\xi). \tag{1.2}$$

This infinite product converges only if $m_0(0) = 1$; i.e., if $\sum_n c_n = 2$. In this case, the infinite products in (2) converge uniformly and absolutely on compact sets, so that $\hat{\phi}$ is a well-defined C^∞ function. Obviously, $\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2)$, or, equivalently, $\phi(t) = \sum_n c_n \phi(2t - n)$ at least in the sense of distributions. From Lemma 3.1 in [2], ϕ has compact support.

We now consider the simplest possible masks $m_0(\xi)$ with the following form.

Definition 1.1 *Denote by Φ the set of all B-spline type scaling functions $\phi(t)$ that have Fourier transform $\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2)$. Here the filter*

$$m_0(\xi) = 2^{-1} \sum_n c_n e^{-in\xi}$$

is in the set M that contains all filters with the form

$$m_0^N(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^N F(\xi), \tag{1.3}$$

where

$$F(\xi) = e^{-ik'\xi} \sum_{j=0}^{k} a_j e^{-ij\xi}. \tag{1.4}$$

Here, all coefficients of $F(\xi)$ are real, $F(0) = 1$; N and k are positive integers; and $k' \in \mathbb{Z}$. Hence, the corresponding ϕ can be written as

$$\hat{\phi}(\xi) = \left(\frac{1 + e^{-i\xi/2}}{2}\right)^N F(\xi/2)\hat{\phi}(\xi/2). \tag{1.5}$$

Clearly, ϕ is a B-spline of order N if $F(\xi) = 1$. Thus, we call ϕ as defined by Def. 1.1 a B-spline type scaling function. The vanishing moments of ϕ are completely controlled by the exponents of its "spline factor," $\left(\frac{1+e^{-i\xi}}{2}\right)^N$. In addition, the regularity of ϕ is justified by the factors $F(\xi)$, and is independent of its vanishing moment.

Let both ϕ and $\tilde{\phi}$ be B-spline type scaling functions defined by Def. 1.1. Then

$$\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2), \quad \hat{\tilde{\phi}}(\xi) = \tilde{m}_0(\xi/2)\hat{\tilde{\phi}}(\xi/2),$$

or, equivalently,

$$\phi(t) = \sum_n c_n \phi(2t - n), \quad \tilde{\phi}(t) = \sum_n \tilde{c}_n \phi(2t - n),$$

at least in the sense of distributions. Here, both $m_0(\xi) \in M$ and $\tilde{m}_0(\xi) \in M$ satisfy Eqs. (1.3) and (1.4). Thus,

$$m_0(\xi) = m_0^N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N F(\xi) \tag{1.6}$$

$$\tilde{m}_0(\xi) = \tilde{m}_0^{\tilde{N}}(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^{\tilde{N}} \tilde{F}(\xi), \tag{1.7}$$

where

$$F(\xi) = e^{-ik'\xi} \sum_{j=0}^{k} a_j e^{-ij\xi} \quad \tilde{F}(\xi) = e^{-i\tilde{k}'\xi} \sum_{j=0}^{\tilde{k}} \tilde{a}_j e^{-ij\xi}.$$

From Lemma 3.1 in [2], ϕ and $\tilde{\phi}$ have compact support.

We also define the corresponding ψ and $\tilde{\psi}$ by

$$\hat{\psi}(\xi) = e^{i\xi/2}\overline{\tilde{m}_0(\xi/2+\pi)}\hat{\phi}(\xi/2), \quad \hat{\tilde{\psi}}(\xi) = e^{i\xi/2}\overline{m_0(\xi/2+\pi)}\hat{\phi}(\xi/2), \tag{1.8}$$

or, equivalently,

$$\psi(x) = \sum_n (-1)^{n-1}\tilde{c}_{-n-1}\phi(2x-n). \quad \tilde{\psi}(x) = \sum_n (-1)^{n-1} c_{-n-1}\phi(2x-n). \tag{1.9}$$

Since vanishing moment conditions $\int x^\ell \psi(x)dx = 0$, $\ell = 0, 1, \cdots, L$, are equivalent to $\frac{d^\ell}{d\xi^\ell}\hat{\psi}|_{\xi=0} = 0$, $\ell = 0, 1, \cdots, L$, we immediately know that the maximum number of vanishing moments for ψ and $\tilde{\psi}$ are $N-1$ and $\tilde{N}-1$, respectively. Therefore, the vanishing moments of ϕ and $\tilde{\phi}$ are completely determined by the exponent of their "spline factors" $\left(\frac{1+e^{-i\xi}}{2}\right)^N$ and $\left(\frac{1+e^{-i\xi}}{2}\right)^{\tilde{N}}$. In addition, as we pointed out before, the regularities of ϕ and $\tilde{\phi}$ will be justified by factors $F(\xi)$ and $\tilde{F}(\xi)$ and are independent of their vanishing moments.

Ingrid Daubechies, in her book "Ten Lectures on Wavelets," wrote: "What is more important, vanishing moments or regularity? The answer depends on the application, and is not always clear." She also pointed out that achieving higher regularity by increasing vanishing moments is not efficient, because 80% of the zero moments are wasted. In [He1998], we gave a method for constructing biorthogonal wavelets with the largest possible regularities and required vanishing moments based on the established condition. In the next section, we give a condition for the coefficients a_j ($j = 0, 1, \cdots, k$) of $F(\xi)$ such that the corresponding ϕ is in $L_2(\mathbb{R})$. By using the condition we improve the results in [6]. In Section 3, we will give a method for constructing a sequence of B-spline type scaling and wavelets from either an orthogonal sacaling function or a pair of biorthogonal scalings by using their convolutions with certain B-splines. In particular, if both the method and the lifting scheme of Sweldens (see [11]) are applied, then all of the pairs of biorthogonal spline type scaling functions shown in references [2] and [4] can be constructed from the Haar scaling function.

2 Construction of biorthogonal B-spline type wavelets

Denote

$$\sum_{k;even\ j} b_j \equiv \sum_{0\le even\ j\le k} b_j \qquad\qquad \sum_{k,odd\ j} b_j \equiv \sum_{0<odd\ j\le k} b_j.$$

We now establish the following main results of this section.

