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Abstract
We present here the definition of Padé spline functions, their expressions, and the estimate of the remainders of padé spline expansions. Some algorithms are also given.

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1 Introduction

Padé approximation is derived by expanding a function as a ratio of two power series and both the numerator and denominator coefficients are thus determined (*cf.* Baker [1-2], Baker and Graves-Morris [3], and Brent, Gustavson, and Yun [4]). In this paper, we shall use two points Padé approximation to construct Padé spline functions. The main idea initially came from the author's talk at the Joint U.S.- China Workshop on Approximation Theory that took place in April, 1985, Hangzhou, China ([5]).

Let

$$\triangle : a = x_0 < x_1 < \cdots < x_n = b$$

be an arbitrary partition on the interval $[a, b]$, and let f be a kth differentiable function defined on $[a, b]$ with function value and dervatives at each node x_i $(i = 0, 1, \cdots, n)$

$$y_i^{(m)} = f^{(m)}(x_i), \quad m = 0, 1, \cdots, k-1; i = 0, 1, \cdots, n.$$

Denote by π_k the collection of all polynomials of degree less than or equal to k. We now give the definition of Padé spline functions.

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Definition 1.1 *We call $R^{(k)}_{r,\ell}(\triangle)$ the set of Padé spline function of order k with nodes x_i $(i = 0, 1, \ldots, n)$, if any function $R(x) \in R^{(k)}_{r,\ell}$ satisfies the following coditions for all $x \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$,*

(i) $R(x) = \frac{P_i(x)}{Q_i(x)}$, $P_i(x) \in \pi_r$, $Q_i(x) \in \pi_\ell$,

(ii) $\sum_{m=0}^{k-1} y^{(m)}_{i-1} \frac{(x-x_{i-1})^m}{m!} - \frac{P_i(x)}{Q_i(x)} = O\left((x - x_{i-1})^k\right)$,

(iii) $\sum_{m=0}^{k-1} y^{(m)}_{i} \frac{(x-x_{i})^m}{m!} - \frac{P_i(x)}{Q_i(x)} = O\left((x - x_{i})^k\right)$,

(iv) $r + \ell = 2k - 1$.

From Definition 1.1, we immediately know $R^{(k)}_{r,\ell}(\triangle) \in C^k$. In addition, the rational Hermite interpolation and rational contact interpolation can be easily obtained by using the padé spline functions.

Although Definition 1.1 only gives the piecewise expression of Padé spline functions, we might discuss its globe expression as follows.

Suppose $Q_i(x_i) \neq 0$. Denote

$$G_i(x) = \sum_{m=0}^{k-1} R^{(m)}(x_i) \frac{(x - x_i)^m}{m!}.$$

Thus

$$R(x) - G_i(x) = \frac{P_i(x) - G_i(x)Q_i(x)}{Q_i(x)}$$

has multiple roots at x_i of order k; i.e.,

$$P_i(x) - G_i(x)Q_i(x) = (x - x_i)^k F_i(x),$$

where *deg* $F_i(x) \leq \max\{r, k + \ell\} - k = \max\{r - k, \ell\}$. Thus, the expressions of $R(x)$ on $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ are respectively

$$\frac{P_i(x)}{Q_i(x)} = \sum_{m=0}^{k-1} R^{(m)}(x_i) \frac{(x - x_i)^m}{m!} + (x - x_i)^k \frac{F_i(x)}{Q_i(x)} \tag{1.1}$$

and

$$\frac{P_{i+1}(x)}{Q_{i+1}(x)} = \sum_{m=0}^{k-1} R^{(m)}(x_i) \frac{(x - x_i)^m}{m!} + (x - x_i)^k \frac{F_{i+1}(x)}{Q_{i+1}(x)}. \tag{1.2}$$