Theorem 2.1 *Let $\phi \in \Phi$ be defined as in Definition 1.1; i.e.,*

$$\phi = \Pi_{j=1}^{\infty} m_0^N(2^{-j}\xi),$$

where $m_0^N(\xi) \in M$ is defined by (1.3) and (1.4):

$$m_0^N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N F(\xi)$$

and $F(\xi) = e^{-ik'\xi}\sum_{j=0}^{k} a_j e^{-ij\xi}$, $N, k \in \mathbb{Z}_+$ and $k' \in \mathbb{Z}$, where $F(0) = 1$. If $F(\pi) \neq -1$ and the coefficients of $F(\xi)$ satisfy

$$a[k] \equiv \left(\sum_{k;even\ j} |a_j|\right)^2 + \left(\sum_{k;odd\ j} |a_j|\right)^2 < 2^{2N-1}, \tag{2.1}$$

then ϕ is in $L_2(\mathbb{R})$.

Remark 2.1 Consider the example shown in [2] (see PP. 542-544 in [2]), we find that the function $\phi = \Pi_{j=1}^{\infty} m_0^N(2^{-j}\xi)$ with the mask

$$m_0^1(\xi) = e^{i\xi}\left(\frac{1+e^{-i\xi}}{2}\right)\left(-\frac{1}{2} + 2e^{-i\xi} - \frac{1}{2}e^{-2i\xi}\right)$$

is not in $L_2(\mathbb{R})$.

It is easy to check that $m_0^1(\xi)$ satisfies

$$\left(\sum_{k;even\ j} |a_j|\right)^2 + \left(\sum_{k;odd\ j} |a_j|\right)^2 = 5 = 2^{2N-1} + 3,$$

where $N = 1$. This example shows that a necessary condition for $\phi \in \Phi$ being square integrable is

$$\left(\sum_{k;even\ j} |a_j|\right)^2 + \left(\sum_{k;odd\ j} |a_j|\right)^2 \geq 2^{2N-1} + 3,$$

where Φ is the set defined in Definition 1.1.

Condition (2.1) can be replaced by stronger conditions,

$$\sum_{j=0}^{k} |a_j| < 2^{N-\frac{1}{2}},$$

due to the obvious inequality $a^2 + b^2 \leq (a+b)^2$ for all $a, b \geq 0$.

Proof. It is sufficient to prove the boundedness of the following integral

$$\begin{aligned}
&\int_{|\xi|\geq\pi} |\hat{\phi}(\xi)|^2 d\xi \\
= & \sum_{\ell=1}^{\infty} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^\ell\pi} \Pi_{j=1}^{\infty} \left| \left(\frac{1+e^{-i2^{-j}\xi}}{2} \right)^N F(2^{-j}\xi) \right|^2 d\xi \\
= & \sum_{\ell=1}^{\infty} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^\ell\pi} \left| \frac{1-e^{-i\xi}}{i\xi} \right|^{2N} \Pi_{j=1}^{\infty} |F(2^{-j}\xi)|^2 d\xi \\
\leq & C \sum_{\ell=1}^{\infty} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^\ell\pi} \frac{1}{|\xi|^{2N}} \Pi_{j=1}^{\infty} |F(2^{-j}\xi)|^2 d\xi \\
\leq & C \sum_{\ell=1}^{\infty} \frac{1}{2^{2\ell N}} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^\ell\pi} \Pi_{j=1}^{\ell} |F(2^{-j}\xi)|^2 \Pi_{j=1}^{\infty} |F(2^{-\ell-j}\xi)|^2 d\xi \\
\leq & C \sum_{\ell=1}^{\infty} 4^{-\ell N} \int_{2^{\ell-1}\pi\leq|\xi|\leq 2^\ell\pi} \Pi_{j=1}^{\ell} |F(2^{-j}\xi)|^2 d\xi. \qquad (2.2)
\end{aligned}$$

We now prove the boundedness of the last integral in inequality (2.2). Denote

$$Tf(\xi) = \left| F\left(\frac{\xi}{2}\right) \right|^2 f\left(\frac{\xi}{2}\right) + \left| F\left(\frac{\xi}{2}+\pi\right) \right|^2 f\left(\frac{\xi}{2}+\pi\right).$$

Hence, for any 2π-periodic continuous function f, we have

$$\begin{aligned}
&\int_{2^{\ell-1}\pi\leq|\xi|\leq 2^\ell\pi} f(2^{-\ell}\xi) \Pi_{j=1}^{\ell} \left| F(2^{-j}\xi) \right|^2 d\xi \\
= & \int_{-\pi}^{\pi} T^\ell f(\xi) d\xi \leq \sqrt{2\pi} ||T^\ell f||_{L^2} \leq \sqrt{2\pi} ||f||_{L^2} ||T^\ell||.
\end{aligned}$$

Let $\rho(T)$ be the spectral radius of the opertor T. Since $F(0) = 1$ and $F(\pi) \neq -1$, it can be shown that $\rho(T) > 0$ (see also [1]). For every $\epsilon > 0$, there is an integer $\ell(\epsilon)$ such that

$$||T^\ell|| \leq (\rho(T)+\epsilon)^\ell, \qquad \ell > \ell(\epsilon).$$

It follows from (2.2) that

$$\int_{|\xi|\geq\pi} |\hat{\phi}(\xi)|^2 d\xi \leq C \sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell} ||T^\ell|| + C \sum_{\ell=\ell(\epsilon)+1}^{\infty} 4^{-N\ell} \left(\rho(T)+\epsilon\right)^\ell ;$$

so $\rho(T)$ must be estimated if we are to choose an $\epsilon > 0$ small enough for the series to converge. Regardless of how small an $\epsilon > 0$ is chosen, the contribution

$$C \sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell} ||T^\ell|| \leq C \sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell} ||T||^\ell$$

is finite, although possibly large.

To evaluate $\rho(T)$, we consider the conjugate operator, T^*, of T. It is easy to find that

$$T^* f(\xi) = 2 \left|F(\xi)\right|^2 f(2\xi).$$

In fact, for any 2π-periodic continuous functions f and g,

$$\begin{aligned}
&\langle Tf, g\rangle \\
= \;& \int_{-\pi}^{\pi} |F\left(\frac{\xi}{2}\right)|^2 f\left(\frac{\xi}{2}\right) \bar{g}(\xi) d\xi + \int_{-\pi}^{\pi} |F\left(\frac{\xi}{2}+\pi\right)|^2 f\left(\frac{\xi}{2}+\pi\right) \bar{g}(\xi) d\xi \\
= \;& 2\left(\int_{-\pi/2}^{\pi/2} |F\left(\xi\right)|^2 f\left(\xi\right) \bar{g}(2\xi) d\xi + \int_{\pi/2}^{3\pi/2} |F\left(\xi\right)|^2 f\left(\xi\right) \bar{g}(2\xi) d\xi\right) \\
= \;& 2\int_{-\pi/2}^{3\pi/2} |F\left(\xi\right)|^2 f\left(\xi\right) \bar{g}(2\xi) d\xi \\
= \;& 2\int_{-\pi}^{\pi} |F\left(\xi\right)|^2 f\left(\xi\right) \bar{g}(2\xi) d\xi = \langle f, T^* g\rangle.
\end{aligned}$$

If we consider the Fourier expression

$$|F(\xi)|^2 = \sum_{\ell=-k}^{k} b_\ell e^{i\ell\xi},$$

then the matrix of T^* restricted to $E_k = \{\sum_{\ell=-k}^{k} c_\ell e^{i\ell\xi}, (c_{-k}, \cdots, c_k) \in C^{2k+1}\}$ is given by

$$2H = 2\left(b_{j-2i}\right)_{i,j=-k,\cdots,k}.$$

It is clear that b_t can be written as

$$b_t = \sum_{j=0}^{k-|t|} a_{k-|t|-j}a_{k-j}, \quad t=-k,\cdots,k. \tag{2.3}$$

Hence, $b_t = b_{-t}$, for all $t=-k,\cdots,k$. It is also obvious that b_k is an eigenvalue of H with multiplicity 2. To estimate bounds of the eigenvalues of H, we consider the maximum column sum matrix norm $\|\cdot\|_1$ of $H \equiv H_k$ for $k=2m$ and $2m+1$:

$$\|H_{2m}\|_1 = \max_{-k\le j\le k}\sum_{i=-k}^{k}|b_{j-2i}| = \max\left\{\sum_{t=-m}^{m}|b_{2t}|, \sum_{t=-m}^{m-1}|b_{2t+1}|\right\}, \tag{2.4}$$

where

$$\begin{aligned}\sum_{t=-m}^{m}|b_{2t}| &= 2\sum_{t=1}^{m}|b_{2t}| + b_0\\ &= 2\sum_{t=1}^{m}\left|\sum_{j=0}^{2m-2t}a_{2m-2t-j}a_{2m-j}\right| + b_0\\ &\le 2\sum_{t=1}^{m}\sum_{j=0}^{2m-2t}|a_{2m-2t-j}a_{2m-j}| + b_0.\end{aligned} \tag{2.5}$$

In (2.5) we substitute $\ell = 2m-j$ and obtain

$$\begin{aligned}
\sum_{t=-m}^{m} |b_{2t}| &\le 2\sum_{t=1}^{m}\sum_{\ell=2t}^{2m} |a_{\ell-2t}a_\ell| + \sum_{t=0}^{2m} a_t^2 \\
&= 2\sum_{t=1}^{m}\sum_{u=t}^{m} |a_{2u-2t}a_{2u}| + 2\sum_{t=1}^{m}\sum_{u=t}^{m-1} |a_{2u-2t+1}a_{2u+1}| \\
&\quad + \sum_{t=0}^{m} a_{2t}^2 + \sum_{j=0}^{m-1} a_{2t+1}^2 \\
&= \sum_{t=0}^{m}\sum_{u=t}^{m} |a_{2u-2t}a_{2u}| + \sum_{t=1}^{m}\sum_{u=t}^{m} |a_{2u-2t}a_{2u}| \\
&\quad + \sum_{t=0}^{m-1}\sum_{u=t}^{m-1} |a_{2u-2t+1}a_{2u+1}| + \sum_{t=1}^{m-1}\sum_{u=t}^{m-1} |a_{2u-2t+1}a_{2u+1}| \\
&= \sum_{u=0}^{m}\sum_{t=0}^{u} |a_{2u-2t}a_{2u}| + \sum_{u=1}^{m}\sum_{t=1}^{u} |a_{2u-2t}a_{2u}| \\
&+ \sum_{u=0}^{m-1}\sum_{t=0}^{u} |a_{2u-2t+1}a_{2u+1}| + \sum_{u=1}^{m-1}\sum_{t=1}^{u} |a_{2u-2t+1}a_{2u+1}| . \quad (2.6)
\end{aligned}$$

In the rightmost equality of (2.6), substituting $u - t = j$ yields

$$\begin{aligned}
\sum_{t=-m}^{m} |b_{2t}| &\le \sum_{u=0}^{m}\sum_{j=0}^{u} |a_{2j}||a_{2u}| + \sum_{u=1}^{m}\sum_{j=0}^{u-1} |a_{2j}||a_{2u}| \\
&\quad + \sum_{u=0}^{m-1}\sum_{j=0}^{u} |a_{2j+1}||a_{2u+1}| + \sum_{u=1}^{m-1}\sum_{j=0}^{u-1} |a_{2j+1}||a_{2u+1}|.
\end{aligned}$$