Consequently,

$$\begin{aligned} \frac{P_{i+1}(x)}{Q_{i+1}(x)} - \frac{P_i(x)}{Q_i(x)} &= \left[\frac{F_{i+1}(x)}{Q_{i+1}(x)} - \frac{F_i(x)}{Q_i(x)}\right](x - x_i)^k \\ &= \frac{M_i(x)}{Q_i(x)Q_{i+1}(x)}(x - x_i)^k, \end{aligned}$$

where $M_i(x) = Q_i(x)F_{i+1}(x) - F_i(x)Q_{i+1}(x)$ is in $\pi_{r+\ell-k}$. Therefore, for $x \in [x_i, x_{i+1}]$, we have

$$\begin{aligned} \frac{P_{i+1}(x)}{Q_{i+1}(x)} - \frac{P_1(x)}{Q_1(x)} &= \sum_{j=1}^{i} \left(\frac{P_{j+1}(x)}{Q_{j+1}(x)} - \frac{P_j(x)}{Q_j(x)} \right) \\ &= \sum_{j=1}^{i} \frac{M_j(x)}{Q_j(x)Q_{j+1}(x)} (x - x_j)_+^k, \end{aligned} \tag{1.3}$$

where $M_j(x) \in \pi_{r+\ell-k}$ and

$$(x - x_j)_+ := \begin{cases} x - x_j & \text{if } x \geq x_j \\ 0 & \text{if } x < x_j. \end{cases}$$

From Eq. (1.3) we obtain the globe expression of Padé spline function $R(x)$ as follows.

$$R(x) = \frac{P_1(x)}{Q_1(x)} + \sum_{j=1}^{n-1} \frac{M_j(x)}{Q_j(x)Q_{j+1}(x)} (x - x_j)_+^k, \tag{1.4}$$

where $M_j(x) \in \pi_{r+\ell-k}$ is completely determined by $P_1(x)$, $Q_j(x)$ $(j = 1, 2, \ldots, n)$ as well as the values and the first k derivatives of $R(x)$ at x_j $(j = 1, 2, \ldots, n)$.

We can also show that if any real-valued function $R(x)$ defined on $[a, b]$ can be written as in Eq. (1.4) with $M_j(x) \in \pi_{r+\ell-k}$, then $R(x) \in R_{r,\ell}^{(k)}(\triangle)$; i.e., $R(x)$ is a Padé spline function defined as in Definition 1.1. Indeed, assume that $R(x)$ shown as in (1.4) is given, where $M_j(x) \in \pi_{r+\ell-k}$, $P_1(x) \in \pi_r$, $Q_i(x) \in \pi_\ell$ $(i = 1, 2, \ldots, n)$, and the greatest common divisor $(P_1(x), Q_1(x)) = 1$, there exist $p(x)$ and $q(x)$ such that

$$p(x)Q_1(x) + P_1(x)q(x) \equiv 1.$$

By multiplying $\phi_1(x) = (x - x_1)^k M_1(x)$ on the both sides of the last equation, we obtain

$$p(x)\phi(x)Q_1(x) + P_1(x)q(x)\phi(x) = \phi(x). \tag{1.5}$$

Since we can write

$$p(x)\phi(x) = P_1(x)r_1(x) + s_1(x)$$

and

$$q(x)\phi(x) = Q_1(x)r_2(x) + s_2(x),$$

Eq. (1.5) can be changed to

$$\left[P_1(x)r_1(x) + s_1(x)\right] Q_1(x) + P_1(x) \left[Q_1(x)r_2(x) + s_2(x)\right] = \phi(x). \tag{1.6}$$

If $(x-x_1) \not| s_2(x)$, we set $Q_2(x) = -s_2(x)$ and

$$P_2(x) = P_1(x)\,[r_1(x) + r_2(x)] + s_1(x).$$

If $(x-x_1)|s_2(x)$, then $(x-x_1) \not| Q_1(x)$ because of $(P_1(x), Q_1(x)) = 1$. We thus denote $Q_1(x) = -s_2(x) - Q_1(x)$ and

$$P_2(x) = P_1(x)\,[r_1(x) + r_2(x) - 1] + s_1(x).$$

Therefore in either case, we can write Eq. (1.6) as

$$P_2(x)Q_1(x) - P_1(x)Q_2(x) = (x-x_1)^k M_1(x),$$

where $(x-x_1) \not| Q_2(x)$.

Similarly, we cane decompose $(x-x_j)^k M_j(x)$ into

$$(x-x_j)^k M_j(x) = P_{j+1}(x)Q_j(x) - P_j(x)Q_{j+1}(x), \tag{1.7}$$

where $Q_j(x_j) \neq 0$, $Q_{j+1}(x_j) \neq 0$, and $j = 1, 2, \ldots, n-1$. Since $M_j(x) \in \pi_{r+\ell-k}$, $P_j(x) \in \pi_r$ for all $j = 1, 2, \ldots, n-1$.