Then, we switch the two sums of the second summation and the fourth summation on the right-hand side of the above equality and combine the new second summation with the first summation and the new fourth summation with the third summation. Thus, we obtain

$$\begin{aligned}
\sum_{t=-m}^{m} |b_{2t}| &\leq \sum_{u=0}^{m}\sum_{j=0}^{u} |a_{2u}||a_{2j}| + \sum_{j=0}^{m-1}\sum_{u=j+1}^{m} |a_{2u}||a_{2j}| \\
&\quad + \sum_{u=0}^{m-1}\sum_{j=0}^{u} |a_{2u+1}||a_{2j+1}| + \sum_{j=0}^{m-2}\sum_{u=j+1}^{m-1} |a_{2u+1}||a_{2j+1}| \\
&= \sum_{u=0}^{m}\sum_{j=0}^{u} |a_{2u}||a_{2j}| + \sum_{u=0}^{m-1}\sum_{j=u+1}^{m} |a_{2u}||a_{2j}| \\
&\quad + \sum_{u=0}^{m-1}\sum_{j=0}^{u} |a_{2u+1}||a_{2j+1}| + \sum_{u=0}^{m-2}\sum_{j=u+1}^{m-1} |a_{2u+1}||a_{2j+1}| \\
&= \sum_{u=0}^{m}\sum_{j=0}^{m} |a_{2u}||a_{2j}| + \sum_{u=0}^{m-1}\sum_{j=0}^{m-1} |a_{2u+1}||a_{2j+1}| \\
&= \left(\sum_{j=0}^{m} |a_{2j}|\right)^2 + \left(\sum_{j=0}^{m-1} |a_{2j+1}|\right)^2. \qquad (2.7)
\end{aligned}$$

Similarly, we have

$$\sum_{t=-m}^{m-1} |b_{2t+1}| \leq \left(\sum_{j=0}^{m} |a_{2j}|\right)^2 + \left(\sum_{j=0}^{m-1} |a_{2j+1}|\right)^2. \qquad (2.8)$$

From (2.4), (2.7) and (2.8) we find the spectral radius of H_{2m} as

$$\rho(H_{2m}) \leq \|H_{2m}\|_1 \leq \left(\sum_{j=0}^{m} |a_{2j}|\right)^2 + \left(\sum_{j=0}^{m-1} |a_{2j+1}|\right)^2.$$

By using the same argument, we find that

$$\rho(H_{2m+1}) \leq \|H_{2m+1}\|_1 \leq \left(\sum_{j=0}^{m} |a_{2j}|\right)^2 + \left(\sum_{j=0}^{m-1} |a_{2j+1}|\right)^2.$$

It follows that

$$\rho(T) = \rho(T^*) = 2\rho(H) \leq 2\left(\sum_{j=0}^{m} |a_{2j}|\right)^2 + 2\left(\sum_{j=0}^{m-1} |a_{2j+1}|\right)^2.$$

If (2.1) holds, i.e.,

$$\left(\sum_{j=0}^{m}|a_{2j}|\right)^2+\left(\sum_{j=0}^{m-1}|a_{2j+1}|\right)^2\leq 2^{2N-1},$$

then $\rho(T)<2^{2N}$. So we choose

$$\epsilon=\frac{1}{2}\left(2^{2N}-\rho(T)\right).$$

Therefore

$$\rho(T)+\epsilon<2^{2N},$$

and we obtain the estimation

$$\int_{|\xi|\geq\pi}\left|\hat{\phi}(\xi)\right|^2 d\xi\leq C\sum_{\ell=1}^{\ell(\epsilon)}4^{-N\ell}||T||^{\ell}+\sum_{\ell=\ell(\epsilon)+1}^{\infty}\left(\frac{\rho(T)+\epsilon}{4^N}\right)^{\ell}.$$

The tail of the series is a convergent geometric series, thus completing the proof of the theorem.

By using Theorem 2.1 and noting Remark 2.1, we immediately have the following improvement of Theorem 2 in [6].

Theorem 2.2 *Let $\phi=\Pi_{j=1}^{\infty}m_0^N(2^{-j}\xi)$ and $\tilde{\phi}=\Pi_{j=1}^{\infty}\tilde{m}_0^{\tilde{N}}(2^{-j}\xi)$ be two B-spline type scaling functions defined by 1.1, and let $m_0^N(\xi)$ and $\tilde{m}_0^{\tilde{N}}(\xi)$ be in the form of (1.6) and (1.7), respectively. If*

(i)

$$a[k]\equiv\left(\sum_{k;even\ j}|a_j|\right)^2+\left(\sum_{k;odd\ j}|a_j|\right)^2<2^{2N-1},$$

$$\tilde{a}[\tilde{k}]\equiv\left(\sum_{\tilde{k};even\ j}|\tilde{a}_j|\right)^2+\left(\sum_{\tilde{k};odd\ j}|\tilde{a}_j|\right)^2<2^{2\tilde{N}-1},\ \textit{and}$$

(ii) $\sum_{j=\mu}^{\nu}\sum_{\ell=0}^{k}\sum_{\tilde{\ell}=0}^{\tilde{k}}\binom{\tilde{N}}{j-\tilde{\ell}-\tilde{k}'}\binom{N}{j+2n-\ell-k'}\tilde{a}_{\tilde{\ell}}a_{\ell}=2^{N+\tilde{N}-1}\delta_{n0},$

where $\mu = min\{k', \tilde{k}'\}$; $\nu = max\{N + k + k', \tilde{N} + \tilde{k} + \tilde{k}'\}$; δ_{n0} *is the Kronecker symbol; and* $n = 0, \pm 1, \pm 2, \cdots$. *Then we have* $\phi, \tilde{\phi} \in L^2(R)$ *and* $\langle \phi(t), \tilde{\phi}(t-i)\rangle = \delta_{i,0}$ *for all* $i \in Z$. *The corresponding* ψ *and* $\tilde{\psi}$ *define biorthogonal wavelets (biorthogonal Riesz bases) and are in* C^α *and* $C^{\tilde{\alpha}}$, *respectively. Here* α *and* $\tilde{\alpha}$ *are more than*

$$N - \frac{1}{2}\log_2(2a[k]) \text{ and } \tilde{N} - \frac{1}{2}\log_2\left(2\tilde{a}[\tilde{k}]\right),$$

respectively.

Proof. By using Theorem 2.1 and arguments similar to that in the proofs of Lemmas 4, 5, and 7 in [6], we can complete the proof of the theorem.