For $x \in [x_i, x_{i+1}]$ $(j = 0, 1, \ldots, n-1)$, from Eqs. (1.4) and (1.7), we have

$$R(x) = \frac{P_1(x)}{Q_1(x)} + \sum_{j=1}^{i} \left(\frac{P_{j+1}(x)}{Q_{j+1}(x)} - \frac{P_j(x)}{Q_j(x)} \right) = \frac{P_{i+1}(x)}{Q_{i+1}(x)}.$$

In addition, since

$$\left[M_i(x)(x-x_i)^k\right]^{(m)}\Big|_{x=x_i} = 0$$

for $i = 1, 2, \ldots, n-1$ and $m = 0, 1, \ldots, k-1$, by using the Lemma shown as in [6], we obtain

$$\left[\frac{(x-x_i)^k M_i(x)}{Q_i(x)Q_{i+1}(x)}\right]^{(m)}\Bigg|_{x=x_i} = 0.$$

Consquently,

$$\left[\frac{P_{i+1}(x)}{Q_{i+1}(x)}\right]^{(m)}\Bigg|_{x=x_i} = \left[\frac{P_i(x)}{Q_i(x)}\right]^{(m)}\Bigg|_{x=x_i}.$$

It follows that $R(x) \in R^{(k)}_{r,\ell}$. Thus we have established the following result.

Theorem 1.2 *Function $R(x)$ defined on $[a, b]$ and shown as in Eq. (1.4) is in $R^{(k)}_{r,\ell}(\triangle)$ if and only if $M_j(x) \in \pi_{r+\ell-k}$.*

2 Algorithm

In this section, we will give two algorithms for constructing Padé spline functions. Our first algorithm is to construct the functions piece by piece by using continued fractions. The second algorithm is based on the general expression of Padé spline functions shown as in (1.4). To describe the algorithms clearly, we only consider the Padé spline function set $R^{(k)}_{k-1,k}$, which is the most important set in the Padé approximation. The first algorithms is also an improvement of [7-8].

For $x \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$, write $R(x) = \frac{P_i(x)}{Q_i(x)}$ as its continued fraction form:

$$\begin{aligned} \frac{P_i(x)}{Q_i(x)} &= a_{i,0} + \frac{x-x_i|}{|a_{i,1}} + \cdots + \frac{x-x_i|}{|a_{i,k-1}} + \frac{x-x_i|}{|a_{i+1,0}} \\ &\quad + \frac{x-x_{i+1}|}{|a_{i+1,1}} + \cdots + \frac{x-x_{i+1}|}{|a_{i+1,k-1}}. \end{aligned} \tag{2.1}$$

Denote

$$\frac{S_{i,0}(x)}{T_{i,0}(x)} = a_{i,0} + \frac{x-x_i|}{|a_{i,1}} + \cdots + \frac{x-x_i|}{|a_{i,k-1}}. \tag{2.2}$$

It is easy to find that

$$\left[\frac{S_{i,0}(x)}{T_{i,0}(x)}\right]^{(m)}\Bigg|_{x=x_i} = \left[\frac{P_i(x)}{Q_i(x)}\right]^{(m)}\Bigg|_{x=x_i} = y_i^{(m)}$$

for $m = 0, 1, \ldots, k-1$, which implies by the Lemma in [6]

$$S^{(m)}_{i,0}(x_i) = [f(x)T_{i,0}(x)]^{(m)}\Big|_{x=x_i}, \tag{2.3}$$

where $m = 0, 1, \ldots, k-1$. From Eq. (2.3) we can find the coefficients of $S_{i,0}(x)$ and $T_{i,0}(x)$. Then, by using the following relations, (2.4) and (2.5), we can determine the coefficient set $\{a_{i,0}, a_{i,1}, \ldots, a_{i,k-1}\}$ of the continued fraction (2.1).

$$\begin{aligned} S_{i,0} &= \Pi_{j=0}^{k-1} a_{i,j} \left(1 + \sum_{j=0}^{k-2} \frac{x-x_i}{a_{i,j}a_{i,j+1}} + \sum_{0\le j<\ell\le k-3} \frac{(x-x_i)^2}{a_{i,j}a_{i,j+1}a_{i,\ell+1}a_{i,\ell+2}} \right. \\ &\quad \left. + \sum_{0\le j<\ell<m\le k-4} \frac{(x-x_i)^3}{a_{i,j}a_{i,j+1}a_{i,\ell+1}a_{i,\ell+2}a_{i,m+2}a_{i,m+3}} + \cdots \right), \end{aligned} \tag{2.4}$$

$$T_{i,0}(x) = \Pi_{j=1}^{k-1} a_{i,j} \left(1 + \sum_{j=1}^{k-2} \frac{x-x_i}{a_{i,j}a_{i,j+1}} \right.$$

$$+\sum_{1\le j<\ell\le k-3}\frac{(x-x_i)^2}{a_{i,j}a_{i,j+1}a_{i,\ell+1}a_{i,\ell+2}}+\cdots\Bigg). \tag{2.5}$$