The algorithm shown in [6] for constructing biorthogonal scaling functions ϕ and $\tilde{\phi}$ with the largest possible regularity and the required vanishing moments can be improved by the following optimization problem of finding suitable $F(\xi)$ and $\tilde{F}(\xi)$, or, equivalently, suitable coefficient sets, $\mathbf{a} = \{a_0, \cdots, a_k\}$ and $\tilde{\mathbf{a}} = \{\tilde{a}_0, \cdots, \tilde{a}_k\}$, of $F(\xi)$ and $\tilde{F}(\xi)$, respectively, such that $a[k]$ and $\tilde{a}[\tilde{k}]$ are the minimum under conditions (i) and (ii) of Theorem 2.2. For wavelet analysis of spline approximation, we usually assume $F(\xi) = 1$; i.e., the corresponding ϕ is the B-spline of order N. Consequently, the optimization problem can be written as follows.

$$\min_{\tilde{\mathbf{a}}} \qquad \tilde{a}[\tilde{k}] \equiv \left(\sum_{\tilde{k};even\ j} |\tilde{a}_j|\right)^2 + \left(\sum_{\tilde{k};odd\ j} |\tilde{a}_j|\right)^2, \tag{2.9}$$

$$\text{subject to} \qquad \left(\sum_{j=0}^{\tilde{k}} (-1)^j \tilde{a}_j + 1\right)^2 > 0, \tag{2.10}$$

$$\tilde{a}[\tilde{k}] < 2^{2\tilde{N}-1}, \tag{2.11}$$

$$\sum_{\ell=0}^{\tilde{k}} \sum_{j=\tilde{k}'}^{\max\{N, \tilde{N}+\tilde{k}+\tilde{k}'\}} \binom{\tilde{N}}{j-\ell-\tilde{k}'}\binom{N}{j+2n}\tilde{a}_\ell = 2^{N+\tilde{N}-1}\delta_{n0}, \tag{2.12}$$

where object (2.9) will give the largest possible regularity, condition (2.10) is from the definition of $\tilde{F}$, and conditions (2.11) and (2.12) come from conditions (i) and (ii) of Theorem 2.2.

As examples, we consider $m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)$; i.e., ϕ is the B-spline of order 1. If we choose $\tilde{N} = 1$ and $\tilde{k} = \tilde{k}' = 0$, then the solution of problem (2.9)-(2.12) is $\tilde{a}_0 = 1$ and we arrive at the Haar function. If we choose $\tilde{N} = 2$, $\tilde{k} = 1$, and $\tilde{k}' = 0$, then the solutions are $\tilde{a}_0 = 3/2$ and $\tilde{a}_1 = -1/2$. Hence, the corresponding $\tilde{\phi}$ is defined by $\hat{\tilde{\phi}}(\xi) = \Pi_{j=1}^{\infty}\tilde{m}_0^2(2^{-j}\xi)$, where $\tilde{m}_0^2(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^2\left(\frac{3-e^{-i\xi}}{2}\right)$. The regularities of ϕ and $\tilde{\phi}$ are more than 0.5 and $2 - \log_2(5)/2 = 0.839036$, respectively.

If we choose $N = 2$, $\tilde{N} = 2$, $\tilde{k} = 2$, and $\tilde{k}' = -1$, then the solutions of the optimization problem are $\tilde{a}_0 = -1/2$, $\tilde{a}_2 = 2$, and $\tilde{a}_3 = -1/2$. Hence, the corresponding ϕ is the B-spline of order 2, and $\tilde{\phi}$ is defined by $\hat{\tilde{\phi}}(\xi) = \Pi_{j=1}^{\infty}\tilde{m}_0^2(2^{-j}\xi)$, where $\tilde{m}_0^2(\xi) = e^{i\xi}\left(\frac{1+e^{-i\xi}}{2}\right)^2\left(-\frac{1}{2} + 2e^{-i\xi} - \frac{1}{2}e^{-2i\xi}\right)$. The regularities of ϕ and $\tilde{\phi}$ are more than 1.5 and $2 - \log_2(5)/2 = 0.339036$, respectively.

3 Construction of sequences of biorthogonal B-spline type scaling functions

In this section, we will give a a method for constructing a sequence of B-spline type scaling functions and wavelets from either an orthogonal sacaling function or a pair of biorthogonal scaling functions by using their convolutions with certain B-splines.

Denote $B_n(t)$ the B-spline of degree $n-1$ having nodes at $0, 1, \cdots, n$; i.e., $B_n(t) = Q_n(t) = M(t; 0, 1, \cdots, n)$, which is defined by (1.1) on page 11 of Schoenberg [12]. The Fourier transform of $B_n(t)$ is

$$\widehat{B_n}(\xi) = e^{-in\xi/2}\left(\frac{\sin(\xi/2)}{\xi/2}\right)^n = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^n. \tag{3.1}$$

Another type of B-spline functions we need are denoted by $C_n(t)$ ($n = 1, 2, \cdots$) that are defined as

$$C_n(t) = M(t; -n, \cdots, -1, 0), \tag{3.2}$$

where function M is given in (1.1) of [12]. Therefore, the Fourier transform of $C_n(t)$ is

$$\widehat{C_n}(\xi) = e^{in\xi/2}\left(\frac{\sin(\xi/2)}{\xi/2}\right)^n. \tag{3.3}$$

Comparing the above expression with Eq. (3.1) yields

$$\overline{\widehat{C_n}}(\xi) = \widehat{B_n}(\xi). \tag{3.4}$$

Eq. (3.4) can also be derived from the symmetry of B-splines. The symmetry $C_n(t) = B_n(-t)$ leads to the Fourier transform of $\widehat{C_n(t)}(\xi) = \widehat{B_n(-t)}(\xi) = \overline{\widehat{B_n(t)}}(\xi)$.

In this section, we denote by $\phi(t)$ an orthogonal scaling function that satisfies the dilation equation (or refinement equation)

$$\phi(t) = \sum_k c_k \phi(2t-k), \tag{3.5}$$

where the constant coefficients c_k satisfy the following four properties.

(i) $c_k = 0$ for $k \notin \{0, 1, \cdots, 2p-1\}$;

(ii) $\sum_k c_k = 2$;

(iii) $\sum_k (-1)^k k^m c_k = 0$ for $0 \le m \le p-1$;

(iv) $\sum_k c_k c_{k-2m} = 2\delta_{0m}$ for $1-p \le m \le p-1$.

If ϕ and $\tilde{\phi}$ are biorthogonal scaling functions with refinement expressions

$$\phi(t) = \sum_n c_n \phi(2t-n), \quad \tilde{\phi}(t) = \sum_n \tilde{c}_n \phi(2t-n), \tag{3.6}$$

then the coefficients c_k and $\tilde{c}_k$ satisfy

(i)' $c_k = \tilde{c}_{\tilde{k}} = 0$ for $k \notin \{0, 1, \cdots, 2p-1\}$ and $\tilde{k} \notin \{0, 1, \cdots, 2\tilde{p}-1\}$;

(ii)' $\sum_k c_k = \sum_k \tilde{c}_k = 2$;

(iii)' $\sum_k (-1)^k k^m c_k = \sum_k (-1)^k k^{\tilde{m}} \tilde{c}_k = 0$ for $0 \le m \le p-1$ and $0 \le \tilde{m} \le \tilde{p}-1$;

(iv)' $\sum_k c_k \tilde{c}_{k-2m} = 2\delta_{0m}$ for $1-\widetilde{p} \le m \le \widetilde{p}-1$.

Let $\phi(t) = \sum_{k=0}^{2p-1} c_k \phi(2t-k)$ and $\phi(t) \in L^2(\mathbb{R})$. We define $\overline{\phi}_n := \phi * B_n$. Clearly ϕ_n is in $L^2(\mathbb{R})$. We also have the following results on ϕ_n.

Theorem 3.1 *Let $\phi(t) = \sum_{k=0}^{2p-1} c_k \phi(2t-k)$, and let $\phi(t) \in L^2(\mathbb{R})$ satisfy (i)-(iv). Then $\overline{\phi}_n := \phi * B_n$ satisfies the dilation equation*

$$\overline{\phi}_n = \sum_{k=0}^{2p+n-1} h_k^{(n)} \overline{\phi}_n(2t-k), \tag{3.7}$$

where

$$h_k^{(n)} = \frac{1}{2^n} \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} c_{k-j}. \tag{3.8}$$

In addition, $h^{(n)}$ as shown in (3.8) possesses the following properties.

(i)” $h_k^{(n)} = 0$ for $k \notin \{0, 1, \cdots, 2p+n-1\}$;

(ii)” $\sum_k h_k^{(n)} = 2$;

(iii)” $\sum_k (-1)^k k^m h_k^{(n)} = 0$ for $0 \le m \le p+n-1$.

Proof. Denote the mask of ϕ by $m_\phi(\xi)$. Then from (3.1), the mask of $\overline{\phi}_n$ is

$$m_{\overline{\phi}_n}(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^n m_\phi(\xi).$$

Consequently,

$$\begin{aligned} m_{\overline{\phi}_n}(\xi) &= \frac{1}{2^n} \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} e^{-ij\xi} \frac{1}{2} \sum_{k=0}^{2p-1} c_k e^{-ik\xi} \\ &= \frac{1}{2} \sum_{k=0}^{2p+n-1} \left(\frac{1}{2^n} \sum_{j=0}^{k} \begin{pmatrix} n \\ j \end{pmatrix} c_{k-j} \right) e^{-ik\xi}. \end{aligned}$$

Noting $\begin{pmatrix} n \\ j \end{pmatrix} = 0$ for $j > n$, we have Eqs. (3.7) and (3.8).

From items (i) and (3.8), we immediately obtain (i)”. Item (ii)” also holds because

$$\sum_{k=0}^{2p+n-1} h_k^{(n)} = \sum_{k=0}^{2p+n-1} \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} c_{k-j}$$
$$= \sum_{j=0}^{n} \frac{1}{2^n} \binom{n}{j} \sum_{k'=-j}^{2p+n-j-1} c_{k'} = \sum_{j=0}^{n} \frac{1}{2^n} \binom{n}{j} \sum_{k'=0}^{2p-1} c_{k'} = 2.$$

Finally, we prove item (iii)". For $m = 0, 1, \cdots, p+n-1$,

$$\sum_{k=0}^{2p+n-1} (-1)^k k^m h_k^{(n)}$$
$$= \frac{1}{2^n} \sum_{k=0}^{2p+n-1} \sum_{j=0}^{n} (-1)^k k^m \binom{n}{j} c_{k-j}$$
$$= \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} \sum_{k=-j}^{2p+n-j-1} (-1)^{k+j} (k+j)^m c_k$$
$$= \frac{1}{2^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{k=0}^{2p-1} (-1)^k c_k \sum_{i=0}^{m} \binom{m}{i} k^i j^{m-i}$$
$$= \frac{1}{2^n} \sum_{i=0}^{m} \binom{m}{i} \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^{m-i} \right) \left(\sum_{k=0}^{2p-1} (-1)^k k^i c_k \right).$$

Hence, for $0 \le i \le p-1$, the sum in the second parenthesis in the last line of the above equation is equal to zero (see item (iii)). For $p \le i \le m$, we have $0 \le m-i \le m-p \le p+n-1-p = n-1$. Therefore, from the identity

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^\ell = (-1)^n n! \delta_{\ell n}$$

($\ell = 0, 1, \cdots, n$), the sum in the first parenthesis in the last line of the above equation becomes zero. It follows that the right-hand side of the last equation is equal to zero. This completes the proof of theorem.

From Theorem 3.1, we know that $\overline{\phi}_n$ is a scaling function with the approximation degree $p+n-1$. However, it does not satisfy orthogonal

condition (iv). Thus, we need to define a biorthogonal scaling function, $\widetilde{\phi}_n$, with respect to $\overline{\phi}_n$ by

$$\left(C_n * \widetilde{\phi}_n\right)(t) = \phi(t). \tag{3.9}$$

Taking Fourier transforms on both sides of Eq. (3.9) while noting Eq. (3.3) leads to

$$\widehat{\widetilde{\phi}_n}(\xi) = \frac{\widehat{\phi}(\xi)}{\widehat{C_n}(\xi)} = e^{-in\xi/2}\left(\frac{\xi/2}{\sin(\xi/2)}\right)^n \widehat{\phi}(\xi). \tag{3.10}$$

Theorem 3.2 *The function $\widetilde{\phi}_n$ is well defined by Eq. (3.9), and it satisfies the following dilation equation.*

$$\widehat{\widetilde{\phi}_n}(\xi) = \widetilde{m}_{\widetilde{\phi}_n}\left(\frac{\xi}{2}\right)\widehat{\widetilde{\phi}_n}\left(\frac{\xi}{2}\right), \tag{3.11}$$

where

$$\widetilde{m}_{\widetilde{\phi}_n}(\xi) = \left(\frac{2}{1+e^{i\xi}}\right)^n m_\phi(\xi)$$

and m_ϕ is the mask of ϕ.