Denote

$$\frac{S_{i+1,0}(x)}{T_{i+1,0}(x)}=a_{i+1,0}+\frac{x-x_{i+1}|}{|a_{i+1,1}}+\cdots+\frac{x-x_{i+1}|}{|a_{i+1,k-1}} \tag{2.6}$$

and

$$\frac{S_{i,-1}(x)}{T_{i,-1}(x)}=a_{i,0}+\frac{x-x_i|}{|a_{i,1}}+\cdots+\frac{x-x_i|}{|a_{i,k-2}}. \tag{2.7}$$

Then,

$$\frac{P_i(x)}{Q_i(x)}=\frac{S_{i,0}S_{i+1,0}+(x-x_i)S_{i,-1}T_{i+1,0}}{T_{i,0}S_{i+1,0}+(x-x_i)T_{i,-1}T_{i+1,0}}.$$

Similar to Eq. (2.3), from [6] we have

$$\begin{aligned} & \left[S_{i,0}S_{i+1,0}+(x-x_i)S_{i,-1}T_{i+1,0}\right]^{(m)}\Big|_{x=x_{i+1}} \\ = \quad & \left\{f(x)\left[T_{i,0}S_{i+1,0}+(x-x_i)T_{i,-1}T_{i+1,0}\right]\right\}^{(m)}\Big|_{x=x_{i+1}}. \end{aligned} \tag{2.8}$$

In Eq. (2.8), since $S_{i,-1}$ and $T_{i,-1}$ have been determined from (2.3), we thus find $S_{i+1,0}(x)$ and $T_{i+1,0}(x)$. We can also establish the relations between the functions $S_{i+1,0}(x)$ and $T_{i+1,0}(x)$ and the coefficient set $\{a_{i+1,j}: j=0,1,\ldots,k-1\}$, which is as the same as Eqs. (2.4) and (2.5) except an index change of $i\to i+1$. From the relations we finally determine the set $\{a_{i+1,j}: j=0,1,\ldots,k-1\}$.
Example 2.1. As an example, we now consider the case of $k=2$. Obviously, we have

$$\begin{aligned} & S_{i,0}(x)=a_{i,0}a_{i,1}+x-x_i,\ T_{i,0}(x)=a_{i,1}, \\ & S_{i+1,0}(x)=a_{i+1,0}a_{i+1,1}+x-x_{i+1},\ T_{i+1,0}=a_{i+1,1}, \\ & S_{i,-1}(x)=a_{i,0},\ T_{i,-1}(x)=1. \end{aligned}$$

Thus, (2.3) is reduced to

$$\left[a_{i,0}a_{i,1}+x-x_i\right]^{(m)}\Big|_{x=x_i}=a_{i,1}y_i^{(m)},\quad m=0,1.$$

Assume that $y_i'\neq 0$, we solve $a_{i,0}=y_i$ and $a_{i,1}=1/y_i'$.

From Eq. (2.8) we have

$$\left.\left[\left(\frac{y_i}{y_i'}+x-x_i\right)(a_{i+1,0}a_{i+1,1}+x-x_{i+1})+y_i a_{i+1,1}(x-x_i)\right]^{(m)}\right|_{x=x_{i+1}}$$

$$=\left.\left\{f(x)\left[\frac{1}{y_i'}(a_{i+1,0}a_{i+1,1}+x-x_{i+1})+a_{i+1,1}(x-x_i)\right]\right\}^{(m)}\right|_{x=x_{i+1}}$$

for $m=0,1$. From the last equation it can be found that

$$\begin{aligned} a_{i+1,0} &= \frac{y_i'(x_{i+1}-x_i)(y_{i+1}-y_i)}{y_i'(x_{i+1}-x_i)-(y_{i+1}-y_i)}, \\ a_{i+1,1} &= \frac{[y_{i+1}-y_i-y_i'(x_{i+1}-x_i)]^2}{y_i'\left[y_i'y_{i+1}'(x_{i+1}-x_i)^2-(y_{i+1}-y_i)^2\right]}. \end{aligned}$$