In addition, let $\overline{\phi}_n$ be defined as in Theorem 3.1. Then $\{\overline{\phi}_n(t-k)\}_{k\in\mathbb{Z}}$ and $\{\widetilde{\phi}_n(t-k)\}_{k\in\mathbb{Z}}$ are biorthogonal sets; i.e., $\overline{\phi}_n$ and $\widetilde{\phi}_n$ satisfy $\langle\overline{\phi}_n(t), \widetilde{\phi}_n(t-k)\rangle = \delta_{0k}$.

Proof. To derive dilation equation (3.11), we start from Eq. (3.10) and apply Eqs. (3.4) and (3.9) to give

$$\begin{aligned}
\frac{\widehat{\widetilde{\phi}_n}(\xi)}{\widehat{\widetilde{\phi}_n}(\xi/2)} &= \frac{\widehat{C_n}(\xi/2)}{\widehat{C_n}(\xi)}\frac{\phi(\xi)}{\phi(\xi/2)} = \frac{\overline{\widehat{B_n}(\xi/2)}}{\widehat{B_n}(\xi)} m_\phi\left(\frac{\xi}{2}\right) \\
&= \left(\frac{2}{1+e^{i\xi}}\right)^n m_\phi\left(\frac{\xi}{2}\right).
\end{aligned}$$

Hence, Eq. (3.11) is established and $\widetilde{\phi}_n$ is well defined by (3.9).

To prove the biorthogonality of $\{\overline{\phi}_n(t-k)\}_{k\in\mathbb{Z}}$ and $\{\widetilde{\phi}_n(t-k)\}_{k\in\mathbb{Z}}$, we use the General Parseval's Relation, substituting expressions shown as in (3.1) and (3.10), and noting relation (3.4), we have

$$
\begin{aligned}
&\langle \overline{\phi}_n(t), \widetilde{\phi}_n(t-k)\rangle = \frac{1}{2\pi}\langle \widehat{\overline{\phi}_n}(\xi), \widehat{\widetilde{\phi}_n}(\xi)e^{-ik\xi}\rangle \\
= \;& \frac{1}{2\pi}e^{ik\xi}\langle \widehat{B_n}(\xi)\widehat{\phi}(\xi), \frac{\widehat{\phi}(\xi)}{\widehat{C_n}(\xi)}\rangle = \frac{1}{2\pi}e^{ik\xi}\langle \widehat{\phi}(\xi), \widehat{\phi}(\xi)\rangle \\
= \;& \langle \phi(t), \phi(t-k)\rangle = \delta_{ok},
\end{aligned}
$$

where the last step is due to the orthonormality of $\{\phi(t-k)\}_{k\in\mathbb{Z}}$.

Similar to Theorem 3.1, we derive the following properties of the dilation coefficients of $\widetilde{\phi}_n$.

Theorem 3.3 *Let $\phi(t) = \sum_{k=0}^{2p-1} c_k\phi(2t-k)$. Then $\widetilde{\phi}_n$, $n \le 2p-1$, defined as in (3.9) satisfies the dilation equation*

$$
\widetilde{\phi}_n = \sum_{k=n}^{2p-1} \widetilde{h}_k^{(n)}\widetilde{\phi}_n(2t-k), \tag{3.12}
$$

where dilation coefficients, $\{\widetilde{h}_k^{(n)}\}_k$, satisfy

$$
c_k = \frac{1}{2^n}\sum_{j=0}^{n}\begin{pmatrix} n \\ j \end{pmatrix}\widetilde{h}_{k+j}^{(n)}
$$

for $k = n, n+1, \cdots, 2p-1$.

More properties on $\widetilde{h}_k^{(n)}$ $(n \le k \le 2p-1)$, the dilation coefficients of $\widetilde{\phi}_n$, and computation of the coefficients will be discussed in a later paper. In the following we shall determine the classes the functions $\widetilde{\phi}_n$ $(1 \le n \le 2p-1)$ may be in which.

Although $\phi \in L^2(\mathbb{R})$ leads to $\overline{\phi}_n = B_n * \phi \in L^2(\mathbb{R})$, it cannot guarantee that $\widetilde{\phi}_n$ is also in $L^2(\mathbb{R})$. However, if $\widetilde{\phi}_n$ is the distribution solution to the dilation equation (3.12), then $\widehat{\widetilde{\phi}_n}(0) = 1$, and from [14], $\widetilde{\phi}_n \in H^{-s}$ for $s > \log_2 \sum_k \left|\widetilde{h}_k^{(n)}\right| - 1/2$. The proof follows from the fact that

$$
\left|\widehat{\widetilde{\phi}_n}(\xi)\right| \le C(1+|\xi|)^M,
$$

where $M = \log_2 \sum_k \left|h_k^{(n)}\right| - 1$, and hence $\widetilde{\phi}_n \in H^{-s}$ for $s > M + 1/2$. Therefore, scaling function $\widetilde{\phi}_n$ generates a multiresolution analysis of H^{-s}.

Using Theorem 2.1, we now give a condition for the orthogonal scaling function, $\phi \in \Phi$, defined in Definition 1.1 such that its corresponding $\widetilde{\phi}_n$ is in $L^2(\mathbb{R})$.

Theorem 3.4 *Let ϕ be defined as*

$$\hat{\phi} = \prod_{j=1}^{\infty} m_0^N(2^{-j}\xi),$$

where $m_0^N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N F(\xi)$ with $F(\xi) = e^{-ik'\xi}\sum_{j=0}^{k} a_j e^{-ij\xi}$ is defined by (1.3) and (1.4). If $F(\pi) \neq -1$ and the coefficients of $F(\xi)$ satisfy

$$\left(\sum_{k;even\ j} |a_j|\right)^2 + \left(\sum_{k;odd\ j} |a_j|\right)^2 < 2^{2N-1}, \tag{3.13}$$

then ϕ is in $L_2(\mathbb{R})$, while both ϕ and $\widetilde{\phi}_n$ $(n < N)$ are in $L^2(\mathbb{R})$ when

$$\left(\sum_{k;even\ j} |a_j|\right)^2 + \left(\sum_{k;odd\ j} |a_j|\right)^2 < 2^{2(N-n)-1}. \tag{3.14}$$

We now extend the results for the orthogonal scaling functions to the the biorthogonal scaling functions.

Theorem 3.5 *Let $f(t) = \sum_{k=0}^{2p-1} c_k f(2t-k)$ and $g(t) = \sum_{k=0}^{2\widetilde{p}-1} d_k g(2t-k)$ be biorthogonal scaling functions (i.e., $\langle f(t), g(t-\ell)\rangle = \delta_{0\ell}$) with approximation degrees p and $\widetilde{p}$, respectively. Then functions $\overline{f}_n$ and $\widetilde{g}_n$ defined by*

$$\overline{f}_n(t) = (B_n * f)(t), \text{ and } (C_n * \widetilde{g}_n)(t) = g(t), \tag{3.15}$$

where B_n and C_n are B-splines, are also biorthogonal scaling functions with approximation degrees $p+n-1$ and $\widetilde{p}-n-1$, respectively.

In addition, if $f \in L^2(\mathbb{R})$, then $\overline{f}_n$ is also in $L^2(\mathbb{R})$. If the Fourier transform of g can be written as

$$\widehat{g}(\xi) = \prod_{j=1}^{\infty} m_0^N(2^{-j}\xi),$$

where $m_0^N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N F(\xi)$ *with* $F(\xi) = e^{-ik'\xi}\sum_{j=0}^k a_j e^{-ij\xi}$ *is defined by (1.3) and (1.4), with* $F(\pi) \neq -1$ *and the coefficients of* $F(\xi)$ *satisfy*

$$\left(\sum_{k;even\ j} |a_j|\right)^2 + \left(\sum_{k;odd\ j} |a_j|\right)^2 < 2^{2(N-n)-1},$$

then both g *and* $\widetilde{g}_n$ *are also in* $L^2(\mathbb{R})$*.*

Example 3.1. Considering the examples given in [2] and [4] (or see the examples shown at the end of Section 2), let ${}_N\phi$ and ${}_{N,\widetilde{N}}\widetilde{\phi}$ be a pair of biorthogonal scaling functions with masks ${}_Nm_0$ and ${}_{N,\widetilde{N}}\widetilde{m}_0$, respectively. It is easy to check that $\overline{{}_N\phi_n}(t) =_{N+n} \phi(t-k)$ and $\widetilde{{}_{N,\widetilde{N}}\widetilde{\phi}_n}(t) =_{N+n,\widetilde{N}-n} \widetilde{\phi}(t-\ell)$ $(n < \widetilde{N})$ for some integers k and ℓ, where $\overline{f}_n$ and $\widetilde{g}_n$ are defined by Eqs. in (3.15). Denote the masks of $\overline{{}_N\phi_n}(t)$ and $\widetilde{{}_{N,\widetilde{N}}\widetilde{\phi}_n}(t)$ by m_{N+n} and $\widetilde{m}_{N+n,\widetilde{N}-n}$, which can be found using Theorems 3.1 and 3.3. We have therefore given an easy way to derive sequences of pairs of biorthogonal spline type scaling functions shown in [2] and [4] (and in Section 2). For instance, the masks of ${}_1\phi$ and ${}_{1,3}\widetilde{\phi}$ are respectively

$${}_1m_0(\xi) = \frac{1+e^{-i\xi}}{2}$$

and

$${}_{1,3}\widetilde{m}_0(\xi) = -\frac{e^{2i\xi}}{16}(1+e^{-i\xi})^3(1-4e^{-i\xi}+e^{-2i\xi}).$$

Then the masks of $\overline{{}_1\phi_n}$ and $\widetilde{{}_{1,3}\widetilde{\phi}_n}$ for $n = 1, 2$ are respectively

$$\begin{aligned}
m_2(\xi) &= \frac{(1+e^{-i\xi})^2}{4},\ m_3(\xi) = \frac{(1+e^{-i\xi})^3}{8},\\
\widetilde{m}_{2,2}(\xi) &= -\frac{e^{i\xi}}{8}(1+e^{-i\xi})^2(1-4e^{-i\xi}+e^{-2i\xi}),\\
\widetilde{m}_{3,1}(\xi) &= -\frac{1}{4}(1+e^{-i\xi})(1-4e^{-i\xi}+e^{-2i\xi}),
\end{aligned}$$

which are respectively ${}_2m_0$, ${}_3m_0$, ${}_{2,2}\widetilde{m}_0$, and ${}_{3,1}\widetilde{m}_0$ (a different factor $e^{ik\xi}$ $(k \in \mathbb{Z})$ merely shifts the function support) shown in papers [3]

and [4]. In addition, by using Sweldens's lifting scheme (see [11]), we can construct masks ${}_{N,\tilde{N}+2}\widetilde{m}_0$ from ${}_{N,\tilde{N}}\widetilde{m}_0$. Therefore, starting from the mask of the Haar scaling function, ${}_1m_0 ={}_{1,1} \widetilde{m}_0$, we can obtain all ${}_nm_0$ ($n = 2, 3, \cdots$) using Theorem 3.1 and all ${}_{n,\ell}\widetilde{m}_0$ using both the lifting scheme and the method supplied by Theorem 3.3. Specifically, from ${}_{1,1}\widetilde{m}_0$ we have all masks ${}_{1,2n+1}\widetilde{m}_0$ ($n = 1, 2, \cdots$) by applying the lifting scheme, and all ${}_{\ell,2n+2-\ell}\widetilde{m}_0$ for $\ell = 2, 3, \cdots, 2n+1$ can be found by using the formulas in Theorem 3.3.

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