Substituting the obtained coefficient set $\{a_{i,0},a_{i,1},a_{i+1,0},a_{i+1,1}\}$ into the expression of the Padé spline function $R(x)\in R_{2,1}^{(2)}$

$$R(x)=a_{i,0}+\frac{x-x_i|}{|a_{i,1}}+\frac{x-x_i|}{|a_{i+1,0}}+\frac{x-x_{i+1}|}{|a_{i+1,1}}$$

yields

$$\begin{aligned} R(x) = {} & [(x-x_i)(x-x_{i+1})+a_{i,0}a_{i,1}(x-x_{i+1})+a_{i+1,0}a_{i+1,1}(x-x_i) \\ & +a_{i,0}a_{i+1,1}(x-x_i)+a_{i,0}a_{i,1}a_{i+1,0}a_{i+1,1}]\,/\,[a_{i,1}(x-x_{i+1}) \\ & +a_{i+1,1}(x-x_i)+a_{i,1}a_{i+1,0}a_{i+1,1}] \end{aligned}$$

for $i=0,1,\ldots,n-1$.

We now discuss the second algorithm. Denote $a_m^{(i)}=y_i^{(m)}/m!$ and $\ell_i=x-x_i$ $(i=0,1,\ldots,n)$, and write

$$\begin{aligned} P_i(x) &= \alpha_0^{(i)}+\alpha_1^{(i)}\ell_i+\cdots+\alpha_{k-1}^{(i)}\ell_i^{k-1} \\ &= \bar{\alpha}_0^{(i)}+\bar{\alpha}_1^{(i)}\ell_{i+1}+\cdots+\bar{\alpha}_{k-1}^{(i)}\ell_{i+1}^{k-1} \qquad (2.9) \\ Q_i(x) &= \beta_0^{(i)}+\beta_1^{(i)}\ell_i+\cdots+\beta_k^{(i)}\ell_i^k \\ &= \bar{\beta}_0^{(i)}+\bar{\beta}_1^{(i)}\ell_{i+1}+\cdots+\bar{\beta}_k^{(i)}\ell_{i+1}^k \qquad (2.10) \end{aligned}$$

From conditions (ii) and (iii) in Definition 1.1, we have

$$\sum_{m=0}^{k-1} y_i^{(m)}\frac{(x-x_i)^m}{m!}-\frac{P_i(x)}{Q_i(x)}=(x-x_i)^k\sum_{j=0}^{\infty}c_j(x-x_i)^j$$

$$\sum_{m=0}^{k-1} y_{i+1}^{(m)}\frac{(x-x_{i+1})^m}{m!}-\frac{P_i(x)}{Q_i(x)}=(x-x_{i+1})^k\sum_{j=0}^{\infty}d_j(x-x_{i+1})^j$$

Substituting expressions (2.9) and (2.10) into the last two equtions yields

$$\sum_{m=0}^{k-1}\sum_{j=0}^{k} a_m^{(i)}\beta_j^{(i)}\ell_i^{m+j} - \sum_{j=0}^{k-1}\alpha_j^{(i)}\ell_i^j = \sum_{j=k}^{2k-1} r_j^{(i)}\ell_i^j \tag{2.11}$$

$$\sum_{m=0}^{k-1}\sum_{j=0}^{k} a_m^{(i+1)}\bar{\beta}_j^{(i)}\ell_{i+1}^{m+j} - \sum_{j=0}^{k-1}\bar{\alpha}_j^{(i)}\ell_{i+1}^j = \sum_{j=k}^{2k-1} \bar{r}_j^{(i)}\ell_{i+1}^j. \tag{2.12}$$

Therefore we obtain

$$\alpha_j^{(i)} = \sum_{\mu=0}^{j} a_{j-\mu}^{(i)}\beta_\mu^{(i)} \tag{2.13}$$

and

$$\bar{\alpha}_j^{(i)} = \sum_{\mu=0}^{j} a_{j-\mu}^{(i+1)}\bar{\beta}_\mu^{(i)} \tag{2.14}$$

for $j = 0, 1, \ldots, k-1$.

Denote $h_i = x_{i+1} - x_i$. From Eq. (2.9) we have

$$\begin{aligned}
\bar{\alpha}_j^{(i)} &= \left.\frac{\partial^j P_i(x)}{j!\partial x^j}\right|_{x=x_{i+1}} \\
&= \frac{1}{j!}\left(j!\alpha_j^{(i)} + \frac{(j+1)!}{1!}\,\alpha_{j+1}^{(i)}\ell_i\Big|_{x=x_{i+1}} + \frac{(j+2)!}{2!}\,\alpha_{j+2}^{(i)}\ell_i^2\Big|_{x=x_{i+1}} \right. \\
&\quad \left. + \cdots + \frac{(k-1)!}{(k-j-1)!}\,\alpha_{k-1}^{(i)}\ell_i^{k-j-1}\Big|_{x=x_{i+1}} \right) \\
&= \sum_{\nu=0}^{k-j-1}\binom{j+\nu}{\nu}\alpha_{j+\nu}^{(i)}h_i^\nu
\end{aligned} \tag{2.15}$$

for $j = 0, 1, \ldots, k-1$. Similarly, from Eq. (2.10) we obtain

$$\bar{\beta}_j^{(i)} = \sum_{\nu=0}^{k-j}\binom{j+\nu}{\nu}\beta_{j+\nu}^{(i)}h_i^\nu \tag{2.16}$$

for $j = 0, 1, \ldots, k$. Substituting (2.15), (2.16), and (2.13) into (2.14) yields

$$\sum_{\mu=0}^{k-j-1}\binom{j+\mu}{\mu}\sum_{\nu=0}^{j+\mu}\alpha_{j+\mu-\nu}^{(i)}\beta_\nu^{(i)}h_i^\mu = \sum_{\mu=0}^{j} a_{j-\mu}^{(i+1)}\sum_{\nu=0}^{k-\nu}\binom{\mu+\nu}{\nu}\beta_{\mu+\nu}^{(i)}h_i^\nu, \tag{2.17}$$

where $j = 0, 1\ldots, k-1$. We separate the left-hand side of Eq. (2.17) into two parts and write them as

$$
\begin{aligned}
& \sum_{\nu=0}^{j} \beta_\nu^{(i)} \left[\sum_{\mu=0}^{k-j-1} \binom{j+\mu}{\mu} a_{j+\mu-\nu}^{(i)} h_i^\mu \right] + \sum_{\nu=1}^{k-j-1} q_{\nu+j}^{(i)} \left[\sum_{\mu=\nu}^{k-j-1} \binom{j+\mu}{\mu} a_{\mu-\nu}^{(i)} h_i^\mu \right] \\
= & \sum_{\nu=0}^{j} \beta_\nu^{(i)} \left[\sum_{\mu=0}^{k-j-1} \binom{j+\mu}{\mu} a_{j+\mu-\nu}^{(i)} h_i^\mu \right] \\
& + \sum_{\nu=j+1}^{k-1} q_\nu^{(i)} \left[\sum_{\mu=\nu-j}^{k-j-1} \binom{j+\mu}{\mu} a_{j+\mu-\nu}^{(i)} h_i^\mu \right]. \qquad (2.18)
\end{aligned}
$$

Similarly, we can change the right-hand side of Eq. (2.17) to

$$
\sum_{\nu=0}^{j} \beta_\nu^{(i)} \left[\sum_{\mu=0}^{\nu} \binom{\nu}{\nu-\mu} a_{j-\mu}^{(i+1)} h_i^{\nu-\mu} \right] + \sum_{\nu=j+1}^{k} q_\nu^{(i)} \left[\sum_{\mu=0}^{j} \binom{\nu}{\nu-\mu} a_{j-\mu}^{(i+1)} h_i^{\nu-\mu} \right]. \qquad (2.19)
$$

Substituting expressions (2.18) and (2.19) into (2.17) yields the following equations for $j = 0, 1, \ldots, k-1$:

$$
\begin{aligned}
& \sum_{\nu=0}^{j} \beta_\nu^{(i)} \left[\sum_{\mu=0}^{k-j-1} \binom{j+\mu}{\mu} a_{j+\mu-\nu}^{(i)} h_i^\mu - \sum_{\mu=0}^{\nu} \binom{\nu}{\nu-\mu} a_{j-\mu}^{(i+1)} h_i^{\nu-\mu} \right] \\
& + \sum_{\nu=j+1}^{k-1} q_\nu^{(i)} \left[\sum_{\mu=\nu-j}^{k-j-1} \binom{j+\mu}{\mu} a_{j+\mu-\nu}^{(i)} h_i^\mu - \sum_{\mu=0}^{j} \binom{\nu}{\nu-\mu} a_{j-\mu}^{(i+1)} h_i^{\nu-\mu} \right] \\
& - \beta_k^{(i)} \left[\sum_{\mu=0}^{j} \binom{k}{k-\mu} a_{j-\mu}^{(i+1)} h_i^{k-\mu} \right] = 0. \qquad (2.20)
\end{aligned}
$$

Eqs. (2.20) is a homogeneous system of $k+1$ unknowns, $\beta_0^{(i)}$, $\beta_1^{(i)}$, $\ldots$, $\beta_k^{(i)}$, consisting of k equations. Hence, it has nontrivial solution. To simplyfy the expression of (2.20), we denote

$$
b_{j,\nu}^{(i)} := \begin{cases}
\sum_{\mu=0}^{k-j-1} \binom{j+\mu}{\mu} a_{j+\mu-\nu}^{(i)} h_i^\mu & \\
\qquad - \sum_{\mu=0}^{\nu} \binom{\nu}{\nu-\mu} a_{j-\mu}^{(i+1)} h_i^{\nu-\mu} & \text{if } 0 \le \nu \le j, \\
\sum_{\mu=\nu-j}^{k-j-1} \binom{j+\mu}{\mu} a_{j+\mu-\nu}^{(i)} h_i^\mu & \\
\qquad - \sum_{\mu=0}^{j} \binom{\nu}{\nu-\mu} a_{j-\mu}^{(i+1)} h_i^{\nu-\mu} & \text{if } j+1 \le \nu \le k-1, \\
- \sum_{\mu=0}^{j} \binom{k}{k-\mu} a_{j-\mu}^{(i+1)} h_i^{k-\mu} & \text{if } \nu = k
\end{cases} \qquad (2.21)
$$

and rewrite (2.20) as

$$
\sum_{\nu=0}^{k} b_{j,\nu}^{(i)} \beta_\nu^{(i)} = 0, \quad j = 0, 1, \ldots, k-1. \qquad (2.22)
$$

After finding

$$
\begin{aligned}
Q_i(x) &= \sum_{j=0}^{k} \beta_j^{(i)} (x-x_i)^j \\
&= \begin{vmatrix}
1 & x-x_i & (x-x_i)^2 & \cdots & (x-x_i)^k \\
b_{0,0}^{(i)} & b_{0,1}^{(i)} & b_{0,2}^{(i)} & \cdots & b_{0,k}^{(i)} \\
\vdots & \vdots & \vdots & & \vdots \\
b_{k-1,0}^{(i)} & b_{k-1,1}^{(i)} & b_{k-1,2}^{(i)} & \cdots & b_{k-1,k}^{(i)}
\end{vmatrix}, \qquad (2.23)
\end{aligned}
$$

by (2.13) we have

$$
P_i(x) = \sum_{\nu=0}^{k-1} \beta_\nu^{(i)} \left(\sum_{j=\nu}^{k-1} a_{j-\nu}^{(i)} (x-x_i)^j \right) + \beta_k^{(i)} \cdot 0 =
$$

$$
\begin{vmatrix}
\sum_{j=0}^{k-1} a_j^{(i)} t_i(x)^j & \sum_{j=1}^{k-1} a_{j-1}^{(i)} t_i(x)^j & \cdots & \sum_{j=k-1}^{k-1} a_{j-k+1}^{(i)} t_i(x)^j & 0 \\
b_{0,0}^{(i)} & b_{0,1}^{(i)} & \cdots & b_{0,k-1}^{(i)} & b_{0,k}^{(i)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
b_{k-1,0}^{(i)} & b_{k-1,1}^{(i)} & \cdots & b_{k-1,k-1}^{(i)} & b_{k-1,k}^{(i)}
\end{vmatrix},
$$

where $t_i(x) = x - x_i$. We now calculate $r_j^{(i)}$ and $\bar{r}_j^{(i)}$ in Eqs. (2.11) and (2.12). First, from (2.11) we obtain

$$
r_\mu^{(i)} = \sum_{j=0}^{k} a_{\mu-j}^{(i)} \beta_j^{(i)}, \qquad (2.24)
$$

where $\mu = k, k+1, \ldots, 2k-1$, and $a_\nu^{(i)} = 0$ for all $\nu \geq k$. Comparing the last equation with (2.23) yields

$$
r_\mu^{(i)} = \begin{vmatrix}
a_\mu^{(i)} & a_{\mu-1}^{(i)} & a_{\mu-2}^{(i)} & \cdots & a_{\mu-k}^{(i)} \\
b_{0,0}^{(i)} & b_{0,1}^{(i)} & b_{0,2}^{(i)} & \cdots & b_{0,k}^{(i)} \\
\vdots & \vdots & \vdots & & \vdots \\
b_{k-1,0}^{(i)} & b_{k-1,1}^{(i)} & b_{k-1,2}^{(i)} & \cdots & b_{k-1,k}^{(i)}
\end{vmatrix}. \qquad (2.25)
$$

Secondly, from (2.12) we have

$$
\bar{r}_\mu^{(i)} = \sum_{j=0}^{k} a_{\mu-j}^{(i+1)} \bar{\beta}_j^{(i)}, \qquad (2.26)
$$

where $\mu = k, k+1, \ldots, 2k-1$, and $a_\nu^{(i)} = 0$ for all $\nu \geq k$. Substituting (2.16) into (2.26) yields

$$
\begin{aligned}
\bar{r}_{\mu}^{(i)} &= \sum_{j=0}^{k}\left[\sum_{\nu=0}^{k-j}\binom{j+\nu}{\nu}\beta_{j+\nu}^{(i)}h_i^{\nu}\right]a_{\mu-j}^{(i+1)} \\
&= \sum_{\nu=0}^{k}\left[\sum_{j=0}^{\nu}\binom{\nu}{\nu-j}a_{\mu-j}^{(i+1)}h_i^{\nu-j}\right]\beta_{\nu}^{(i)}.
\end{aligned} \tag{2.27}
$$

Denoting $c_{\mu,\nu}^{(i+1)} = \sum_{j=0}^{\nu}\binom{\nu}{\nu-j}a_{\mu-j}^{(i+1)}h_i^{\nu-j}$ in (2.27) and using (2.23), we obtain

$$
\bar{r}_{\mu}^{(i)} = \sum_{\nu=0}^{k} c_{\mu,\nu}^{(i+1)}\beta_{\mu}^{(i)} = \begin{vmatrix}
c_{\mu,0}^{(i+1)} & c_{\mu,1}^{(i+1)} & c_{\mu,2}^{(i+1)} & \cdots & c_{\mu,k}^{(i+1)} \\
b_{0,0}^{(i)} & b_{0,1}^{(i)} & b_{0,2}^{(i)} & \cdots & b_{0,k}^{(i)} \\
\vdots & \vdots & \vdots & & \vdots \\
b_{k-1,0}^{(i)} & b_{k-1,1}^{(i)} & b_{k-1,2}^{(i)} & \cdots & b_{k-1,k}^{(i)}
\end{vmatrix}. \tag{2.28}
$$

Therefore, Eqs. (2.11) and (2.11) are evantually obtained as

$$
\begin{aligned}
\sum_{m=0}^{k-1} y_i^{(m)}\frac{(x-x_i)^m}{m!} - \frac{P_i}{Q_i} &= \sum_{j=k}^{2k-1}\frac{r_j^{(i)}}{Q_i}(x-x_i)^j, \\
\sum_{m=0}^{k-1} y_{i+1}^{(m)}\frac{(x-x_{i+1})^m}{m!} - \frac{P_i}{Q_i} &= \sum_{j=k}^{2k-1}\frac{\bar{r}_j^{(i)}}{Q_i}(x-x_{i+1})^j,
\end{aligned}
$$

from which we have the Padé spline function defined on $[a, b]$ with the form

$$
R(x) = \frac{P_0}{Q_0} + \sum_{i=0}^{n-2}\left[\sum_{\mu=k}^{2k-1}\left(\frac{\bar{r}_{\mu}^{(i)}}{Q_i} - \frac{r_{\mu}^{(i)}}{Q_{i+1}}\right)\right](x-x_{i+1})_+^{\mu}, \tag{2.29}
$$

where $r_{\mu}^{(i+1)}$, $\bar{r}_{\mu}^{(i)}$, and Q_i are given by Eqs. (2.25), (2.28), and (2.23), respectively.

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